7 Review of SLODEs

Throughout this section, if y denotes a function (of time, say), then $y^{[k]}$ or $y^{(k)}$ denotes the k'th derivative of the function y,

$$y^{[k]} = \frac{d^k y}{dt^k}$$

In the case of $k = 1$ and $k = 2$, it will be convenient to also use $\dot{y}$ and $\ddot{y}$.

7.1 Linear, Time-Invariant Differential Equations

Often, in this class, we will analyze a closed-loop feedback control system, and end up with an equation of the form

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_n y(t) = v(t) \tag{39}$$

where y is some variable of the plant that is of interest to us, and v is a forcing function, usually either a reference signal ($y_{\text{des}}(t)$) or a disturbance (ie., inclination of hill), or a combination of such signals. One job of the control designer is to analyze the resulting equation, and determine if the behavior of the closed-loop system is acceptable.

The differential equation in (39) is called a *forced, linear, time-invariant* differential equation. For now, associate the fact that the $\{a_i\}_{i=1}^n$ are constants with the term *time-invariant*, and the fact that the left-hand side (which contains all y terms) is a linear combination of y and its derivatives with the term *linear*.

The right-hand side function, v, is called the forcing function. For a specific problem, it will be a given, known function of time.

Sometimes, we are given an initial time t_0 and initial conditions for differential equation, that is, real numbers

$$y_0, \dot{y}_0, \ddot{y}_0, \ldots, y_0^{(n-1)} \tag{40}$$

and we are looking for a *solution*, namely a function y that satisfies both the differential equation (39) and the initial condition constraints

$$y(t_0) = y_0, \quad \dot{y}(t_0) = \dot{y}_0, \quad \ddot{y}(t_0) = \ddot{y}_0, \quad \ldots, y^{(n-1)}(t_0) = y_0^{(n-1)}. \tag{41}$$

For essentially all differential equations (even those that are not linear, and not time-invariant), there is a theorem which says that solutions always exist, and are unique:

Theorem (Existence and Uniqueness of Solutions): Given a forcing function v, defined for $t \geq t_0$, and initial conditions of the form (41). Then, there exists a unique function y which satisfies the initial conditions and the differential equation (39).

7.2 Importance of Linearity

Suppose that y_P is a function which satisfies (39) so that

$$y_P^{(n)}(t) + a_1 y_P^{(n-1)}(t) + \cdots + a_n y_P(t) = v(t) \tag{42}$$

and y_H is a function which, for all t satisfies

$$y_H^{(n)}(t) + a_1 y_H^{(n-1)}(t) + \cdots + a_n y_H(t) = 0 \tag{43}$$

y_H is called a *homogeneous solution* of (39), and the differential equation in (43) is called the homogeneous equation. The function y_P is called a *particular* solution to (39), since it satisfies the equation with the forcing function in place.

The derivative of $y_P + y_H$ is the sum of the derivatives, so we add the equations satisfied by y_P and y_H to get

$$\left[y_P + y_H\right]^{(n)}(t) + a_1 \left[y_P + y_H\right]^{(n-1)}(t) + \cdots a_n \left[y_P + y_H\right](t) = v(t)$$

This implies that the function $y_P + y_H$ also satisfies (39). Hence, adding a particular solution to a homogeneous solution results in a new particular solution.

Conversely, suppose that y_{P_1} and y_{P_2} are two functions, both of which solve (39). Consider the function $y_d := y_{P_1} - y_{P_2}$. Easy manipulation shows that this function y_d satisfies the homogeneous equation. It is a trivial relationship that

$$\begin{aligned} y_{P_1} &= y_{P_2} + (y_{P_1} - y_{P_2}) \\ &= y_{P_2} + y_d \end{aligned}$$

We have shown that any two particular solutions differ by a homogeneous solution. Hence *all* particular solutions to (39) can be generated by taking one specific particular solution, and adding to it every homogeneous solution. In order to get the correct initial conditions, we simply need to add the "right" homogeneous solution.

Remark: The main points of this section rely only on linearity, but not time-invariance.

7.3 Solving Homogeneous Equation

Let's try to solve (43). Take a fixed complex number $r \in \mathbf{C}$, and suppose that the function y_H, defined as

$$y_H(t) = e^{rt}$$

is a solution to (43). Substituting in, using the fact that for any integer $k > 0$, $y_H^{(k)}(t) = r^k e^{rt}$, gives

$$\left(r^n + a_1 r^{n-1} + \cdots + a_n\right) e^{rt} = 0$$

for all t. Clearly, $e^{rt} \neq 0$, always, so it can be divided out, leaving

$$r^n + a_1 r^{n-1} + \cdots + a_n = 0 \tag{44}$$

Thus, if e^{rt} is a solution to the homogeneous equation, *it must be that* the scalar r satisfies (44).

Conversely, suppose that r is a complex number which satisfies (44), then simple substitution reveals that e^{rt} does satisfy the homogeneous equation. Moreover, if r is a repeated root, say l times, then substitution shows that the functions $\left\{e^{rt}, te^{rt}, \ldots, t^{l-1}e^{rt}\right\}$ all satisfy the homogeneous differential equation. This leads to the following nomenclature:

Let $r_1, r_2, \ldots, r_n$ be the roots of the polynomial equation

$$\lambda^n + a_1 \lambda^{n-1} + \cdots + a_n = 0$$

This polynomial is called the *characteristic polynomial* associated with (39).

Fact 1 (requires proof): If the $\{\lambda_i\}_{i=1}^n$ are all distinct from one another , then y_H satisfies (43) if and only if there exist complex constants $c_1, c_2, \ldots, c_n$ such that

$$y_H(t) = \sum_{i=1}^n c_i e^{r_i t}$$

Fact 2 (requires proof): If the $\{\lambda_i\}_{i=1}^n$ are not distinct from one another, then group the roots $\{r_1, r_2, \cdots, r_n\}$ as

$$\underbrace{p_1, p_1, \ldots, p_1}_{l_1}, \underbrace{p_2, p_2, \ldots, p_2}_{l_2}, \cdots, \underbrace{p_f, p_f, \ldots, p_f}_{l_f}$$

Hence, p_1 is a root with multiplicity l_1, p_2 is a root with multiplicity l_2 and so on. Then y_H satisfies (43) if and only if there exist complex constants c_{ij} $(i = 1, \ldots f,\ j = 0, \ldots l_i - 1)$ such that

$$y_H(t) = \sum_{i=1}^{f} \sum_{j=0}^{l_i - 1} c_{ij} e^{p_i t} t^j$$

So, **Fact 2** includes **Fact 1** as a special case. Both indicate that by solving for the roots of the characteristic equation, it is easy to pick n linearly independent functions which form a basis for the set of **all** homogeneous solutions to the differential equation. Here, we explicitly have used the time-invariance (ie., the coefficients of the ODE are constants) to generate a basis (the exponential functions) for all homogeneous solutions. However, the fact that the space of homogenous solutions is n-dimensional only relies on linearity, and not time-invariance.

Basic idea: Suppose there are m solutions to the homogeneous differential equation, labeled $y_{1,H}, y_{2,H}, \ldots, y_{m,H}$, with $m > n$. Then, look at the $n \times m$ matrix of these solutions initial conditions,

$$M := \begin{bmatrix} y_{1,H}^{(0)} & y_{2,H}^{(0)} & \cdots & y_{m,H}^{(0)} \\ y_{1,H}^{(1)} & y_{2,H}^{(1)} & \cdots & y_{m,H}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ y_{1,H}^{(n-1)} & y_{2,H}^{(n-1)} & \cdots & y_{m,H}^{(n-1)} \end{bmatrix}$$

Since $m > n$, this must have linearly dependent columns, so there is a nonzero $m \times 1$ vector α such that $M\alpha = 0_{n\times 1}$. Define $y_z := \alpha_1 y_{1,H} + \alpha_2 y_{2,H} + \cdots + \alpha_m y_{m,H}$. Since this is a sum of homogeneous solutions, it itself is a homogeneous solution. Moreover, y_z satisfies

$$y_z^{(0)}(t_0) = 0,\ y_z^{(1)}(t_0) = 0,\ \cdots,\ y_z^{(n-1)}(t_0) = 0$$

Note that the function $y_I(t) \equiv 0$ for all t also satisfies the same initial conditions as y_z, and it satisfies the homogeneous differential equation as well. By uniqueness of solutions, it must be that $y_z(t) = y_I(t)$ for all t. Hence, y_z is actually the identically zero function. Hence, we have shown that the set of homogeneous solutions is finite dimensional, and of dimension at most n. Moreover, by the simple substitutions above, we already know how to construct n linearly independent solutions to the homogeneous differential equation, so those must form a basis for all homogeneous solutions.

7.4 General Solution Technique

The general technique (conceptually, at least) for finding particular solutions of (39) which also satisfy the initial conditions is a combination of all of the above ideas. It is very important conceptually, and somewhat important in actual use.

1. Find any particular solution y_P
2. Choose constants c_{ij} so that the function

$$y_P(t) + \sum_{i=1}^{f} \sum_{j=0}^{l_i - 1} c_{ij} t^j e^{p_i t}$$

   satisfies the initial conditions.

The resulting function is the solution. Note that in step 2, there are n equations, and n unknowns.

7.5 Behavior of Homogeneous Solutions as $t \to \infty$

In this section, we study the behavior of the class of homogeneous solutions as $t \to \infty$. If we can show that all homogeneous solutions decay to 0 as $t \to \infty$, then it must be that for a give forcing function, all particular solutions, regardless of the initial conditions, approach each other. This will be useful in many contexts, to quickly understand how a system behaves.

Suppose r is a complex number, $r \in \mathbf{C}$, and we decompose it into its real and imaginary parts. Let $\alpha := \text{Real}(r)$ and $\beta := \text{Imag}(r)$. Hence, both α and β are real numbers, and $r = \alpha + j\beta$. We will always use $j := \sqrt{-1}$. The exponential function e^{rt} can be expressed as

$$e^{rt} = e^{\alpha t} \left[\cos \beta t + j \sin \beta t\right]$$

Note that the real part of r, namely α, determines the qualitative behavior of e^{rt} as $t \to \infty$. Specifically,

- if $\alpha < 0$, then $\lim_{t\to\infty} e^{rt} = 0$
- if $\alpha = 0$, then e^{rt} does not decay to 0, and does not "explode," but rather oscillates, with $|e^{rt}| = 1$ for all t
- if $\alpha > 0$, then $\lim_{t\to\infty} e^{rt} = \infty$

Since all homogeneous solutions are (essentially) of the form e^{rt} where r is a root of the characteristic polynomial, we see that the roots of the characteristic polynomial determine the qualitative nature of the homogeneous solutions.

We summarize this as follows:

- If *all* of the roots, $\{r_i\}_{i=1}^n$, of the characteristic polynomial satisfy
$$\text{Real}(r_i) < 0$$
then *every* homogeneous solution decays to 0 as $t \to \infty$
- If *any* of the roots, $\{r_i\}_{i=1}^n$, of the characteristic polynomial satisfy
$$\text{Real}(r_i) \geq 0$$
then there are homogeneous solutions which *do not* decay to 0 as $t \to \infty$

7.6 Response of stable system to constant input (Steady-State Gain)

Suppose the system (input u, output y) is governed by the SLODE

$$\begin{aligned} & y^{[n]}(t) + a_1 y^{[n-1]}(t) + \cdots + a_{n-1} y^{[1]}(t) + a_n y(t) \\ & \quad = b_0 u^{[n]}(t) + b_1 u^{[n-1]}(t) + \cdots + b_{n-1} u^{[1]}(t) + b_n u(t) \end{aligned}$$

Suppose initial conditions for y are given, and that the input u is specified to be a constant, $u(t) \equiv \bar{u}$ for all $t \geq t_0$. What is the limiting behavior of y?

If the system is stable, then this is easy to compute. First notice that the constant function $y_P(t) \equiv \frac{b_n}{a_n}\bar{u}$ is *a particular solution*, although it does not satisfy any of the initial conditions. The actual solution y differs from this particular solution by some homogeneous solution, y_H. Hence for all t,

$$\begin{aligned} y(t) &= y_P(t) + y_H(t) \\ &= \frac{b_n}{a_n}\bar{u} + y_H(t) \end{aligned}$$

Now, take limits, since we know (by the stability assumption) that $\lim_{t\to\infty} y_H(t) = 0$, giving

$$\lim_{t\to\infty} y(t) = \frac{b_n}{a_n}\bar{u}$$

Hence, the **steady-state gain** of the system is $\frac{b_n}{a_n}$.

7.7 Example

Consider the differential equation

$$\ddot{y}(t) + 4\dot{y}(t) + y(t) = 1 \tag{45}$$

subject to the initial conditions $y(0) = y_0$, $\dot{y}(0) = \dot{y}_0$. The characteristic equation is $\lambda^2 + 4\lambda + 1 = 0$, which has roots at

$$\lambda = -2 \pm \sqrt{3} \approx \{-3.7, -0.3\}$$

Hence, all homogeneous solutions are of the form

$$y_H(t) = c_1 e^{-3.7t} + c_2 e^{-0.3t}$$

Terms of the form $e^{-3.7t}$ take about 0.8 time units to decay, while terms of the form $e^{-0.3t}$ take about 10 time units to decay. In general then (though not always - it depends on the initial conditions) homogeneous solutions will typically take about 10 time units to decay.

A particular solution to (45) is simply $y_P(t) = 1$ for all $t \geq 0$. Note that this choice of y_P does not satisfy the initial conditions, but it does satisfy the differential equation.

As we have learned, **all** solutions to (45) are **any** particular solution plus **all** homogeneous solutions. Therefore, the general solution is

$$y(t) = 1 + c_1 e^{(-2-\sqrt{3})t} + c_2 e^{(-2+\sqrt{3})t}$$

which has as its derivative

$$\dot{y}(t) = (-2-\sqrt{3})c_1 e^{(-2-\sqrt{3})t} + (-2+\sqrt{3})c_2 e^{(-2+\sqrt{3})t}$$

Evaluating these at $t = 0$, and equating to the given initial conditions yields

$$\begin{aligned} y(0) &= 1 + c_1 + c_2 \\ &= y_0 \\ \dot{y}(0) &= \left(-2-\sqrt{3}\right) c_1 + \left(-2+\sqrt{3}\right) c_2 \\ &= \dot{y}_0 \end{aligned}$$

In matrix form, we have

$$\begin{bmatrix} 1 & 1 \\ -2-\sqrt{3} & -2+\sqrt{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 - 1 \\ \dot{y}_0 \end{bmatrix}$$

This is easy to invert. recall

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Hence

$$\begin{bmatrix} 1 & 1 \\ -2-\sqrt{3} & -2+\sqrt{3} \end{bmatrix}^{-1} = \frac{1}{2\sqrt{3}} \begin{bmatrix} -2+\sqrt{3} & -1 \\ 2+\sqrt{3} & 1 \end{bmatrix}$$

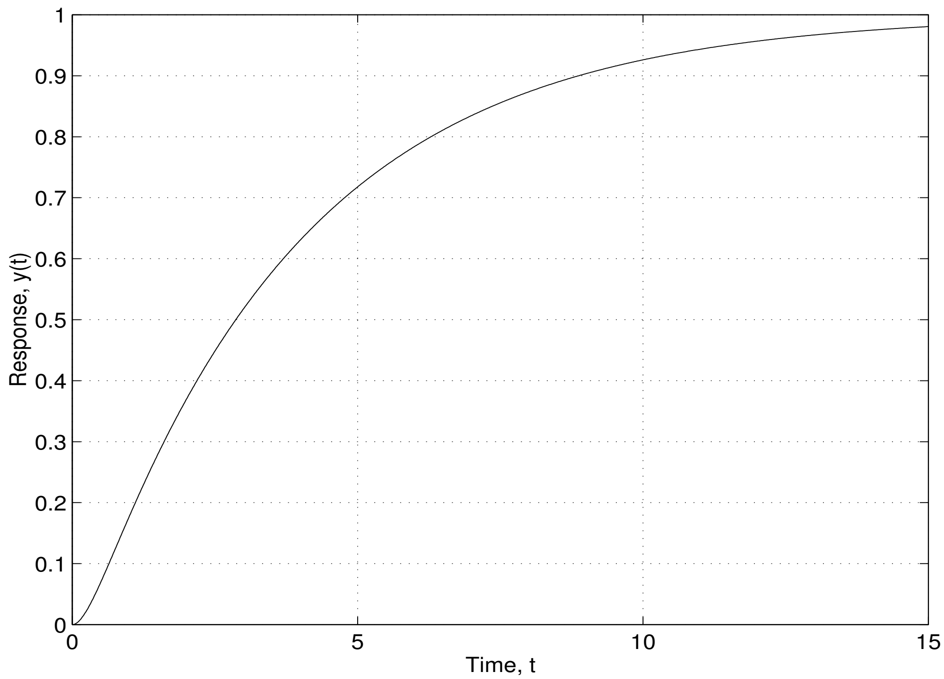
so that

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2-\sqrt{3} & -2+\sqrt{3} \end{bmatrix}^{-1} \begin{bmatrix} y_0 - 1 \\ \dot{y}_0 \end{bmatrix}$$

For the case at hand, let's take $y_0 := 0, \dot{y}_0 := 0$, hence

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{2\sqrt{3}} \begin{bmatrix} 2-\sqrt{3} \\ -2-\sqrt{3} \end{bmatrix}$$

A plot of $y(t)$ is shown in the figure below.



Note that indeed, as we had guessed, the homogeneous solution (which connects the initial values at $t = 0$ to the final behavior at $t \to \infty$ takes about 10 time units to decay.

7.8 Stability Conditions for 2nd order differential equation

Given real numbers a_0, a_1 and a_2, with $a_0 \neq 0$, we wish to determine if the roots of the equation

$$a_0\lambda^2 + a_1\lambda + a_2 = 0$$

have negative real parts. This question is important in determining the qualitative nature (exponential decay versus exponential growth) of the homogeneous solutions to

$$a_0\ddot{y}(t) + a_1\dot{y}(t) + a_2y(t) = 0$$

Since $a_0 \neq 0$ (so that we actually do have a quadratic, rather than linear equation) divide out by it, giving

$$\ddot{y}(t) + \frac{a_1}{a_0}\dot{y}(t) + \frac{a_2}{a_0}y(t) = 0$$

Call $b_1 := \frac{a_1}{a_0}, b_2 := \frac{a_2}{a_0}$. The characteristic equation is $\lambda^2 + b_1\lambda + b_2 = 0$. The roots are

$$\lambda_1 = \frac{-b_1 + \sqrt{b_1^2 - 4b_2}}{2}, \quad \lambda_2 = \frac{-b_1 - \sqrt{b_1^2 - 4b_2}}{2}$$

Consider 4 cases:

1. $b_1 > 0, b_2 > 0$. In this case, the term $b_1^2 - 4b_2 < b_1^2$, hence either
   - $\sqrt{b_1^2 - 4b_2}$ is imaginary
   - $\sqrt{b_1^2 - 4b_2}$ is real, but $\sqrt{b_1^2 - 4b_2} < b_1$.

   In either situation, both $\text{Re}(\lambda_1) < 0$ and $\text{Re}(\lambda_2) < 0$.

2. $b_1 \leq 0$: Again, the square-root is either real or imaginary. If it is imaginary, then $\text{Re}(\lambda_i) = \frac{-b_1}{2} \geq 0$. If the square-root is real, then then $\text{Re}(\lambda_1) = \frac{-b_1+\sqrt{b_1^2-4b_2}}{2} \geq 0$. In either case, at least one of the roots has a nonnegative real part.

3. $b_2 \leq 0$: In this case, the square root is real, and hence both roots are real. However, $\sqrt{b_1^2 - 4b_2} \geq |b_1|$, hence $\lambda_1 \geq 0$. so one of the roots has a non-negative real part.

This enumerates all possibilities. We collect these ideas into a theorem.

Theorem: For a second order polynomial equation $\lambda^2 + b_1\lambda + b_2 = 0$, the roots have negative real parts if and only if $b_1 > 0, b_2 > 0$.

If the leading coefficient is not 1, then we have

Theorem: For a second order polynomial equation $b_0\lambda^2 + b_1\lambda + b_2 = 0$, the roots have negative real parts if and only if all of the b_i are nonzero, and have the same sign (positive or negative).

WARNING: These theorems are not true for polynomials with order higher than 2. We will study simple methods to determine if all of the roots have negative real parts later. For instance;

Theorem: For a third order polynomial equation $\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0$, all of the roots have negative real parts if and only if $b_1 > 0, b_3 > 0$ and $b_1 b_2 > b_3$.

Note that the condition $b_1 b_2 > b_3$, coupled with the first two conditions, certainly implies that $b_2 > 0$. However, that is only necessary, not sufficient, as an example illustrates: the roots of $\lambda^3 + \lambda^2 + \lambda + 3$ are $\{-1.575, 0.2874 \pm j1.35\}$.

Theorem: For a fourth order polynomial equation $\lambda^4 + b_1\lambda^3 + b_2\lambda^2 + b_3\lambda + b4 = 0$, all of the roots have negative real parts if and only if $b_1 > 0, b_4 > 0$, $b_1b_2 > b_3$ and $(b_1b_2 - b_3)b_3 > b_1^2 b_4$.

7.9 Important 2nd order example

It is useful to study a general second order differential equation, and interpret it in a different manner. Start with

$$\ddot{y}(t) + a_1\dot{y}(t) + a_2y(t) = bv(t)$$

with $y(0) = 0, \dot{y}(0) = 0$ and $v(t) = 1$ for all $t \geq 0$. Assume that the homogeneous solutions are exponentially decaying, which is equivalent to $a_1, a_2 > 0$. Rewrite using new variables (instead of a_1 and a_2) ξ, ω_n as

$$\ddot{y}(t) + 2\xi\omega_n\dot{y}(t) + \omega_n^2 y(t) = bv(t)$$

where both $\xi, \omega_n > 0$. In order to match up terms here, we must have

$$2\xi\omega_n = a_1, \qquad \omega_n^2 = a_2$$

which can be inverted to give

$$\xi = \frac{a_1}{2\sqrt{a_2}}, \qquad \omega_n = \sqrt{a_2}$$

Note that since $a_1, a_2 > 0$, we also have $\xi, \omega_n > 0$. With these new variables, the homogeneous equation is

$$\ddot{y}(t) + 2\xi\omega_n\dot{y}(t) + \omega_n^2 y(t) = 0$$

which has a characteristic polynomial

$$\lambda^2 + 2\xi\omega_n\lambda + \omega_n^2 = 0$$

The roots are at

$$\begin{aligned} \lambda &= -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1} \\ &= -\xi\omega_n \pm j\omega_n\sqrt{1 - \xi^2} \end{aligned}$$

There are three cases to consider:

- $\xi > 1$: Roots are distinct, and both roots are real, and negative. The general homogeneous solution is most easily written in the form

$$y_H(t) = c_1 e^{\left(-\xi\omega_n + \omega_n\sqrt{\xi^2-1}\right)t} + c_2 e^{\left(-\xi\omega_n - \omega_n\sqrt{\xi^2-1}\right)t}$$

- $\xi = 1$, which results in repeated real roots, at $\lambda = -\xi\omega_n$, so that the general form of homogeneous solutions are

$$y_H(t) = c_1 e^{-\xi\omega_n t} + c_2 t e^{-\xi\omega_n t}$$

- $0 < \xi < 1$: Roots are distinct, and complex (complex-conjugate pair), with negative real part, and the general homogeneous solution is easily expressable as

$$y_H(t) = c_1 e^{\left[-\xi\omega_n + j\omega_n\sqrt{1-\xi^2}\right]t} + c_2 e^{\left[-\xi\omega_n - j\omega_n\sqrt{1-\xi^2}\right]t}$$

This can be rewritten as

$$y_H(t) = e^{-\xi\omega_n t}\left[c_1 e^{j\omega_n\sqrt{1-\xi^2}t} + c_2 e^{-j\omega_n\sqrt{1-\xi^2}t}\right]$$

Recall $e^{j\beta} = \cos\beta + j\sin\beta$. Hence

$$y_H(t) = e^{-\xi\omega_n t}\left[\begin{array}{rcl} c_1\cos\omega_n\sqrt{1-\xi^2}t & + & jc_1\sin\omega_n\sqrt{1-\xi^2}t \\ +c_2\cos\omega_n\sqrt{1-\xi^2}t & - & jc_2\sin\omega_n\sqrt{1-\xi^2}t \end{array}\right]$$

which simplifies to

$$y_H(t) = e^{-\xi\omega_n t}\left[(c_1 + c_2)\cos\omega_n\sqrt{1-\xi^2}t \; + \; j(c_1 - c_2)\sin\omega_n\sqrt{1-\xi^2}t\right]$$

For a general problem, use the initial conditions to determine c_1 and c_2. Usually, the initial conditions, $y(0), \dot{y}(0)$, are real numbers, the differential equation coefficients (a_1, a_2 or ξ, ω_n) are real, hence the solution must be real. Since $\cos\omega_n\sqrt{1-\xi^2}t$ and $\sin\omega_n\sqrt{1-\xi^2}t$ are linearly independent functions, it will always then work out that

$$\begin{array}{rcl} c_1 + c_2 & = & \text{purely real} \\ c_1 - c_2 & = & \text{purely imaginary} \end{array}$$

In other words, $\mathrm{Im}(c_1) = -\mathrm{Im}(c_2)$, and $\mathrm{Re}(c_1) = \mathrm{Re}(c_2)$, which means that c_1 is the *complex conjugate* of c_2.

Under this condition, call $c := c_1$. The homogeneous solution is

$$y_H(t) = e^{-\xi\omega_n t}\left[2\mathrm{Re}(c)\cos\omega_n\sqrt{1-\xi^2}t \; - \; 2\mathrm{Im}(c)\sin\omega_n\sqrt{1-\xi^2}t\right]$$

Use $A := 2\mathrm{Re}(c)$, and $B := -2\mathrm{Im}(c)$, and the final form of the real homogeneous solution is

$$y_H(t) = e^{-\xi\omega_n t}\left[A\cos\omega_n\sqrt{1-\xi^2}t \; + \; B\sin\omega_n\sqrt{1-\xi^2}t\right]$$

A and B are two, free, real parameters for the real homogeneous solutions when $0 < \xi < 1$.

The solution is made up of two terms:

1. An exponentially decaying envelope, $e^{-\xi\omega_n t}$. Note that this decays to zero in approximately $\frac{3}{\xi\omega_n}$
2. sinosoidal oscillation terms, $\sin\omega_n\sqrt{1-\xi^2}t$ and $\cos\omega_n\sqrt{1-\xi^2}t$. The period of oscillation is $\frac{2\pi}{\omega_n\sqrt{1-\xi^2}}$.

Comparing these times, we expect "alot of oscillations before the homogeneous solution decays" if

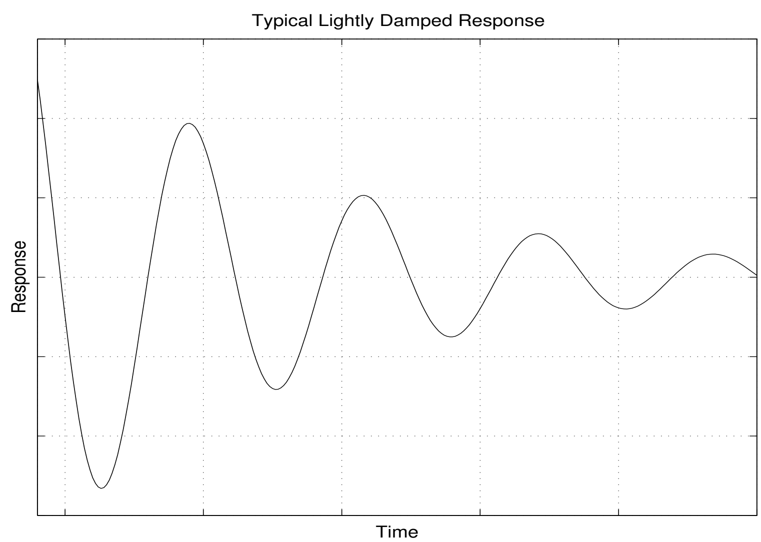
$$\frac{2\pi}{\omega_n\sqrt{1-\xi^2}} << \frac{3}{\xi\omega_n}$$

Clearly ω_n drops out of this comparison, and the above condition is equivalent to

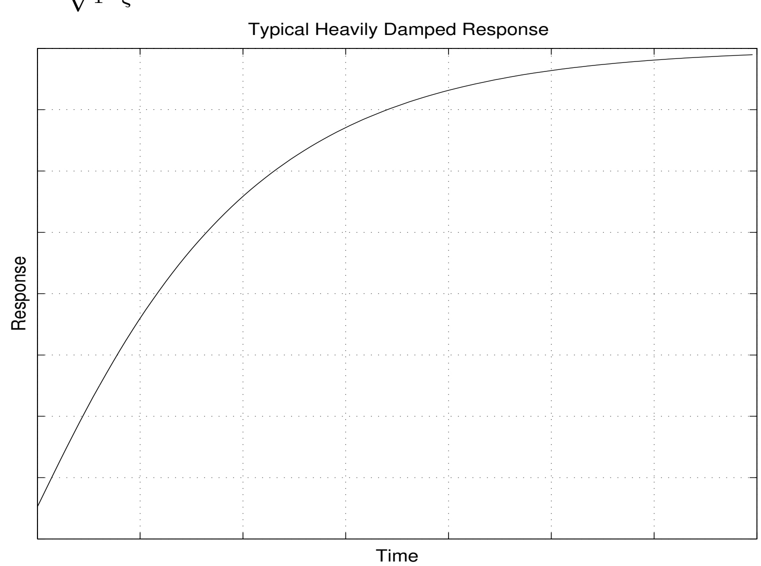
$$\frac{2\pi}{3}\frac{\xi}{\sqrt{1-\xi^2}} << 1$$

This quantity is very small if and only if ξ is very small, in which case there are many oscillations in the homogeneous solutions, and the system associated with the differential equation is called "lightly damped." The quantity ξ is called *the damping ratio*.

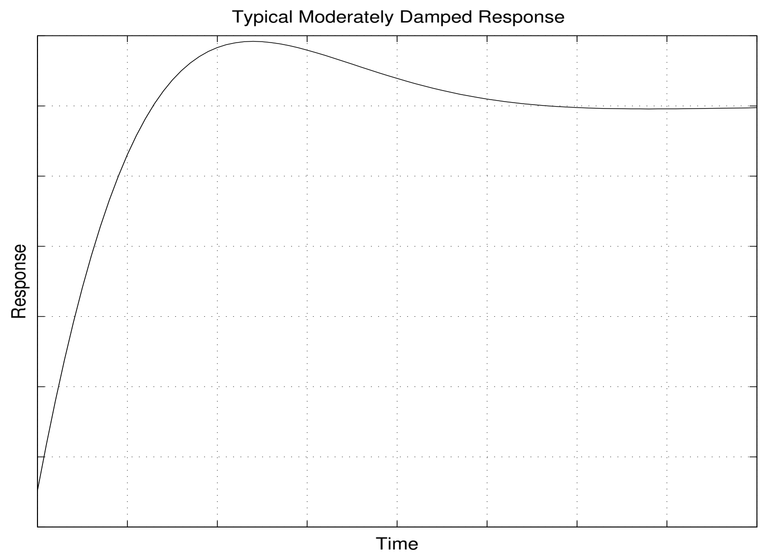
If $\frac{2\pi}{3}\frac{\xi}{\sqrt{1-\xi^2}} << 1$, then homogeneous solutions "look like"



Conversely, if $\frac{2\pi}{3}\frac{\xi}{\sqrt{1-\xi^2}} >> 1$, then then homogeneous solutions "look like"



A moderate value of ξ, say $\xi \approx 0.7$ gives homogeneous responses that "look like" the figure below.



Finally, note that ω_n enters y_H very simply,

$$y_H(t) = e^{-\xi\omega_n t}\left[A\cos\omega_n\sqrt{1-\xi^2}t \;+\; B\sin\omega_n\sqrt{1-\xi^2}t\right]$$

Note, everywhere ω_n <u>or</u> t appear, they appear together in a term $\omega_n t$. Hence, ω_n simply "scales" the response $y_H(t)$ in t. The larger value of ω_n, the faster the response.

For a constant input $v(t) \equiv \bar{v}$, it is easy to write down a particular solution to

$$\ddot{y}(t) + a_1\dot{y}(t) + a_2 y(t) = bv(t)$$

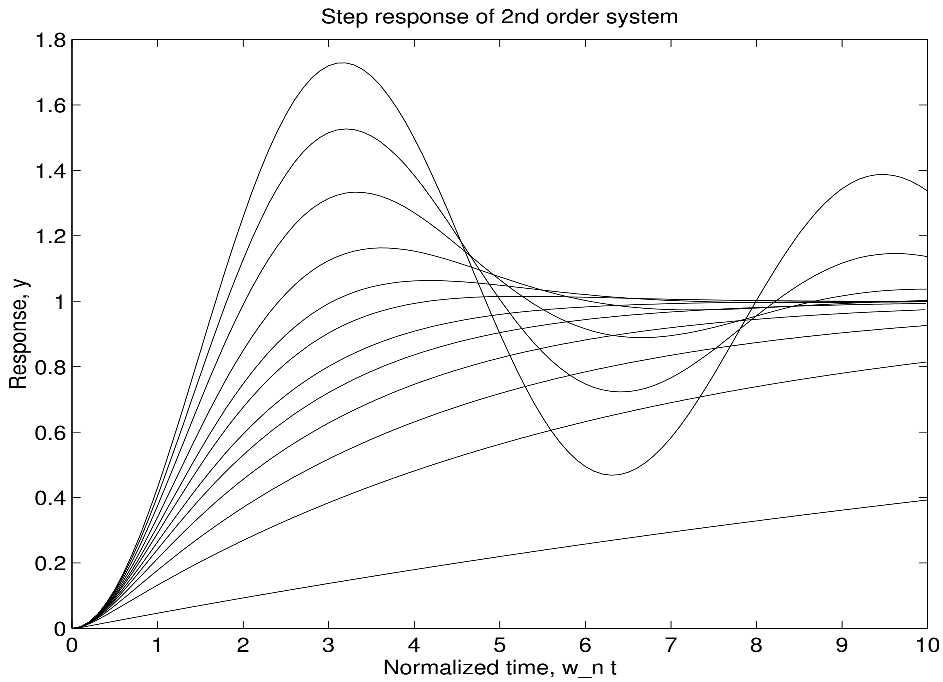
Note that $y_P(t) = \frac{b}{a_2}\bar{v}$ satisfies the differential equation. Hence, the actual solution (which must also satisfy the initial conditions) is of the form

$$\frac{b}{a_2}\bar{v} + e^{-\xi\omega_n t}\left[A\cos\omega_n\sqrt{1-\xi^2}t \;+\; B\sin\omega_n\sqrt{1-\xi^2}t\right]$$

where A and B are chosen suitably to satisfy the initial conditions. So, the homogeneous solutions connect the initial conditions to the final conditions. In this case,

$$\lim_{t\to\infty} y(t) = \frac{b}{a_2}\bar{v}, \quad \lim_{t\to\infty} \dot{y}(t) = 0$$

Shown below are plots for $b = \omega_n^2$ (so that $a_2 = b$), $\bar{v} = 1$, with ξ taking on values between 0.1 and 10.



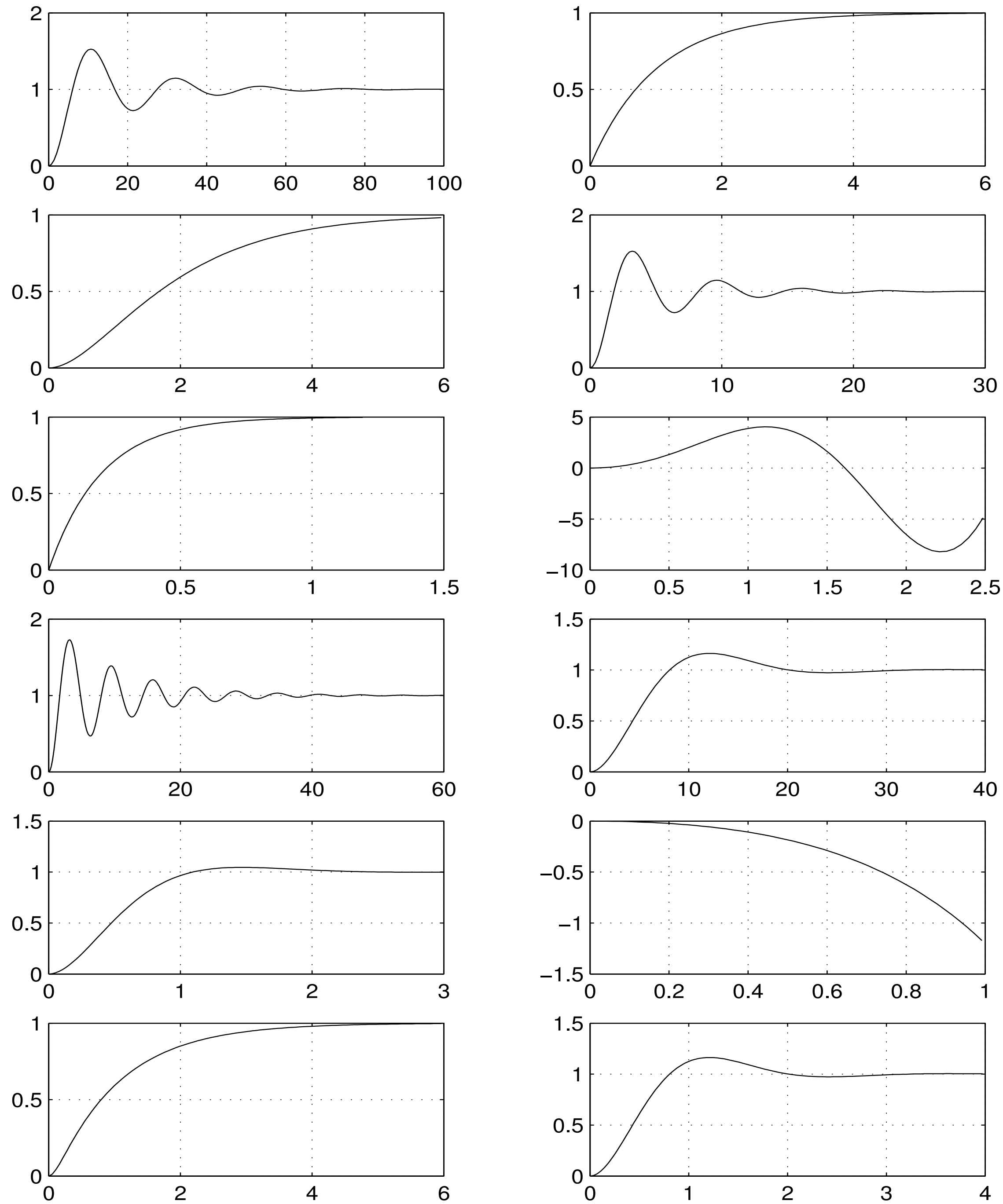
Step response of 2nd order system
Response, y
Normalized time, w_n t
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
1
2
3
4
5
6
7
8
9
10

7.10 Problems

1. Suppose $a < 0$, and consider the function te^{at} for $t \geq 0$.
   (a) For what value of t does the maximum occur?
   (b) At what value(s) of t is the function equal to 0.05 of its maximum value. For comparison, recall that for the function e^{at}, the function is equal to 0.05 of the maximum value at about $\frac{3}{-a}$.

2. Consider the example equation from Section 7.7, $\ddot{y}(t) + 4\dot{y}(t) + y(t) = 1$, with initial conditions $y(0) = y_0, \dot{y}(0) = v_0$. Consider all possible combinations of initial conditions from the lists below:
$$y_0 = \{-2, -1, 0, 1, 2\}$$
and
$$v_0 = \{-2, -1, 0, 1, 2\}$$
(**Now, do part (a) first!**)
   (a) Note that without explicitly solving the differential equation, one can easily compute 4 things: limiting value of y (as $t \to \infty$), time constant of the "slowest" homogeneous solution, initial value (given) and initial slope (given). With these numbers, carefully sketch on graph paper how you think all 25 solutions will look.
   (b) In the notes, the solution is derived for general initial conditions. Use Matlab (or similar) to plot these expressions. Compare to your simplistic approximations in part 2a.

3. The response (with all appropriate initial conditions set to 0) of the systems listed below is shown. Match the ODE with the solution graph. Explain your reasoning.
   (a) $\dot{y}(t) + y(t) = 1$
   (b) $\dot{y}(t) + 5y(t) = 5$
   (c) $\ddot{y}(t) + 2\dot{y}(t) + y(t) = 1$
   (d) $\ddot{y}(t) - 2\dot{y}(t) - y(t) = -1$
   (e) $\ddot{y}(t) - 2\dot{y}(t) + 9y(t) = 9$
   (f) $\ddot{y}(t) + 0.4\dot{y}(t) + y(t) = 1$
   (g) $\ddot{y}(t) + 0.12\dot{y}(t) + 0.09y(t) = 0.09$
   (h) $\ddot{y}(t) + 11\dot{y}(t) + 10y(t) = 10$
   (i) $\ddot{y}(t) + 0.3\dot{y}(t) + 0.09y(t) = 0.09$
   (j) $\ddot{y}(t) + 3\dot{y}(t) + 9y(t) = 9$
   (k) $\ddot{y}(t) + 4.2\dot{y}(t) + 9y(t) = 9$
   (l) $\ddot{y}(t) + 0.2\dot{y}(t) + y(t) = 1$



4. Consider the homogeneous differential equation

$$y^{[3]}(t) + 9\ddot{y}(t) + 24\dot{y}(t) + 20y(t) = 0$$

(a) What is the characteristic polynomial of the ODE?

(b) What are the roots of the characteristic polynomial.

(c) Write the general form of a homogeneous solution. Explain what are the free parameters.

(d) Show, by direct substitution, that $y_H(t) = te^{-2t}$ is a solution.

(e) Show, by direct substitution, that $y_H(t) = t^2 e^{-2t}$ is not a solution.

(f) Find the solution which satisfies initial conditions $y(0) = 3, \dot{y}(0) = 1, y^{[2]}(0) = 0$.

(g) Find the solution which satisfies initial conditions $y(0) = 3, \dot{y}(0) = 1, y^{[3]}(0) = 0$.

5. In section 7.9, we considered differential equations of the form

$$\ddot{y}(t) + a_1\dot{y}(t) + a_2 y(t) = bv(t).$$

If $a_1 > 0$ and $a_2 > 0$, and $a_1 < 2\sqrt{a_2}$, then we chose to write the solution in terms of the (ω_n, ξ) parameters, which are derived from a_1 and a_2.

If the forcing function v is a constant, $v(t) \equiv \bar{v}$, we derived that all particular solutions are of the form

$$\frac{b}{a_2}\bar{v} + e^{-\xi\omega_n t}\left[A\cos\omega_n\sqrt{1-\xi^2}t \;+\; B\sin\omega_n\sqrt{1-\xi^2}t\right]$$

where A and B are free parameters.

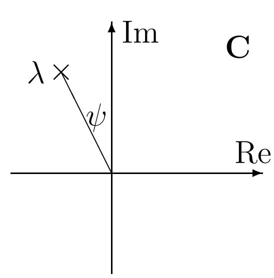
Suppose the initial conditions are $y(0) = y_0$ and $\dot{y}(0) = \dot{y}_0$. Find the correct values for A and B so that the initial conditions are satisfied. Your answer should be in terms of the given initial conditions, and system parameters (ω_n, ξ, b).

6. Suppose that $0 < \xi < 1$, and $\omega > 0$. Let λ be the complex number

$$\lambda := -\xi\omega_n \;+\; j\omega_n\sqrt{1-\xi^2}$$

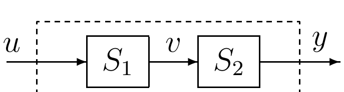
(a) Show that $|\lambda| = \omega_n$, regardless of $0 < \xi < 1$.

(b) The complex number λ is plotted in the complex plane, as shown below.



Express $\sin\psi$ in terms of ξ and ω_n.

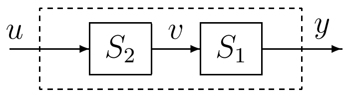
7. The cascade of two systems is shown below. The relationship between the inputs and outputs are given. Differentiate and eliminate the intermediate variable v, obtaining a differential equation relationship between u and y.



$S_2: \; \dot{y}(t) + c_1 y(t) = d_1 v(t)$

$S_1: \; \ddot{v}(t) + a_1\dot{v}(t) + a_2 v(t) = b_1\dot{u}(t) + b_2 u(t)$

Repeat the calculation for the cascade in the reverse order, as shown below.



$$S_2: \ \dot{v}(t) + c_1 v(t) = d_1 u(t)$$
$$S_1: \ \ddot{y}(t) + a_1\dot{y}(t) + a_2 y(t) = b_1\dot{v}(t) + b_2 v(t)$$

8. Compute (by analytic hand calculation) and plot the solutions to the differential equations below. Before you explicitly solve each differential equation, make a table listing

   - each root of the characteristic equation
   - the damping ratio ξ, and natural frequency ω_n for each pair (if there is one) of complex roots.
   - the final value of y, ie., $\lim_{t\to\infty} y(t)$.

   for each case. For the plots, put both cases in part (a) on one plot, and put both cases for part (b) on another plot.

   (a) i. $\frac{d^3}{dt^3}y(t) + \left(1 + 10\sqrt{2}\right)\ddot{y}(t) + \left(100 + 10\sqrt{2}\right)\dot{y}(t) + 100y(t) = 100u(t)$, subject to the initial conditions $\ddot{y}(0) = \dot{y}(0) = y(0) = 0$, and $u(t) = 1$ for all $t > 0$. **Hint:** One of the roots to the characteristic equation is -1. Given that you can easily solve for the other two.

   ii. $\dot{y}(t) + y(t) = u(t)$ subject to the initial conditions $y(0) = 0$, and $u(t) = 1$ for all $t > 0$.

   (b) i. $\frac{d^3}{dt^3}y(t) + 10.6\ddot{y}(t) + 7\dot{y}(t) + 10y(t) = 10u(t)$, subject to the initial conditions $\ddot{y}(0) = \dot{y}(0) = y(0) = 0$, and $u(t) = 1$ for all $t > 0$. **Hint:** One of the roots to the characteristic equation is -10.

   ii. $\ddot{y}(t) + 0.6\dot{y}(t) + y(t) = u(t)$, subject to the initial conditions $\dot{y}(0) = y(0) = 0$, and $u(t) = 1$ for all $t > 0$.

9. We have studied the behavior of the first-order differential equation

$$\begin{aligned} \dot{x}(t) &= -\tfrac{1}{\tau}x(t) + \tfrac{1}{\tau}u(t) \\ v(t) &= x(t) \end{aligned}$$

   which has a "time-constant" of τ, and a steady-state gain (to step inputs) of 1. Hence, if τ is "small," the output v of system follows u quite closely. For "slowly-varying" inputs u, the behavior is approximately $v(t) \approx u(t)$.

   (a) With that in mind, decompose the differential equation in (8)(a)(i) into the cascade of

      i. a "fast" 2nd order system, with steady-state gain equal to 1

      ii. "slow" first order system whose steady-state gain is 1.

      Is one of these decomposed systems similar to the system in (8)(a)(ii)? Are the two plots in (8)(a) consistent with your decomposition?

   (b) Do a similar decomposition for (8)(b)(i), and again think/comment about the response of the 3rd order system in (8)(b)(i) and the 2nd order system's response in (8)(b)(ii).