Ibn Sīnā’s syllogistic: a logic between modal and many-valued

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`www.maths.qmul.ac.uk/~wilfrid/ismodal.pdf`

Components:

(a) The mathematical content (mostly implicit).

(b) The proof theory.

(c) The semantics—describing the world so as to apply the logic.

We will concentrate on (a) using modern tools,
with some briefer remarks on (c).
(See the Street reference below for (b).)

Ibn Sīnā (Avicenna) was born in Uzbekhistan in 981,
and spent most of his life in Persia, writing in Arabic.
He died in 1037.

He wrote one large (c. 600 pages) text of logic and
several shorter ones,
mostly on modal logic and its foundations.

The mathematics: basic idea

We allow that a relation R can hold with different degrees.
For example ‘x is red’ yields:

- x could never be anything other than red.
- x is always red.
- x is red.
- x is sometimes red.
- x is or could be red.

Instead of letting $R(\bar{t})$ hold with different values,
we let copies of R represent the different values.

This device works for any logic, but for Ibn Sīnā the relevant logic is syllogisms.
There are four kinds of syllogistic sentence

$$
\begin{array}{lll}
1. & \forall x(Px \to Qx) & (P\downarrow Q\uparrow) \\
2. & \forall x(Px \to \neg Qx) & (P\downarrow Q\downarrow) \\
3. & \exists x(Px \wedge Qx) & (P\uparrow Q\uparrow) \\
4. & \exists x(Px \wedge \neg Qx) & (P\uparrow Q\downarrow)
\end{array}
$$

The notation on the right indicates whether P or Q occurs positively $\uparrow$ or negatively $\downarrow$.

The theory of syllogisms describes the order-inconsistent sets of *three* syllogistic sentences.

The order-sentence $\forall x(R_j x \to R_i x)$ is $(R_j \downarrow R_i \uparrow)$.

So the theory of syllogisms in σ^+ reduces to the theory of inconsistent sets of *six* syllogistic sentences.

Arbitrary n is just as easy as 6,
and proofs for n will make clearer what are the general logical principles involved.

Formal description:

Let σ be a set of relation symbols (a signature).

Then σ^+ is the same as σ except that each symbol R is replaced by copies $R_1, R_2, \ldots$ of the same arity (the number of copies won't matter).

A σ^+-structure A is *orderly* if it satisfies

$$A \models \forall \bar{x}(R_j\bar{x} \to R_i\bar{x})$$

whenever $i \leqslant j$.

We call these sentences the *order-sentences*.

A σ^+-theory T is *order-consistent* if it has an orderly model; otherwise it's *order-inconsistent*.

So T is order-inconsistent if and only if $T \cup S$ is inconsistent, where S is the set of relevant order-sentences.

Lemma (Laws of Distribution) If T is an inconsistent circular theory then every relation symbol in T occurs once positively and once negatively.

Proof Otherwise say P occurs twice positively (or twice negatively), say in ϕ and in $(T \setminus \{\phi\})$. We have

$$(T \setminus \{\phi\}) \ \vdash \ \neg\phi.$$

By Lyndon Interpolation Theorem there is θ so that

$$(T \setminus \{\phi\}) \ \vdash \ \theta \vdash \neg\phi$$

where every relation symbol positive (resp. negative) in θ is positive (resp. negative) in both $(T \setminus \{\phi\})$ and $\neg\phi$.

So by assumption P doesn't occur in θ.

In ϕ replace P by new symbol P', getting ϕ'. Then

$$(T \setminus \{\phi\}) \ \vdash \ \theta \vdash \neg\phi'$$

and hence

$$(T \setminus \{\phi\}) \cup \{\phi'\}$$

is inconsistent. But this is impossible, because the theory is linear. □

Footnote: Apart from use of Lyndon, this argument is sketched in *Port-Royal Logic*, Arnauld and Nicole 1662.

Henceforth *theory* means set of syllogistic sentences where each relation symbol occurs *at most twice*, and *at most once in each sentence*.

A theory T has a directed graph $\Gamma(T)$:

> The vertices are the relation symbols used in T.
> An edge from P to Q is a sentence in T which has P on left and Q on right.

Ignoring directions, $\Gamma(T)$ falls into connected components, each of which is either linear or circular.
T is inconsistent if and only if at least one connected component is inconsistent.

Lemma Every linear theory T has a 2-element model in which each symbol is interpreted as a singleton.

Proof Domain of A is $\{0, 1\}$. Take a sentence $\phi \in T$.
By inspection we can interpret the symbols of ϕ in A as singletons, to make ϕ true.

If sentence ψ is adjacent to ϕ, one symbol of ψ is already interpreted as a singleton. Again inspection shows we can interpret the other symbol of ψ as a singleton, making ψ true. □

Footnote: Skolemisation, in a form adequate for this argument, appears for the first time in the work of Ibn Sīnā’s successor Suhrawardī, who was murdered in 1183 by order of Saladin.

Suhrawardī used it to bring all theories to universal form, essentially as in applications of the resolution calculus.

Other murdered logicians include Montague, Van Heijenoort, Kurepa and probably Lindenbaum and Gentzen.

So an inconsistent circular theory T of size n has one sentence $(\uparrow\ -)$ and $(n-1)$ sentences $(\downarrow\ -)$.

Hence by Distribution it has one sentence of the form $(-\ \downarrow)$ and $(n-1)$ sentences of the form $(-\ \uparrow)$.

Lemma If T is an inconsistent circular theory, then exactly one sentence in T has the form $(\uparrow\ -)$.

Proof Suppose for example T has just the two existential sentences $\exists x\phi$ and $\exists x\psi$.
Introduce distinct Skolem constants c_ϕ, c_ψ.
Let T^ϕ and T^ψ be as follows.

$$\begin{array}{lllll}
T: & \exists x\phi & \exists x\psi & \forall x\chi & \dots \\
T^\phi: & \phi(c_\phi) & & \chi(c_\phi) & \dots \\
T^\psi: & & \psi(c_\psi) & \chi(c_\psi) & \dots
\end{array}$$

The disjoint union of a one-element model of T^ϕ and a one-element model of T^ψ is a model of T.
But T^ϕ is also a Skolem theory for $T \setminus \{\exists x\psi\}$, which is linear.
So T^ϕ has a model, and hence (being universal) a one-element model.
Likewise T^ψ. So T has a model, contradiction.

Hence T contains at most one existential sentence.

If T has no existential sentences, then any structure in which all relation symbols have empty interpretation is a model of T. $\square$

Theorem The conditions in the lemmas are necessary and sufficient for inconsistency.

Proof of sufficiency If T meets the conditions and contains $\geqslant 3$ sentences, then it contains at least one $(P \downarrow Q \uparrow)$. The other occurrence of P is positive, say in $\chi(P)$. Then

$$\forall x(Px \to Qx), \chi(P) \;\vdash\; \chi(Q).$$

Removing $(P \downarrow Q \uparrow)$ and replacing $\chi(P)$ by $\chi(Q)$ preserves the conditions. So eventually we reduce to a 2-sentence theory meeting the conditions. By inspection all such theories are inconsistent. $\square$

Variant, contracting the righthand side

$$\begin{array}{ll}
 & \exists x(Ax \wedge \neg Ix) \\
\forall x(Ax \to Cx) & \\
\forall x(Cx \to Dx) & \\
\vdots & \\
\forall x(Hx \to Ix) &
\end{array}$$

Typical inconsistent theory (with its graph)

$$\begin{array}{lcl}
 & \exists x(Ax \wedge Bx) & \\
\forall x(Ax \to Cx) & & \forall x(Bx \to Zx) \\
\forall x(Cx \to Dx) & & \forall x(Zx \to Yx) \\
\vdots & & \vdots \\
\forall x(Hx \to Ix) & & \forall x(Qx \to Px) \\
 & \forall x(Px \to \neg Ix) &
\end{array}$$

Typical example

The inconsistent theory

$$\exists x(P_k x \wedge \neg R_\ell x), \forall x(P_i x \to Q_j x), \forall x(Q_m x \to R_n x)$$

yields the sequent

$$\forall x(P_i x \to Q_j x), \forall x(Q_m x \to R_n x) \;\vdash\; \forall x(P_k x \to R_\ell x)$$

This is valid if and only if $i \leqslant k$ and $m \leqslant j$ and $\ell \leqslant n$. Ibn Sīnā gets such calculations right (in his own terminology).

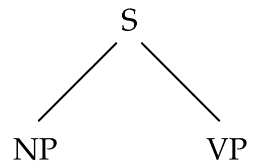
It also excludes the general monotonicity law

$$\forall x(Px \to Qx), \chi(P) \ \vdash \ \chi(Q).$$

where P is positive in $\chi(P)$.

In place of general monotonicity,
Ibn Sīnā uses four specific instances known as the *perfect* (i.e. self-evident) *syllogisms*.
Two of them have $\neg Q$ for Q.

Ibn Sīnā accepts the standard analysis of simple sentences:



NounPhrase and VerbPhrase each carry a criterion for what things they are true of.

This first level of analysis also includes whatever the sentence says about the relation between these criteria, at least so far as it is used in reasoning.

Bug

Ibn Sīnā uses

> $\forall x(Px \to Qx) \wedge \exists x Px$ instead of $\forall x(Px \to Qx)$;
> $\exists x(Px \wedge \neg Qx) \vee \forall x \neg Px$ instead of $\exists x(Px \wedge \neg Qx)$.

This causes only limited changes,
but a lot of extra work to show it.

The main change is that $\exists x(Px \wedge Qx)$ follows from $\forall x(Px \to Qx) \wedge \exists x Px$.

Natural language reasoning

Reasoning is done by processing natural language sentences.

So (for Ibn Sīnā) a single step of reasoning involves only a single step of analysis of each sentence.

This excludes rules like

$$\frac{\forall x \phi(x)}{\phi(c)}$$

(The first examples of ‘deep’ rules in the West are 19th c.)

Hence Ibn Sīnā reads our sentence

Some P is not necessarily-Q.

as

Possibly some P is Q.

This makes it all the more remarkable that he accepts as valid the same modal syllogisms as we do.

Much of Ibn Sīnā's logic is semantic discussion on how to apply syllogisms to sentences containing various 'conditions'.

This involved a study of event structure among other things.

After Ibn Sīnā, 'logic' in the Arab world largely meant cataloguing this semantic work of Ibn Sīnā.

The ginger cat sat on the mat.
My cat has never sat on the mat.

The ginger cat is not my cat.

Hence the first level of analysis includes (a) tense, (b) quantification over times, (c) negation.

It also includes modality, since otherwise we couldn't do one-step modal reasoning.

This extra information uncovered by the first step of analysis is called 'conditions' (*shurūṭ*).

It is *not* included in the VP criterion (which is a black box at this level of analysis).

Ibn Sīnā also argues that any modal condition is outside the scope of the negation, which is why we can put it at the beginning of the sentence ('It's possible that . . . ').

Rescher:

> "Clearly, the Arabic logicians of the Middle Ages were in possession of a complex theory of temporal modal syllogisms, which they elaborated in great and sophisticated detail."

Really? Resourceful, maybe.

References

- Shams C. Inati, *Ibn Sīnā Remarks and Admonitions I: Logic*, Pontifical Institute of Mediaeval Studies, Toronto 1984.
- Tony Street, 'An outline of Avicenna's syllogistic', *Archiv für Geschichte der Philosophie* 84 (2002) 129–160.
- Suhrawardī, *The Philosophy of Illumination*, trans. Wallbridge and Ziai, Brigham Young University Press, Provo Utah 1999.

I'm intending to put the Sprenger translation of Qazwīnī's *Shamsiyya* on my website in the next few days.

Example (illustrating an argument-form in al-Qazwīnī al-Kātibī, died 1276):

- Every writer moves whenever he writes.
- Everything that sometimes moves, sometimes makes a noise while it's moving.
- *Therefore* Every writer sometimes makes a noise.

Put $P_i(x) \equiv P_k(x) \equiv$ 'x is a writer';
$Q_j(x) \equiv$ 'x moves when writing, which he sometimes does';
$Q_m(x) \equiv$ 'x sometimes moves';
$R_n(x) \equiv$ 'x sometimes makes a noise while moving';
$R_\ell(x) \equiv$ 'x sometimes makes a noise'.

Then $i \leqslant k$ and $m \leqslant j$ and $\ell \leqslant n$.

These are exactly the conditions for validity of

$$\forall x(P_i x \to Q_j x), \forall x(Q_m x \to R_n x) \;\vdash\; \forall x(P_k x \to R_\ell x).$$

So the argument is valid.