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DIFFERENTIAL INVARIANTS AND GEOMETRY

The problem of the comparative study of geometries was clearly outlined in a very general form by RIEMANN in his Habilitationschrift in 1854. After explicitly recognizing the possibility of discrete spaces, RIEMANN limited his discourse to continuous manifolds in the sense of Analysis Situs and defined what he meant by such manifolds. This amounted to assuming that the points of any neighborhood can be represented by ordered sets of n coordinates, $(x^1, x^2, \ldots, x^n)$. He also assumed that his discourse was to use the analytic methods which involve differentials. This implies that we admit to our attention only a class of coordinate systems which are related among themselves by analytic transformations — or at least by transformations equipped with a sufficient number of derivatives. He thus had a sufficient basis for the discussion of any phenomena which could be described by means of coordinates and differentials. But his own work narrowed down to an investigation of the measure of distance and, ultimately, to the theory of quadratic differential forms.

The comparative geometry problem was again formulated in 1872 by KLEIN in his Erlanger Programm. With the same presuppositions as RIEMANN regarding the nature of the underlying manifold, KLEIN asked us to consider a group of transformations (not necessarily point transformations) in this manifold and to regard a geometry as the theory of properties of figures in the manifold which are unaltered by the transformations of this group.

This point of view was the dominant one for the first half century after it was enunciated. It effectively took account of subjects like Projective Geometry which the Riemannian point of view seemed to overlook. It was a helpful guide in actual study and research. Geometers felt that it was a correct general formulation of what they were trying to do. For they were all thinking of space as a locus in which figures were moved about and compared. The nature of this mobility was what distinguished between geometries.

With the advent of Relativity we became conscious that space need not be looked at only as a « locus in which », but that it may have a structure, a field-theory, of its own. This brought to attention precisely those Riemannian geometries about which the Erlanger Programm said nothing, namely those whose group is the identity. In such spaces there is essentially only one figure, namely the space structure as a whole. It became clear that in some respects the point of view of RIEMANN was more fundamental than that of KLEIN ([1]).

Nevertheless the hold of the Erlanger Programm upon the imagination of mathematicians is such that attempts were sure to be made to revamp the Programm so as to adapt it to the new order of things. And these attempts have had a considerable degree of success. The concept of infinitesimal parallelism which had been introduced by LEVI-CIVITA was developed and enlarged by WEYL and has been generalized by a number of mathematicians. In particular, CARTAN and SCHOUTEN have shown that there are other ways than those forseen by KLEIN of connecting up the theory of continuous groups with geometry. As CARTAN has said, we may regard a Riemannian space as a non-holonomic Euclidean space, and many of the generalizations of Riemannian spaces can be arrived at in a similar manner.

But while these new relations between group theory and geometry are important and fruitful, each new step in advance makes the whole matter seem more complicated than before. The KLEIN theory of geometry seems to be showing the same symptoms as a physical theory whose heyday is past. More and more complicated devices have to be introduced in order to fit it to the facts of nature. Its fate, I should expect, will be the same as that of a physical theory — it becomes classical and its limitations as well as its merits are recognized.

Once we have recognized that there are geometries which are not invariant theories of groups in the simple sense which we had in mind at first, we are on the way to recognize that a space may be characterized in many other ways than by means of a group. For example, there is the fundamental class of *spaces of paths* studied by EISENHART and some of my other colleagues, which are

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([1]) It should be remarked in passing (partly because this point has been commented on by SCHOUTEN, *Rendiconti del Circolo Matematico di Palermo*, Vol. 50 (1926) and CARTAN, *L'Enseignement Mathématique*, 26e Année (1927) p. 203) that the way in which the Riemannian geometries fit most naturally into the Erlanger Programm is to take as the manifold the set of points $(x^1, x^2, ..., x^n)$ and instead of the group (for it is not, strictly speaking, a group) the set of all analytic transformations *regarded as point transformations*, not as transformations of coordinates. The Riemannian spaces (or the quadratic differential forms) fall into classes of those which are equivalent under these transformations. From this point of view the theory of all Riemannian geometries is a single geometry. There is just one space and in it the various Riemannian spaces are particular figures. This way of looking at the matter is precisely analogous to the way in which Klein himself brought the theory of contact transformations into the Programm as a geometry. It is helpful in connection with the equivalence problem, but it is not a way of characterizing a particular Riemannian space by means of its group. And it was just this sort of a characterization of a projective, an affine, a Euclidean, a non-Euclidean space, that was the significant thing about the Erlanger Programm.

characterized by the presence of a system of curves such that each pair of points is joined by one and only one curve of the system. Whether or not these spaces can be characterized in other ways there can be no doubt of the significance of this way of viewing them.

If we give up the idea of making any one concept — such as the group concept — dominant in geometry, we naturally return to something like the starting point of Riemann's discussion. That is to say, we prescribe only the continuous nature of the manifold to be considered and the analytic character of the operations. There has indeed been an uninterrupted development of the Riemannian geometries along these, so to speak, unprejudiced lines. I mean the work of LIPSCHITZ, CHRISTOFFEL, RICCI and, more recently, the mathematical physicists. This work seemed to most mathematicians to be extremely formal and narrow in outlook. But it was continually developing the ideas of differential invariant theory. The definitions and terminology were at first modelled as nearly as possible on those current in algebraic invariant theory, but the growth of the subject, particularly since the applications to relativity have emphasized the importance of the systematic methods of RICCI, has led to a conception of a differential invariant which is well suited to the comparative study of geometries.

Such an invariant is an abstract object which has in each coordinate system a unique set of *components*, each component being a function of the coordinates and their differentials (1). For example a quadratic differential form is an invariant which has a single component in each coordinate system, this component being a function which is a homogeneous polynomial of degree two in the differentials and an analytic function of the coordinates. The theory of one or more such invariants is what we call a geometry.

In some cases the geometries at which we arrive by this definition will be geometries in the sense of the Erlanger Programm or one of its generalizations, and in some cases they will not. I do not regard this definition of the term geometry as anything definitive, because I regard any attempt to make a sharp definition of such a term as savoring of pedantry. I would rather say that a theory is a geometry when it is sufficiently like the classical geometry to deserve this name — and let it go at that.

Moreover the family of transformations of coordinates which underlies the definition of a differential invariant is not the only one we should consider. There are other transformations of the frame of reference, such as contact transformations, which have a right to consideration. But the definition of a differential invariant which we have adopted is sufficiently general so that with whatever descriptive idea of a space you may choose to begin, you are likely

(1) This conception of a differential invariant is discussed at greater length in Chap. II of my recent Cambridge Tract, *Invariants of Quadratic Differential Forms*, Cambridge, 1927.

to find in working it out that you must come to grips with the theory of a particular differential invariant.

Let us consider some of the differential invariants which go with the classical geometry. First of all there is a quadratic differential form. In each coordinate system this invariant has one component, namely a function

$$g_{ij}\, dx^i\, dx^j \tag{1}$$

of the coordinates and their differentials. If there is a coordinate system in which the component is simply the sum of the squares of the differentials, the differential form is said to be Euclidean. In the neighborhood of any point this differential form determines a unique Euclidean space, but it also determines a unit of length. So it is not quite accurate to say that the Euclidean geometry is the theory of this quadratic differential form. The Euclidean space and the unit of length together determine a unique quadratic differential form. The Euclidean space by itself determines an infinite class of differential forms such that in each coordinate system they have components,

$$\sigma\, g_{ij}\, dx^i\, dx^j, \tag{2}$$

one for each choice of the function σ of the coordinates.

In each coordinate system we may choose a unique one of the components (2) by the requirement that the determinant of the n^2 quantities $\sigma\, g_{ij}$ shall be equal to unity. This detemines for each coordinate system a unique function

$$G_{ij}\, dx^i\, dx^j$$

and therefore another invariant which has this function as its component in each coordinate system. This invariant is *a relative quadratic form of weight* $-2/n$. Its components in any two coordinate systems x and $\bar{x}$ are related by the formula

$$\bar{G}_{ij}\, d\bar{x}^i\, d\bar{x}^j = \left| \frac{\partial x}{\partial \bar{x}} \right|^{-2/n} G_{ij}\, dx^i\, dx^j. \tag{3}$$

The Euclidean geometry uniquely determines this invariant, but it would not be correct to say that the Euclidean geometry is the theory of this invariant. For, as was first remarked by T. Y. THOMAS (1), the theory of a relative quadratic form of weight $-2/n$ is conformal geometry. In the case before us the Euclidean conformal group is the group of all transformations between coordinate systems in which the component of our relative differential form is the sum of the squares of the differentials. The Euclidean group (of similarity transformations) is the subgroup of linear transformations of this group. In other words, we cannot have Euclidean geometry until we distinguish between circles and straight lines.

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(1) *Proceedings of the National Academy of Sciences*, Vol. 11 (1925) p. 722.

The differential equations of the straight lines are

$$\frac{d^2x^i}{dt^2}=0 \tag{4}$$

in cartesian coordinates and

$$\frac{d^2x^i}{dt^2}+\Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}=0 \tag{5}$$

in arbitrary coordinates. In each coordinate system there is one and only one set of funtions Γ^i_{jk} and the sets of functions in any two coordinate systems are connected by a simple law of transformation. The functions Γ are terefore the components of an invariant, which is called *an affine connection*, the theory of this invariant being *affine geometry*. If the components of an affine connection vanish identically in one coordinate systems, they vanish identically in all coordinate systems related to this one by linear transformations.

The Euclidean geometry may now be characterized exactly as the simultaneous theory of a particular relative quadratic form of weight $-2/n$ and a particular affine connection. There must be a coordinate system in which the components of affine connection are all zero. The Euclidean geometry is what is common to this conformal, and this affine, geometry.

A geometer cannot help remarking at this point that we may replace affine by projective geometry in the above statement. Projective geometry is the theory of the straight lines free from some of the restrictions imposed by the affine treatment. One of these restrictions is that the differential equations (4) imply a particular assignment of the parameter t to the points of the line ([1]). If the parameter is to be assigned arbitrarily, the differential equations become

$$\frac{d^2x^i}{dt^2}\Big/\frac{dx^i}{dt}=\varphi\left(x,\frac{dx}{dt}\right), \tag{5}$$

where φ is an arbitrary function, homogeneous of degree one in the quantities dx^i/dt. This amounts to changing the components of affine connection from Γ^i_{jk} into

$$\Gamma^i_{jk}+\delta^i_j\,\varphi_k+\delta^i_k\,\varphi_j$$

where φ_j is homogeneous of degree zero in dx^i/dt. None of these changes affect the quantities

$$\Pi^i_{jk}=\Gamma^i_{jk}-\frac{1}{n+1}\left(\Gamma^a_{aj}\,\delta^i_k+\Gamma^a_{ak}\,\delta^i_j\right),$$

which are thus uniquely determined by the system of straight lines. These quan-

([1]) The question of the parametrization of systems of paths is very clearly discussed by J. DOUGLAS, *Annals of Math.*, Vol. 29 (1928) p. 143.

tities (1) are the components of an invariant, which may be called a *projective connection*, with a law of transformation which is somewhat more complicated than that of an affine connection. If the components of a projective connection vanish identically in one coordinate system, they vanish identically in all coordinate systems related to this one by linear fractional tranformations. The classical projective geometry is the theory of a projective connection for which there exists a coordinate system in which its components are identically zero.

Beside the affine and the projective connections we must place another invariant called the *conformal connection* (2), whose components can be given in terms of the conformal relative tensor, G_{ij}, by the formula for Christoffel symbols of the second kind. If its components are identically zero in one coordinate system they are identically zero in all coordinate systems related to this one by a set of transformations (3) which contains the conformal group as sub-group and, indeed, is related to the conformal group in much the same way that the affine group is related to the Euclidean group.

I have now mentioned five invariants connected in an intimate way with the Euclidean geometry, (1) an absolute quadratic differential form, (2) a relative quadratic differential form of weight $-2/n$, (3) an affine connection, (4) a projective connection, (5) a conformal connection. Each of these invariants is specialized in an obvious way: the first two so that in some coordinate system their components are sums of squares of the differentials, the last three so that all their components shall be zero in some coordinate system.

In each case, if we drop the restriction imposed by its application to the Euclidean geometry, we obtain a class of invariants each of which has a theory which is a geometry in the generalized sense. In the first case we obtain the

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(1) These quantities were introduced by T. Y. Thomas, *Proc. Nat. Ac. of Sc.,* Vol. 11 (1925) p. 199. Their law of transformation is

$$\bar{\Pi}^i_{jk} = \Pi^a_{bc} \frac{\partial \bar{x}^i}{\partial x^a} \frac{\partial x^b}{\partial \bar{x}^j} \frac{\partial x^c}{\partial \bar{x}^k} + \frac{\partial^2 x^a}{\partial \bar{x}^j \partial \bar{x}^k} \frac{\partial \bar{x}^i}{\partial x^a} - \frac{1}{n+1} \left(\delta^i_j \frac{\partial \log \left| \frac{\partial x}{\partial \bar{x}} \right|}{\partial \bar{x}^k} + \delta^i_k \frac{\partial \log \left| \frac{\partial x}{\partial \bar{x}} \right|}{\partial \bar{x}^j} \right).$$

(2) The conformal connection was introduced by J. M. Thomas, *Proc. Nat. Ac. of Sc.*, Vol. 11 (1925) p. 257. It has the law of transformation,

$$\bar{K}^i_{jk} = K^a_{bc} \frac{\partial \bar{x}^i}{\partial x^a} \frac{\partial x^b}{\partial \bar{x}^j} \frac{\partial x^c}{\partial \bar{x}^k} + \frac{\partial^2 x^a}{\partial \bar{x}^j \partial \bar{x}^k} \frac{\partial \bar{x}^i}{\partial x^a} - \frac{1}{n} \left(\delta^i_j \frac{\partial \log \left| \frac{\partial x}{\partial \bar{x}} \right|}{\partial \bar{x}^k} + \delta^i_k \frac{\partial \log \left| \frac{\partial x}{\partial \bar{x}} \right|}{\partial \bar{x}^j} - \bar{G}_{jk} \bar{G}^{ip} \frac{\partial \log \left| \frac{\partial x}{\partial \bar{x}} \right|}{\partial \bar{x}^p} \right).$$

The formula for its components in terms of the G's is due to T. Y. Thomas.

(3) *Note added 3 May, 1929*: In the paper refened to in the last footnote below, I called this set the enlarged conformal gronp. But as Professor Weyl has comteonaly pointed ont, it is not a gronp and my argument did not actually essume that it wasone.

Riemannian geometries, in the second and fifth cases the generalized conformal geometries, in the third case the generalized affine geometries, in the fourth case the generalized projective geometries.

These are some, but by no means all, of the geometries that arise by the process which we are considering, namely, to find a differential invariant which is significant for an aspect of elementary geometry and then to remove the restrictions which tie this invariant to the elementary geometry.

It would be interesting to compare these geometries with those studied by CARTAN, SCHOUTEN, and others. But this would hardly be possible in a short address, and besides it would involve questions of interpretation about which I am not perfectly sure. In any case, my point is merely that the differential invariant approach to these geometries is a significant one, not that it is a unique or a dominant one.

It has among other merits that of determining a straight-forward method of working out each geometry in detail. We know how this has been done in the affine case. The first step is to determine a suitable class of invariants in terms of which to state the properties of particular affine geometries. These invariants are the tensors. They have a law of transformation characterized by an isomorphism between the totality of analytic transformations at any point and the group of linear homogeneous transformations,

$$X^i = u^i_j X^j,$$

which we have already seen to be associated intimately with an affine connection. The isomorphism is determined by the equations,

$$\text{(A)} \qquad u^i_j = \frac{\partial x^i}{\partial \bar{x}^j}.$$

The second step is to find a tensor, the curvature tensor, which is an invariant of the basic invariant, and the third step to find a recursive process (such as covariant differentiation or the process of forming extensions by the method of normal coordinates) for generating a complete sequence of tensor invariants of the basic invariant.

These steps can all be paralleled in the projective and the conformal cases. In the projective case we first discover a unique process of associating a linear fractional transformation

$$X^i = \frac{u^i_j X^j}{1 + u^0_j X^j}$$

at each point with each analytic transformation. This amounts to defining the quantities u^α_β by the equations (A) and

$$\text{(B)} \qquad u^0_i = \frac{\partial \log u}{\partial \bar{x}^i}, \qquad u = \left| \frac{\partial x}{\partial \bar{x}} \right|.$$

The $(n+1)$-rowed square matrices of the coefficients u^α_β of these transformations can be used in exactly the same way as the n-rowed matrices u^i_j of (A) to define invariants with linear laws of transformation. The invariants thus defined are formally analogous to the classical affine tensors, and so may be called *projective tensors* (1). A projective tensor has $(n+1)^k$ components in each coordinate system, instead of n^k. The next step is to find a process of projective differentiation analogous to covariant differentiation which gives rise to an infinite sequence of projective tensors. In this process we use an invariant called the *extended projective connection* with $(n+1)^3$ components which is in a simple relationship with the original projective connection. By a suitable elimination between the law of transformations of this invariant and that of the derivatives of the components of a projective tensor we find a formula which leads from any given projective tensor to another projective tensor with one more covariant index. This is the process of projective differentation. Since it can be repeated indefinitely, it leads from any projective tensor to an infinite sequence of projective tensors.

By forming the integrability conditions of the law of transformation of the extended projective connection we obtain a projective tensor analogous to the curvature tensor. Its components include those of the curvature tensor for projective geometry discovered by WEYL (2). With this tensor and the recursive projective differentiation process we have a method of getting a complete set of invariants for generalized projective geometry in a form that is accessible to analysis.

In conformal geometry also an analogous theory can be developed. The conformal connection determines a special set of transformations just as the affine and projective connections determine the affine and projective groups respectively, and an isomorphism between this set and the totality of transformations of coordinates determines a class of invariants with $(n+2)^k$ components. These are the *conformal tensors.* There is also an extended conformal connection and conformal differentiation. In this case the extended conformal connection has $(n+2)^2(n+1)$ components, and in order to complete the conformal differentiation process we have to determine some of the components of the conformal derivative by imposing a further invariant condition,

$$G_{\alpha\beta}\, T^\alpha\, T^\beta = 0$$

for example. As in the projective case we arrive at formulas which include and

(1) The projective tensors were introduced by T. Y. THOMAS, *Math. Zeitschrift*, Vol. 25 (1926) p. 723, and also have been used implicitly by the writers on Five-dimensional Relativity, cf. O. KLEIN, *Zeitschrift für Physik*, Vol. 46 (1927) p. 188. For the developments referred to in the text, cf. VEBLEN, *Proc. Nat. Ac. of Sc.* Vol. 14 (1928) p. 154.

(2) H. WEYL, Göttinger Nachrichten, 1921, p. 99.

clarify those obtained by the geometers who have been studying the question from the point of view of infinitesimal displacements. But there is no time in a short address like this to give details. I must refer you to the papers in which some of them have been worked out (1).

The main point which I wish to make is that there is still vitality in the generalized Riemannian view of geometry, and that there are invariants, as yet but little known, which have simple laws of transformation and applications to geometry of a quite elementary type.

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(1) On the conformal geometry see my paper in *Proc. Nat. Ac. of Sc.*, Vol. 14 (1928) p. 735, and the earlier papers by T. Y. THOMAS and J. M. THOMAS which are cited there.