HOMOTOPY GROUPS AND ALGEBRAIC HOMOLOGY THEORIES

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This note will present certain topological results obtained by Saunders MacLane and the author. Some of the algebraic aspects of these questions are presented by MacLane in another note in these Proceedings.

Let X be an arcwise connected topological space with base point x_0 and with vanishing homotopy groups $\pi_i(X)$ for $1 \leqq i < n$. The singular homology and cohomology groups of X may then be derived from the singular complex $S_n(X)$ consisting of the singular simplexes whose faces of dimension less than n all degenerate to the point x_0. A q-simplex T of $S_n(X)$ determines a system of "labels" consisting of elements of $\pi_n = \pi_n(X)$ attached to each n-dimensional face of T. The alternating sum of the labels on the faces of an $(n + 1)$-face of T are zero. Such a system of labels may be regarded as an abstract simplex of a complex $K(\pi_n, n)$; this is a purely algebraic construction on the group $\Pi = \pi_n$ and the integer n. The function which to each simplex of $S_n(X)$ assigns its system of labels yields a simplicial mapping $\kappa: S_n(X) \rightarrow K(\pi_n, n)$. Each n-dimensional simplex of $K(\pi_n, n)$ consists of a single label; i.e., of an element of π_n. This yields the basic cohomology class $b^n \in H^n(\pi_n, n; \pi_n)$ of the complex $K(\pi_n, n)$ with coefficients in π_n, and the basic cohomology class $s^n = \kappa^* b^n \in H^n(X; \pi_n)$ of X with coefficients in π_n.

We shall further assume that the homotopy groups $\pi_i(X)$ vanish also for $n < i < q$. Then every simplex of $K(\pi_n, n)$ of dimension less than or equal to q can be realized geometrically in $S_n(X)$ and this yields an *inverse* simplicial mapping $\bar{\kappa}: K(\pi_n, n) \rightarrow S_n(X)$ defined in dimensions less than or equal to q. Using this map we have shown[2] that the homology and cohomology groups of X in dimensions less than q (and also partially in dimension q) are those of $K(\pi_n, n)$. In attempting to extend $\bar{\kappa}$ to the dimension $q + 1$ one encounters an obstruction which is a cohomology class $k_n^{q+1} \in H^{q+1}(\pi_n, n; \pi_q)$ of the complex $K(\pi_n, n)$ with coefficients in $\pi_q = \pi_q(X)$.

Let K be a (possibly infinite) simplicial complex with ordered vertices and $f: K^n \rightarrow X$ a continuous mapping of the n-skeleton of K. Without loss of generality we may assume that $f(K^{n-1}) = x_0$. If the map f is extendable to a map $K^{n+1} \rightarrow X$, then the cohomology class $f^* s^n \in H^n(K^n; \pi_n)$ determines uniquely a cohomology class $f^{\blacktriangle} s^n \in H^n(K; \pi_n)$. If further $g: K^n \rightarrow X$ is another such map which agrees with f on a subcomplex L of K, then a relative cohomology class $(f - g)^{\blacktriangle} s^n \in H^n(K, L; \pi_n)$ is uniquely determined.

Let $f: K^n \cup L \rightarrow X$ be a map extendable to a map $K^{n+1} \cup L \rightarrow X$. For each

[1] John Simon Guggenheim Memorial Fellow.

[2] S. Eilenberg and S. MacLane, ***Relations between homology and homotopy groups of spaces* I *and* II**, **Ann. of Math. vol. 46 (1945) pp. 480–509 and vol. 51 (1950) pp. 514–533.**

simplex Δ of K and each n-face of Δ, f determines a label which is an element of π_n. These labels form a simplex of $K(\pi_n, n)$, thus yielding a simplicial map $\tilde{f}$: $K \to K(\pi_n, n)$. It can also be shown that f is extendable to a map f': $K^q \cup L \to X$. The obstruction $c^{q+1}(f') \in H^{q+1}(K, L; \pi_q)$ is independent of the choice of f' and is called the secondary obstruction $z^{q+1}(f)$ of f.

THEOREM 1. *Let K be a simplicial complex, L a subcomplex, f, g: $K^n \cup L \to X$ maps such that $f(K^{n-1}) = g(K^{n-1}) = x_0$ and $f \mid L = g \mid L$. If both f and g are extendable to maps $K^{n+1} \cup L \to X$, then their secondary obstructions satisfy*

$$z^{q+1}(f) - z^{q+1}(g) = (\tilde{f} - \tilde{g})^* k_n^{q+1}.$$

Here $(\tilde{f} - \tilde{g})^*$ is the difference homomorphism $H^{q+1}(\pi_n, n; \pi_q) \to H^{q+1}(K, L, \pi_q)$ induced by the maps $\tilde{f}$ and $\tilde{g}$ which agree on L.

The above theorem yields interesting results only if one has more information about the element k_n^{q+1} and about the group $H^{q+1}(\pi_n, n; \pi_q)$ in which it lies. The latter is closely related with the homology theory of abelian groups based on the complexes $A(\Pi)$ and $A^1(\Pi)$ of the aforementioned note of MacLane.

We limit our attention here to the case $q = n + 1$. In this case we have natural isomorphisms

$$H^4(\Pi, n; G) \approx H^3(A^1(\Pi); G)$$

$$H^{n+2}(\Pi, n; G) \approx H^3(A(\Pi); G), \qquad n > 2.$$

Each element χ of $H^{n+2}(\Pi, n; g)$ yields a trace t which is a function defined on Π with values in G and which satisfies the conditions

(1) $t(x) - t(-x) = 0$

(2) $t(x + y + z) - t(y + z) - t(x + z) - t(x + y) + t(x) + t(y) + t(z) = 0$ for $n = 2$. For $n > 2$, (2) is replaced by the stronger condition

(2′) $t(x + y) - t(y) - t(x) = 0$.

In particular, we consider the element $k_n^{n+2} \in H^{n+2}(\pi_n, n; \pi_{n+1})$, and prove that the trace of k_n^{n+2} is J. H. C. Whitehead's function η: $\pi_n \to \pi_{n+1}$ obtained by combining each map $S^n \to X$ with a map $S^{n+1} \to S^n$ that yields a generator of $\pi_{n+1}(S^n)$.[3]

The correspondence $\chi \to t$ yields an isomorphic mapping of the group $H^{n+2}(\Pi, n; G)$ onto the group of all functions satisfying conditions (1) and (2) (or (1) and (2′) if $n > 2$). If the abelian group Π is finitely generated, then the inverse mapping may be described as follows.

Let K be a complex with sufficient simplicial structure, L a subcomplex, and let t: $\Pi \to G$ be a function satisfying (1) and (2). Then if Π is finitely generated, one may define a "Pontrjagin square"

$$P_t: H^q(K, L; \Pi) \to H^{2q}(K, L; G), \qquad q \text{ even},$$

[3] Cf. G. W. Whitehead, *On spaces with vanishing low-dimensional homotopy groups*, Proc. Nat. Acad. Sci. U. S. A. vol. 34 (1948) pp. 207–211; Theorem 5.

which itself satisfies (1) and (2). If t satisfies (1) and (2′), then one has a "Steenrod square"

$$Sq_t^2 \colon H^q(K, L; \Pi) \to H^{q+2}(K, L; G)$$

which itself satisfies (1) and (2′). If $b^n \in H^n(\Pi, n; \Pi)$ is the basic cohomology class of $K(\Pi, n)$, then the elements $P_t(b^2)$ and $Sq_t^2(b^n)$ for $n > 2$ of $H^{n+2}(\Pi, n; G)$ have precisely the trace t.

These results combined with some formal properties of P_t and Sq_t^2 and with Theorem 1 yield the following theorem.

THEOREM 2. *Let X be an arcwise connected topological space with $\pi_i(X) = 0$ for $i < n$ ($n > 1$) and with $\pi_n(X)$ finitely generated. Let K be a simplicial complex, L a subcomplex, and $f, g\colon K^n \cup L \to X$ two maps extendable to maps $K^{n+1} \cup L \to X$ which agree on L. Then their secondary obstructions satisfy*

$$z^{n+2}(f) - z^{n+2}(g) = \begin{cases} P_\eta(\lambda^2) + g^\blacktriangle(s^2) \cup \lambda^2, & n = 2, \\ f^\blacktriangle(s^2) \cup \lambda^2 - P_\eta(\lambda^2), & n = 2, \\ Sq_\eta^2(\lambda^n), & n > 2, \end{cases}$$

where $s^n \in H^n(X; \pi_n)$ is the basic cohomology class of X, $\lambda^n = (f - g)^\blacktriangle s^n$, and $\cup$ denotes the ordinary cup product relative to the pairing $[x, y] = \eta(x + y) - \eta(x) - \eta(y)$.

In order to deduce from this theorem a classification theorem for maps $K^{n+1} \to X$ we need the "Postnikov square" which is a homomorphism

$$\overline{P}_t\colon H^q(K, L; \Pi) \to H^{2q+1}(K, L; G), \qquad q \text{ odd},$$

defined for each t satisfying conditions (1) and (2).

THEOREM 3. *Let X be as in Theorem 2 and let $f, g\colon K^{n+1} \to X$ be two maps which agree on $K^n \cup L$. Let $d^{n+1}(f, g) \in H^{n+1}(K, L; \pi_{n+1})$ be the cohomology class measuring the difference between f and g. Then f and g are homotopic relative to L if and only if there is a cohomology class $e^{n-1} \in H^{n-1}(K, L; \pi_n)$ such that*

$$d^{n+1}(f, g) = \begin{cases} \overline{P}_\eta(e^1) + f^*(s^2) \cup e^1, & n = 2, \\ Sq_\eta^2(e^{n-1}), & n > 2. \end{cases}$$

Theorems 2 and 3 constitute a generalization of Steenrod's results[4] for the case $X = S^n$. It should be noted that the method also is in a sense a generalization

[4] N. E. Steenrod, *Products of cocycles and extensions of mappings*, Ann. of Math. vol. 48 (1947) pp. 290–320.

of that of Steenrod: the complexes M^n are replaced here by the algebraic complexes $K(\pi_n, n)$. Our results are almost identical with results recently obtained by J. H. C. Whitehead by a different method and include earlier results of Whitney[5] and Postnikov.[6]

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[5] H. Whitney, *Classification of the mappings of a 3-complex into a simply connected space*, Ann. of Math. vol. 50 (1949) pp. 270–284.

[6] M. M. Postnikov, *Classification of the continuous mappings of an arbitrary n-dimensional polyhedron into a connected topological space which is aspherical in dimensions greater than unity and less than n*, C. R. (Doklady) Acad. Sci. URSS N.S. vol. 67 (1949) pp. 427–430