The geometry of music

1. Music takes place in a two-dimensional space, where the dimensions are time and pitch. Usually we write time from left to right, and higher pitch is higher on the page. (So time is on the x axis and pitch on the y axis.) Conventional western music uses only certain pitches and rhythms.

Edward Elgar, 'Enigma Variations: xi, GRS', EMI 1983.
George Gershwin, 'Rhapsody in Blue', with Paul Whiteman, Gemm 1991.
Fairuz, 'Ya mersal el marassil', Chahine 1993.

2. Since the time of Felix Klein (1849–1925), geometers have classified spaces and their contents by applying transformations and seeing what stays fixes and what moves. We multiply transformations S and T by doing S first and then T; we invert T by doing T backwards. A *transformation group* consists of a nonempty set of transformations of a space, closed under multiplication and inverse. For musical space one of the most important groups is the *Klein Four-group* with multiplication table

$\times$	I	M_h	M_v	R
I	I	M_h	M_v	R
M_h	M_h	I	R	M_v
M_v	M_v	R	I	M_h
R	R	M_v	M_h	I

3. The transformation M_h moves points in time, as if they were reflected in a mirror placed vertically. Applying M_h to a piece of music runs it backwards in time. Some pieces are the same forwards as backwards, so M_h leaves them unchanged; we say M_h is a *symmetry* of these pieces.

Georg Handel, 'Messiah: Hallelujah Chorus', Naxos 1992.

4. The transformation M_v turns pitches upside down, as if in a mirror laid horizontally. It's not always obvious when one piece is the same as another but upside down.

Nicolò Paganini, 'Capriccio 24 for Violin', DG 1978.
Sergei Rachmaninov, 'Rhapsody on a theme of Paganini: Variation 18', Vox Box 1991.

5. Rotation R turns a thing around by 180 degrees, so that it's upside down and faces backwards. Multiplying transformations, R equals $M_h \times M_v$. Rimsky-Korsakov uses a theme with symmetry R (but not M_h or M_v) for a signal that can be sent two ways.

Nikolai Rimsky-Korsakov, 'The golden cockerel', DCA 1991.

6. Pieces whose group of symmetries is the entire Klein Four-group are very hard to find. It seems most composers avoid them.

John Tavener, 'The Lamb', Naxos 2000.

7. The only other groups that transform a musical theme within a bounded area of musical space involve rotations through other angles, and these don't seem to be musically significant. So to go beyond the Klein Four-group we need transformations of infinite order;

these generate groups that take a theme arbitrarily far away from its original position. The simplest example is a translation T_h moving everything through a fixed distance in time. Doing T_h once gives a *repeat*.

8. Doing T_h once and then twice gives the same theme three times in a row. Like '$\cdots$', this suggests an infinite repetition, or something that sounds over and over again like a church bell.

Benjamin Britten, 'Peter Grimes'.
Arvo Pärt, 'Cantus in memoriam Benjamin Britten', EMI Classics for Pleasure 2002.

9. Multiplying T_h by M_v gives an upside down repetition (called a *glide reflection*).

Judith Weir, 'King Harald's Saga'.

10. Horizontal dilation D_h stretches the time in some fixed ratio r. (If $r > 1$ it slows down the music, if $0 < r < 1$ it speeds it up.) An example of D_h combined with a translation is at a high point of Brahms' Requiem, where he chose his own words 'You are sad now, but I will see you again; I will comfort you', apparently in memory of his mother. The theme on 'I will see you again' is repeated lower and slower, as if by Brahms himself responding to his mother.

Johannes Brahms, 'Ein Deutsches Requiem: Ihr habt nun Traurichkeit', EMI Classics 1993.

11. Multiple applications of D_h together with translations generate two striking pieces of music. Nancarrow takes the ratio very close to 1, so the themes get closer towards the middle of the piece, raising the tension continuously, and then drift apart. Finer makes the separate voices repeat too, and fixes the ratios so that the voices come into alignment exactly once every thousand years.

Conlon Nancarrow, 'Studies for Player Piano 36', Wergo 1990.
Jem Finer, `http://longplayer.org`

12. We started by classifying the musical themes within a given framework of pitches and rhythms (a subspace of the full musical space). By the end we were classifying the frameworks too. Since around 1900, many western composers have built their own frameworks instead of accepting one given by the style of the time. As an example of choosing a framework, I give two new scales that contain exactly the same intervals the same number of times, but are not either the same scale or inversions of each other. (This last is work in progress, joint with Patrick Ozzard-Low.)

- J. Fauvel. R. Flood and R. Wilson eds., *Music and Mathematics: From Pythagoras to Fractals*, Oxford University Press 2003.

- L. Harkleroad, *The Math behind the Music*, Cambridge University Press 2006.

- D. Lewin, *Generalized Musical Intervals and Transformations*, Oxford University Press 2007. (For the serious scholar. Full of musical insights, but I think it makes the mathematics needlessly heavy.)

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