THE DOLD-KAN CORRESPONDENCE

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original: October 26, 2013
updated: January 17, 2015

1. Introduction

1.1. Definition. A *simplicial object* in a category $\mathcal{C}$ is a contravariant functor from the simplex category Δ to $\mathcal{C}$. We denote the category $\mathcal{C}^{\Delta^{\mathrm{op}}}$ of simplicial objects in $\mathcal{C}$ by s$\mathcal{C}$. E.g., sSet is the category of *simplicial sets* and sAb is the category of *simplicial abelian groups*.

1.2. Recall we have a functor $\mathrm{Sing}\colon \mathrm{Top} \longrightarrow \mathrm{sSet}$, sending $X \longmapsto \hom_{\mathrm{Top}}(|\Delta^\bullet|, X)$. Lately we've been talking about Sing for two reasons:

(a) It's a right adjoint to geometric realisation $|-|\colon \mathrm{sSet} \longrightarrow \mathrm{Top}$.

(b) $\mathrm{Sing}(X)$ is a Kan complex for all $X \in \mathrm{Top}$—this was the start of the slogan "Kan complexes are like spaces".

But this isn't the first place one sees Sing, probably. Indeed, the singular homology functors are essentially defined by a composition

$$H_n(-;\mathbb{Z})\colon \mathrm{Top} \xrightarrow{\mathrm{Sing}} \mathrm{sSet} \xrightarrow{\mathbb{Z}} \mathrm{sAb} \xrightarrow{\sum(-1)^i d_i} \mathrm{Ch}_{\geq 0} \xrightarrow{H_n} \mathrm{Ab}.$$

Here $\mathbb{Z}$ denotes the functor which takes free abelian groups level-wise, from which we get the *singular chain complex* by letting the boundary map be given by $\partial := \sum(-1)^i d_i$.

This was just to remind us that we've seen a natural functor $\mathrm{sAb} \longrightarrow \mathrm{Ch}$ relating simplicial abelian groups and chain complexes. We'll look at it a bit more carefully in a second, and develop this relationship much further.

2. Stating the correspondence

We fix $\mathcal{A}$ any abelian category—but we'll probably be imagining $\mathcal{A} = \mathrm{Ab}$, or more generally $\mathcal{A} = \mathrm{Mod}(R)$ for some commutative ring R.[1]

2.1. Notation. We denote the category of nonnegatively graded chain complexes in $\mathcal{A}$ (and chain maps) by $\mathrm{Ch}_{\geq 0}(\mathcal{A})$.

Let's now make precise the $\partial := \sum(-1)^i d_i$ business with which we started this discussion.

2.2. Definition. Let $A \in \mathrm{s}\mathcal{A}$ a simplicial object in $\mathcal{A}$ (e.g., a simplicial abelian group). We define the *associated chain complex* $C(A) \in \mathrm{Ch}_{\geq 0}(\mathcal{A})$ by

$$C_n(A) := A_n, \quad \partial_n := \textstyle\sum_{i=0}^n (-1)^i d_i \colon C_n(A) \longrightarrow C_{n-1}(A)$$

for $n \geq 0$. Note that the simplicial identities clearly imply $\partial^2 = 0$, so $C(A)$ is indeed a chain complex. Moreover, this evidently defines a functor $C\colon \mathrm{s}\mathcal{A} \longrightarrow \mathrm{Ch}_{\geq 0}(\mathcal{A})$.

This is perhaps the most natural—or familiar, at least—functor $\mathrm{s}\mathcal{A} \longrightarrow \mathrm{Ch}_{\geq 0}(\mathcal{A})$, but it turns out not to be the cleanest to use when discussing the relationship between the two categories. In fact, we will want to use the following alternative.

[1] In a couple of arguments, we won't be totally categorical and will use elements here to make things clearer. But it's not all that hard to see how one would get rid of them, and in any case we can appeal to Freyd-Mitchell.

2.3. Definition. Again let $A \in \mathrm{s}\mathcal{A}$ a simplicial object in $\mathcal{A}$. We define the *normalised chain complex* $N(A) \in \mathrm{Ch}_{\geq 0}(\mathcal{A})$ by $N_0(A) := A_0$ and

$$N_n(A) := \bigcap_{i=0}^{n-1} \ker(d_i) \subseteq A_n, \quad \partial_n := (-1)^n d_n \colon N_n(A) \longrightarrow N_{n-1}(A)$$

for $n \geq 1$. The simplicial identities imply both that $d_n(N_n(A)) \subseteq N_{n-1}(A)$, assumed in the above definition of ∂_n, and that $\partial^2 = 0$. Again this gives a functor $N \colon \mathrm{s}\mathcal{A} \longrightarrow \mathrm{Ch}_{\geq 0}(\mathcal{A})$.

What is this unmotivated nonsense? Well, let's at least see an example.

2.4. Example. Recall there is a functor $B \colon \mathrm{Ab} \longrightarrow \mathrm{sAb}$ which associates to an abelian group G it's "classifying space" BG, which is constructed as the nerve of the groupoid with one object and morphisms G. In particular, we have

- $BG_n \simeq G^n$ for $n \geq 0$;
- the face map $d_i \colon BG_n \longrightarrow BG_{n-1}$ sends

  $$(g_1, \ldots, g_n) \longmapsto (g_1, \ldots, g_{i-1}, g_i + g_{i+1}, g_{i+2}, \ldots, g_n),$$

  where we have let $g_0 := 0$ and $g_{n+1} := 0$.

Let's compute the normalised chain complex $N(BG)$.

- Of course $N_0(BG) \simeq BG_0 \simeq 0$ is the trivial group.
- Thus $N_1(BG) = \ker(d_0 \colon BG_1 \longrightarrow BG_0) = BG_1 \simeq G$.
- Let $n \geq 2$. Let $g := (g_1, \ldots, g_n) \in BG_n$. Observe that by definition $g \in \ker(d_0) \implies g_2 = \cdots = g_n = 0$ and thus $g \in \ker(d_1) \implies g_1 + g_2 = 0 \implies g_1 = 0$. So then $N_n(BG) \simeq 0$.

It follows also of course that the homology of $N(BG)$ is just G concentrated in degree 1. (Perhaps this reminds you of the homotopy groups of BG! We will see why this is so in §4.)

Ok that's an example, but maybe the definition of N still seems crazy. Have no fear, for C and N are intimately related! For instance we can note immediately from the definitions that the natural inclusion $N(A) \longrightarrow C(A)$ is in fact a chain map for $A \in \mathrm{s}\mathcal{A}$. But there's more!

2.5. Definition. Let $A \in \mathrm{s}\mathcal{A}$. We define the *degenerate subcomplex* $D(A)$ of $C(A)$ by $D_0(A) := 0$ and $D_n(A) := \sum_{i=0}^{n-1} \mathrm{im}(s_i)$ for $n \geq 1$. That is, $D(A)$ is generated by the images of the degeneracy maps. Note that by the simplicial identities

$$\partial s_j = \sum_{i=0}^{n}(-1)^i d_i s_j = \sum_{i=0}^{j-1}(-1)^i s_{j-1} d_i + \sum_{i=j+2}^{n}(-1)^i s_j d_i,$$

so $D(A)$ is indeed a subcomplex.

2.6. Proposition. Let $A \in \mathrm{s}\mathcal{A}$. For $n \geq 0$ the natural map

$$\phi \colon N_n(A) \oplus D_n(A) \longrightarrow A_n = C_n(A)$$

induced by the inclusions is an isomorphism. Therefore we have a natural isomorphism $N(A) \simeq C(A)/D(A)$. Furthermore, the inclusion $N(A) \longrightarrow C(A)$ is a natural chain homotopy equivalence.

Proof. When $n = 0$ we have by definition that $D_0(A) \simeq 0$ and $N_0(A) \hookrightarrow A_0$ an isomorphism, so the claim is tautological. So fix $n \geq 1$. We first show $N_n(A) \cap D_n(A) \simeq 0$ (we use elements here). Suppose $y \in N_n(A) \cap D_n(A)$. We will show inductively that we can write $y = \sum_{j=i}^{n-1} s_j(x_j)$. The base case is the assumption $y \in D_n(A)$ and when we reach $i = n$ we will have $y = 0$, as desired. Assume the claim holds for i. We first reduce to the case that $d_i x_j = 0$ for $j > i$. This reduction is tautological if $n = 1$. If $n \geq 2$ we have a canonical splitting $A_{n-1} \simeq \ker(d_i) \oplus \mathrm{im}(s_i)$

as a consequence of the simplicial identity $d_i s_i = \mathrm{id}_{A_{n-2}}$. Writing $x_j = u_j + v_j$ in this splitting, with $v_j = s_i w_j$, we have by the simplicial identities that

$$s_j x_j = s_j u_j + s_j s_i w_j = s_j u_j + s_i s_j w_j.$$

Moving all the $s_i s_j w_j$ into the $s_i x_i$ term proves the reduction step. Now since $d_i x_j = 0$ for $j > i$ and $y \in \ker(d_i) \subset N_n(A)$, the simplicial identities give

$$0 = d_i y = d_i s_i x_i + \sum_{j>i} d_i s_j x_j = x_i + \sum_{j>i} s_j d_i x_j = x_i,$$

which proves the induction step.

Now we're just left to show that ϕ is surjective. We prove by downward induction on $0 \leq j \leq n-1$ that

$$\mathrm{im}(\phi) \supseteq N_j := \textstyle\bigcap_{i=0}^{j} \ker(d_i).$$

The base case $j = n-1$ is tautological and the final case $j = 0$ will proved the desired splitting. Now consider the map $\psi := \mathrm{id}_{A_n} - s_j d_j \colon A_n \longrightarrow A_n$. Observe by the simplicial identities that

$$d_j \psi = d_j - d_j s_j d_j = d_j - d_j = 0 \quad \text{and} \quad d_i \psi = d_i - d_i s_j d_j = d_i - s_{j-1} d_{j-1} d_i$$

for $i < j$, implying that $\psi(N_j) \subseteq N_{j+1}$. By induction $\mathrm{im}(\phi) \supseteq N_{j+1}$, and since $\mathrm{im}(s_j d_j) \supseteq D_n(A)$ it follows that $\mathrm{im}(\phi) \supseteq N_j$.

The proof of the last statement regarding the chain homotopy equivalence is omitted here; see [GJ99] or [Wei94]. □

So there's the relationship between C and N: the normalised chain complex somehow tells us the nondegenerate information of the associated chain complex, and moreover loses no homological information. With these definitions in hand, we can now state our main goal, the *Dold-Kan correspondence*.

2.7. Theorem (Dold-Kan)**.** The functor $N \colon \mathrm{s}\mathcal{A} \longrightarrow \mathrm{Ch}_{\geq 0}(\mathcal{A})$ is an equivalence of categories.

3. Proving the correspondence

3.1. Definition. Let $C \in \mathrm{Ch}_{\geq 0}(\mathcal{A})$. Define

$$\Gamma_n(C) := \bigoplus_{[n] \twoheadrightarrow [k]} C_k,$$

where the direct sum is over all surjections $\sigma \colon [n] \twoheadrightarrow [k]$ in the category Δ.

Let $\nu \colon [m] \longrightarrow [n]$ a morphism in Δ. Let $\tau \colon [n] \twoheadrightarrow [k]$ an indexing surjection. We can factor $\tau\nu$ as a composition $[m] \twoheadrightarrow [j] \hookrightarrow [k]$ of a surjection σ and an injection ι. Then we define a map[2]

$$C_k \longrightarrow C_j \quad \text{as} \quad \begin{cases} \mathrm{id}_{C_n}, & \text{if } j = k; \\ (-1)^n \partial_n, & \text{if } j = k-1 \text{ and } \iota = d^k; \\ 0, & \text{otherwise.} \end{cases}$$

Then composition with the inclusion $C_j \longrightarrow \Gamma_m(C)$ of the factor with index $\sigma \colon [m] \twoheadrightarrow [j]$ gives a map $C_k \longrightarrow \Gamma_m(C)$. Finally, the direct sum of these maps gives us an induced morphism $\nu^* \colon \Gamma_n(C) \longrightarrow \Gamma_m(C)$.

[2]This definition is not so random: compare it with our definition of the normalised chain complex N.

Suppose $\mu\colon [l] \longrightarrow [m]$ is another morphism in Δ. Factoring $\sigma\mu$ as $\rho\theta\colon [l] \twoheadrightarrow [i] \hookrightarrow [j]$, we have a commutative diagram

$$\begin{array}{ccccc} [l] & \xrightarrow{\mu} & [m] & \xrightarrow{\nu} & [n] \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\tau} \\ [i] & \xhookrightarrow{\theta} & [j] & \xhookrightarrow{\iota} & [k]. \end{array}$$

It's easy to see then that $(\nu\mu)_* = \mu_*\nu_*$.

It is also evident that a chain map $C \longrightarrow D$ in $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ gives rise to a simplicial map $\Gamma(C) \longrightarrow \Gamma(D)$ in $\mathrm{s}\mathcal{A}$ via factor-wise application. So finally we have constructed a functor

$$\Gamma\colon \mathrm{Ch}_{\geq 0}(\mathcal{A}) \longrightarrow \mathrm{s}\mathcal{A},$$

which to each chain complex in $\mathcal{A}$ gives an *associated simplicial object in* $\mathcal{A}$.

To prove Dold-Kan, we're going to show that Γ is a quasi-inverse to $N\colon \mathrm{s}\mathcal{A} \longrightarrow \mathrm{Ch}_{\geq 0}(\mathcal{A})$.

3.2. Definition. Observe that there is a natural transformation $\Phi\colon \Gamma \circ N \longrightarrow \mathrm{id}_{\mathrm{s}\mathcal{A}}$ defined by the maps

$$\Phi_n(A)\colon \Gamma_n(N(A)) = \bigoplus_{[n] \twoheadrightarrow [k]} N_k(A) \longrightarrow A_n$$

for $A \in \mathrm{s}\mathcal{A}$ and $n \geq 0$, which restrict to the factor indexed by $\sigma\colon [n] \twoheadrightarrow [k]$ as the composition

$$N_k(A) \hookrightarrow A_k \xrightarrow{\sigma^*} A_n.$$

(It is clear this defines a simplicial map $\Gamma(N(A)) \longrightarrow A$ which is natural in A.)

3.3. Lemma. In fact $\Phi\colon \Gamma \circ N \longrightarrow \mathrm{id}_{\mathrm{s}\mathcal{A}}$, defined above, is a natural isomorphism.

Proof. Fix $A \in \mathrm{s}\mathcal{A}$. We will prove by induction on $n \geq 0$ that

$$\Phi_n(A)\colon \Gamma_n(N(A)) \longrightarrow A_n$$

is an isomorphism, and then we will be done. Since the only surjection $[0] \twoheadrightarrow [k]$ in Δ is $\mathrm{id}_{[0]}$, and the inclusion $N_0(A) \hookrightarrow A_0$ is an isomorphism, the base case $n = 0$ is tautological.

First, surjectivity. Recall from (2.6) that we have a splitting $A_n \simeq N_n(A) \oplus D_n(A)$. From the factor $\mathrm{id}_{[n]}\colon [n] \twoheadrightarrow [n]$ of $\Gamma_n(N(A))$ we $\mathrm{im}(\Phi_n(A)) \supseteq N_n(A)$. By induction $\Phi_{n-1}(A)$ is surjective, so we must have $\mathrm{im}(\Phi_n(A)) \supseteq D_n(A)$ by definition. Hence $\Phi_n(A)$ is surjective.

Next, injectivity (we use elements here). Assume we have $(x_\sigma) \in \ker(\Phi_n(A))$. Fix $0 \leq k < n$. Observe that to each surjection $\sigma\colon [n] \twoheadrightarrow [k]$ we can assign a section of σ,

$$\nu_\sigma\colon [k] \hookrightarrow [n], \quad \nu_\sigma(i) := \max\{j \in [n] \mid \sigma(j) = i\}.$$

If we have $\sigma, \sigma'\colon [n] \twoheadrightarrow [k]$, we say

$$\sigma \leq \sigma' \iff \nu_\sigma(i) \leq \nu_{\sigma'}(i) \text{ for all } i \in [k].$$

In particular, $\sigma'\nu_\sigma = \mathrm{id}_{[k]} \implies \sigma \leq \sigma'$. If there exists $\tau\colon [n] \twoheadrightarrow [k]$ such that $x_\tau \neq 0$, choose a *maximal* such τ (with respect to the ordering just defined). By definition of the simplicial structure on $\Gamma(N(A))$, it follows that the component of $\nu_\tau^*(x_\sigma)$ in the factor of $\Gamma_k(N(A))$ indexed by $\mathrm{id}_{[k]}$ is precisely x_τ. But then, by induction,

$$(x_\sigma) \in \ker(\Phi_n(A)) \implies \nu_\tau^*(x_\sigma) \in \ker(\Phi_k(A)) \implies x_\tau = 0,$$

contradiction.

So we must have $x_\sigma = 0$ for all $\sigma \neq \mathrm{id}_{[n]}$. But the restriction of $\Phi_n(A)$ to the factor indexed by $\mathrm{id}_{[n]}$ is just the inclusion $N_n(A) \hookrightarrow A_n$. So then $x_{\mathrm{id}_{[n]}} = 0$ too, and hence $\Phi_n(A)$ is injective. □

3.4. Lemma. Let $C \in \mathrm{Ch}_{\geq 0}(\mathcal{A})$. For $n \geq 0$, the natural inclusion

$$\Psi_n(C)\colon N_n(\Gamma(C)) \hookrightarrow C_n(\Gamma(C)) = \Gamma_n(C) = \bigoplus_{[n] \twoheadrightarrow [k]} C_k$$

has image the factor C_n indexed by $\mathrm{id}_{[n]}$. This of course gives a natural isomorphism

$$\Psi\colon N \circ \Gamma \longrightarrow \mathrm{id}_{\mathrm{Ch}_{\geq 0}(\mathcal{A})}.$$

Proof. By definition of the simplicial structure on $\Gamma(C)$ we have

$$C_n \subseteq \textstyle\bigcap_{i=0}^{n-1} \ker(d_i) = \mathrm{im}(\Psi_n(C)).$$

Now note for $\sigma\colon [n] \twoheadrightarrow [k]$ with $k < n$, we must have a factorisation of σ as a composition

$$[n] \xrightarrow{s^i} [n-1] \longrightarrow [k],$$

so it follows that the factor of $\Gamma_n(C)$ indexed by σ lies in the image of the degeneracies $D_n(\Gamma(C))$. Then we're done, since by (2.6) we have a splitting

$$\Gamma_n(C) \simeq N_n(\Gamma(C)) \oplus D_n(\Gamma(C)).$$ □

Proof of (2.7). The natural isomorphisms $\Phi\colon \Gamma \circ N \longrightarrow \mathrm{id}_{\mathrm{s}\mathcal{A}}$ and $\Psi\colon N \circ \Gamma \longrightarrow \mathrm{id}_{\mathrm{Ch}_{\geq 0}(\mathcal{A})}$ of Lemmas 3.3 and 3.4 exhibit N (and Γ) as an equivalence of categories, thus proving the Dold-Kan correspondence. □

4. Applying the correspondence

4.1. Notation. For the remainder we will fix some commutative ring R and actually set $\mathcal{A} := \mathrm{Mod}(R)$.

4.2. Recall that if $X \in \mathrm{sSet}$ is *fibrant* (i.e., a *Kan complex*) with basepoint $v \in X_0$ then we can define its homotopy groups for $n \geq 0$,

$$\pi_n(X, v) := [(\Delta^n, \partial\Delta^n), (X, v)],$$

that is, homotopy classes $[\alpha]$ of maps $\alpha\colon \Delta^n \longrightarrow X$ which fit in the commutative diagram

$$\begin{array}{ccc} \Delta^n & \xrightarrow{\alpha} & X \\ \uparrow & & \uparrow{\scriptstyle v} \\ \partial\Delta^n & \longrightarrow & \Delta^0. \end{array}$$

We now state a couple of facts about the homotopy groups.

4.3. Lemma. Let $X \in \mathrm{sSet}$ fibrant, $v \in X_0$, and $n \geq 0$. Denote all degeneracies of v also by v, so that $[v] \in \pi_n(X, v)$ is the identity element. Let $[\alpha] \in \pi_n(X, v)$ represented by $\alpha\colon \Delta^n \longrightarrow X$. Then $[\alpha] = [v]$ if and only if there exists $\omega \in X_{n+1}$ such that

$$d_{n+1}\omega = \alpha \quad \text{and} \quad d_i\omega = v \text{ for } 0 \leq i \leq n.$$

Proof. Omitted. □

4.4. Lemma. Let $G \in \mathrm{sGrp}$ a simplical group and $v \in G_0$. Then the following hold.

(a) The simplicial set underlying G is fibrant, so the homotopy groups $\pi_n(G, v)$ are well-defined.

(b) The group structure on G_n induces a natural group structure on $\pi_n(G, v)$ which agrees with the homotopy group structure.

Proof. Omitted. See [GJ99] for (1), and (2) can be proved by an Eckmann-Hilton argument. ☐

We can now state a relationship between the homotopy theory of $\mathrm{s}\mathcal{A}$ and that of $\mathrm{Ch}_{\geq 0}(\mathcal{A})$.

4.5. Proposition. Let $A \in \mathrm{s}\mathcal{A}$ a simplicial R-module and $0 \in A_0$ the identity element. Then for $n \geq 0$ we have natural isomorphisms

$$\pi_n(A, 0) \simeq H_n(N(A)) \simeq H_n(C(A)).$$

In particular, the functors N and Γ correspond quasi-isomorphisms in $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ with weak equivalences in $\mathrm{s}\mathcal{A}$.

Proof. The second isomorphism is immediate from (2.6), so we just prove the first. Let $x \in N_n(A) \subseteq A_n$. For $n \geq 1$ we have by definition that

$$x \in \ker(\partial_n) \iff x \in \textstyle\bigcap_{i=0}^{n} \ker(d_n),$$

which says precisely that x represents an element $[x] \in \pi_n(A, 0)$. And by (refnull-homotopic) we have for all $n \geq 0$ that $[x] = 0 \in \pi_n(A, 0)$ precisely when $x \in \mathrm{im}(\partial_{n+1})$. Finally by (4.4) we can take the group structure on $\pi_n(A, 0)$ to be the one induced by the R-module structure of A_n, whence the above discussion immediately gives us a (manifestly natural) isomorphism $\pi_n(A, 0) \simeq H_n(N(A))$. ☐

4.6. Remark. Recall that in (2.4) we computed the homology of $N(BG)$ for an abelian group G to be G concentrated in degree 1. Then (4.5) gives, as expected,

$$\pi_n(BG, 0) \simeq \begin{cases} G, & \text{if } n = 1; \\ 0, & \text{otherwise.} \end{cases}$$

The same observation motivates more generally the following definition.

4.7. Definition. Let A an R-module and $n \geq 0$. Define $A[n] \in \mathrm{Ch}_{\geq 0}(\mathcal{A})$ the complex with A concentrated in degree n. Define the *Eilenberg-Maclane space* $K(A, n) := \Gamma(A[n]) \in \mathrm{s}\mathcal{A}$.

4.8. Remark. If one remembers that geometric realisation preserves homotopy groups, in the sense that $\pi_n(|X|, v) \simeq \pi_n(X, v)$ for fibrant $X \in \mathrm{sSet}$, then $|K(A, n)|$ with the definition above really is an Eilenberg-Maclane space, in the ordinary topological sense.

We'll end with an interesting consequence of (4.5). Assume that R is a PID.

4.9. Lemma. Let $C \in \mathrm{Ch}_{\geq 0}(\mathcal{A})$. Then there is a quasi-isomorphism

$$C \simeq \prod_{n \geq 0} H_n(C)[n].$$

Proof. Let $n \geq 0$. Let $Z_n := \ker(\partial_n)$ and $B_n := \mathrm{im}(\partial_{n+1})$ the cycles and boundaries, respectively, in C_n. Let F_n a free R-module with a surjection $F_n \twoheadrightarrow Z_n$. Let

$$F_n' := \ker(F_n \twoheadrightarrow Z_n \twoheadrightarrow H_n(C)),$$

which is free since it is a submodule of a free R-module and R is a PID. There is then an induced map $F_n' \longrightarrow B_n$, which lifts to a map $F_n' \longrightarrow C_{n+1}$ since F_n' is free and $\partial_{n+1} \colon C_{n+1} \longrightarrow B_n$ is surjective. Define $F_n(C) \in \mathrm{Ch}_{\geq 0}(\mathcal{A})$ the complex with just F_n in degree n and F_n' in degree $n + 1$. The maps $F_n \longrightarrow Z_n \hookrightarrow C_n$ and $F_n' \longrightarrow C_{n+1}$ give a chain map $F_n(C) \longrightarrow C$ which induces an isomorphism

$H_n(F_n(C)) \simeq F_n/F_n' \simeq H_n(C)$. And the map $F_n \longrightarrow H_n(C)$ induces a quasi-isomorphism $F_n(C) \simeq H_n(C)[n]$.

Since homology commutes with products, there are then quasi-isomorphisms

$$C \simeq \prod_{n \geq 0} \mathbb{F}_n(C) \simeq \prod_{n \geq 0} H_n(C)[n]. \qquad \square$$

4.10. Proposition. Let $A \in \mathrm{s}\mathcal{A}$ a simplicial R-module. Then A is weakly equivalent to a product of Eilenberg-Maclane spaces:

$$A \simeq \prod_{n \geq 0} K(\pi_n(A, 0), n).$$

Proof. In light of (4.5) and the fact that Γ preserves products (as an equivalence of categories), this is immediate from applying Γ to (4.9) with $C := N(A)$. $\square$

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