PRIME IDEALS IN NOETHERIAN RINGS: A SURVEY

ROGER WIEGAND AND SYLVIA WIEGAND

Abstract. We consider the structure of the partially ordered set of prime ideals in a Noetherian ring. The main focus is Noetherian two-dimensional integral domains that are rings of polynomials or power series.

0. Introduction

The authors have been captivated by the partially ordered set of prime ideals for about four decades. Their initial motivation was interest in Kaplansky's question, phrased about 1950: "What partially ordered sets occur as the set of prime ideals of a Noetherian ring, ordered under inclusion?" This has turned out to be an extremely difficult question, perhaps a hopeless one.

Various mathematicians have studied Kaplansky's question and related questions. In 1971, M. Hochster [12] characterized the topological spaces X such that $X \cong \operatorname{Spec}(R)$ for some commutative ring R, where $\operatorname{Spec}(R)$ is considered as a topological space with the Zariski topology. In this topology, the sets of the form $\mathrm{V}(I) := \{P \in \operatorname{Spec}(R) \mid P \supseteq I\}$, where I is an ideal of R, are the closed sets. Of course the topology determines the partial ordering, since $P \subseteq Q$ if and only if $Q \in \overline{\{P\}}$.

In 1973, W. J. Lewis showed that every finite partially ordered set is the prime spectrum of a commutative ring R, and, in 1976, Lewis and J. Ohm found necessary and sufficient conditions for a partially ordered set to be the prime spectrum of a Bézout domain [19],[20]. In [42], S. Wiegand showed that for every *rooted tree* U, there is a Bézout domain R having prime spectrum order-isomorphic to U and such that each localization $R_{\mathbf{m}}$ of R at a maximal ideal $\mathbf{m}$ of R is a maximal valuation domain. (A rooted tree is a finite poset U, with unique minimal element, such that for each $x \in U$ the elements below x form a chain.) The construction in [42] was motivated by another problem of Kaplansky: Characterize the commutative rings

Date: November 8, 2009 filename: RSSpec2009-11-8submit [final version].

1991 *Mathematics Subject Classification.* 13E05 Noetherian rings and modules, 13F20 Polynomial rings and ideals; rings of integer-valued polynomials, 13F25 Formal power series rings;" 13J15 Henselian rings .

Key words and phrases. prime ideals, Noetherian.

The authors thank the University of Nebraska for its support of our research projects.

for which every finitely generated module is a direct sum of cyclic modules. The solution, which makes heavy use of the prime spectrum, is in [40].

In general, the topology carries more information than the partial ordering. For example, one can build a non-Noetherian domain R with non-zero Jacobson radical $\mathcal{J}(R)$, but whose spectrum is order-isomorphic to $\operatorname{Spec}(\mathbb{Z})$. The partial ordering does not reveal the fact that the radical is non-zero, but the topology does: For this domain R, the set of maximal elements of $\operatorname{Spec}(R)$ is closed, whereas in $\operatorname{Spec}(\mathbb{Z})$ it is not. On the other hand, if a ring is Noetherian, the partial order determines the topology. To see this, we recall that for every ideal I of a Noetherian ring R there are only finitely many prime ideals minimal with respect to containing I; if $P_1, \dots, P_n$ are those primes, then $\mathrm{V}(I) = \bigcup_{i=1}^n \mathrm{V}(P_i) = \bigcup_{i=1}^n \overline{\{P_i\}}$. Therefore the closed subsets of $\operatorname{Spec}(R)$ are exactly the finite unions of the sets $\overline{\{P\}}$, as P ranges over $\operatorname{Spec}(R)$.

We establish some notation and terminology for *posets* (partially ordered sets).

Notation 0.1. The *height* of an element u in a poset U is $\operatorname{ht}(u) := \sup\{n \mid$ there is a chain $x_0 < x_1 < \cdots < x_n = u$ in $U\}$. The *dimension* of U is $\dim(U) = \sup\{\operatorname{ht}(u) \mid u \in U\}$. For a subset S of U, $\min(S)$ denotes the set of minimal elements of S, and $\max(S)$ its set of maximal elements. For $u \in U$, we define

$$u^{\uparrow} := \{v \mid u \leq v\} \quad \text{and} \quad u_{\downarrow} := \{v \mid v \leq u\}.$$

The *exactly less than set* for a subset $S \subseteq U$ is $\mathrm{L}_e(S) := \{v \in U \mid v^{\uparrow} - \{v\} = S\}$. For elements $u, v \in U$, their *minimal upper bound set* is the set $\operatorname{mub}(u, v) := \min(u^{\uparrow} \cap v^{\uparrow})$ and their *maximal lower bound set* is $\operatorname{Mlb}(u, v) := \max(u_{\downarrow} \cap v_{\downarrow})$.

We say v *covers* u (or v is a *cover* of u) and write "$u << v$" provided $u < v$ and there are no elements of U strictly between u and v. A chain $u_0 < u_1 < \cdots < u_n$ is *saturated* provided u_{i+1} covers u_i for each i.

To return to Kaplansky's problem, we begin by listing some well-known properties of a partially ordered set U if U is order-isomorphic to $\operatorname{Spec}(R)$, for R a Noetherian ring:

Proposition 0.2. *Let R be a Noetherian commutative ring and let U be a poset order-isomorphic to* $\operatorname{Spec}(R)$ *for some Noetherian ring R. Then*

1. *U has only finitely many minimal elements,*
2. *U satisfies the ascending chain condition.*
3. *Every element of U has finite height; in particular, U satisfies the descending chain condition.*

(4) $\mathrm{mub}(u, v)$ *is finite, for every pair of elements* $u, v \in U$.

(5) *If* $u < v < w$, *then there exist infinitely many* v_i *with* $u < v_i < w$.

Proof. Items (1) and (2) are clear, and (3) comes from the Krull Height Theorem [24, Theorem 13.5], which says that in a Noetherian ring a prime ideal minimal over an n-generated ideal has height at most n. For (4), let P and Q be prime ideals of R, and note that $\mathrm{mub}(P, Q)$ is the set of minimal prime ideals of the ideal $P + Q$. To prove (5), suppose we have a chain $P < V < Q$ of prime ideals in a Noetherian ring R, but that there are only finitely many prime ideals $V_1, \ldots, V_n$ between P and Q. By localizing at Q and passing to R_Q/PR_Q, we may assume that R is a local domain of dimension at least two, with only finitely many non-zero prime ideals V_i properly contained in the maximal ideal Q. By "prime avoidance" [2, Lemma 1.2.2], there is an element $r \in Q - (V_1 \cup \cdots \cup V_n)$. But then Q is a minimal prime of the principal ideal (r), and Krull's Principal Ideal Theorem (the case $n = 1$ of the Krull Height Theorem) says that $\mathrm{ht}(Q) \leq 1$, a contradiction. □

In 1976 [39], the present authors characterized those partially ordered sets that are order-isomorphic to the *j-spectrum* of some countable Noetherian ring. (The *j-spectrum* is the set of primes that are intersections of maximal ideals.) A poset U arises in this way if and only if

(1) U is countable and has only finitely many minimal elements,
(2) U has the ascending chain condition,
(3) every element of U has finite height,
(4) $\mathrm{mub}(u, v)$ is finite for each $u, v \in U$, and
(5) $\min(u^{\uparrow} - \{u\})$ is infinite for each non-maximal element $u \in U$.

An equivalent way of stating the theorem is: A topological space X is homeomorphic to the maximal ideal space of some countable Noetherian ring if and only if

(1) X has only countably many closed sets,
(2) X is T_1 and Noetherian, and
(3) for every $x \in X$ there is a bound on the lengths of chains of closed irreducible sets containing x.

It is still unknown whether or not the theorem is true if all occurrences of "countable" are removed.

1. Bad Behavior

Recall that a Noetherian ring R is *catenary* provided, for every pair of primes P and Q, with $P \subset Q$, all saturated chains of primes between P and Q have the same length. Every ring finitely generated as an algebra over a field, or over $\mathbb{Z}$, is catenary. More generally, excellent rings are, by definition, catenary, and the class of excellent rings is closed under the usual operations of passage to homomorphic images, localizations, and finitely generated algebras, cf. [24, p. 260]. Since fields, complete rings (e.g. rings of formal power series over a field), and the ring of integers are all excellent, the rings one encounters in nature are all catenary. Perhaps the first indicator of the rich pathology that can occur in a Noetherian ring was Nagata's example [29] of a Noetherian ring that is not catenary. Every two-dimensional integral domain is catenary, and so Nagata's example is a Noetherian local domain of dimension three; it has saturated chains of length two and length three between (0) and the maximal ideal. Later, in 1979, R. Heitmann [11] showed that every finite poset admits a saturated (i.e., cover-preserving) embedding into $\mathrm{Spec}(R)$ for some Noetherian ring R.

The catenary condition has a connection with the representation theory of local rings. As Hochster observed in 1972 [13], the existence of a maximal Cohen-Macaulay module (a finitely generated module with depth equal to $\dim(R)$) and with support equal to $\mathrm{Spec}(R)$ forces R to be *universally* catenary, that is, every finitely generated R-algebra is catenary. In particular, an integral domain with a maximal Cohen-Macaulay module must be universally catenary. G. Leuschke and R. Wiegand used this connection in [18] to manufacture a two-dimensional domain R with *no* maximal Cohen-Macaulay modules but whose completion $\widehat{R}$ has infinite Cohen-Macaulay type. (This gave a negative answer to a conjecture of Schreyer [34] on ascent of finite Cohen-Macaulay type to the completion.) For other connections between prime ideal structure and representation theory we refer the reader to the survey paper [41] by the present authors.

In Nagata's example, the catenary condition fails because a height-one prime has a cover that has height three. A theorem of McAdam [25] guarantees that such behavior cannot be too widespread:

Theorem 1.1. [25] *Let P be a prime of height n in a Noetherian ring. Then all but finitely many covers of P have height $n+1$.*

In response to a question raised by Hochster in 1974 [14], Heitmann [10] and S. McAdam [26] showed independently that there exists a two-dimensional Noetherian domain R with maximal ideals P and Q of height two such that $P \cap Q$ contains no height-one prime ideal. Later, in 1983, S. Wiegand [43] combined Heitmann's procedure with a method for producing non-catenary rings due to A. M. de Souza-Doering and I. Lequain [3], to prove the following:

Theorem 1.2. [43, Theorem 1] *Let F be an arbitrary finite poset. There exist a Noetherian ring R and a saturated embedding $\varphi : U \to \mathrm{Spec}(R)$ such that φ preserves minimal upper bounds sets and maximal lower bound sets. In detail, for $u, v \in U$, we have*

(i) $u < v$ *if and only if* $\varphi(u) < \varphi(v)$*;*
(ii) v *covers* u *if and only if* $\varphi(v)$ *covers* $\varphi(u)$*;*
(iii) $\varphi(\mathrm{mub}(u, v)) = \mathrm{mub}(\varphi(u), \varphi(v))$*; and*
(iv) $\varphi(\mathrm{Mlb}(u, v)) = \mathrm{Mlb}(\varphi(u), \varphi(v))$*.*

Using this theorem, one can characterize the spectra of two-dimensional semi-local Noetherian domains:

Corollary 1.3. [43, Theorem 2] *Let U be a countable poset of dimension two. Assume that U has a unique minimal element and* $\max(U)$ *is finite. Then* $U \cong \mathrm{Spec}(R)$ *for some Noetherian domain R if and only if* $\mathrm{L}_e(u)$ *is infinite for each element u with* $\mathrm{ht}(u) = 2$*.*

Conjecture 1.4. Let U be a two-dimensional poset in which both $\min(U)$ and $\max(U)$ are finite. Then $U \cong \mathrm{Spec}(R)$ for some Noetherian ring R if and only if

(1) $\mathrm{L}_e(u)$ is infinite for each element u with $\mathrm{ht}(u) = 2$, and
(2) $\mathrm{mub}(u, v)$ is finite for all $u, v \in \min(U)$.

2. Affine Domains of Dimension Two

We begin with an example that illustrates the effect of the ground field on delicate properties of the prime spectrum.

Example 2.1. Let k be an algebraically closed field, let $R = k[X, Y]$, let $P = (X^3 - Y^2)$, and let $\mathbf{m}$ be a maximal ideal containing P. There exists a height-one prime ideal Q such that $P^{\uparrow} \cap Q^{\uparrow} = \{\mathbf{m}\}$ if and only if either

(i) $\mathbf{m} = (X, Y)$, or
(ii) $\mathrm{char}(k) \neq 0$.

A geometric interpretation is helpful. Let C be the cuspidal curve $y^2 = x^3$, and let $p \in C - \{(0, 0)\}$. Then there is an irreducible plane curve D with $D \cap C = \{p\}$ (set-theoretically) if and only k has non-zero characteristic.

Proof. The curve C is parametrized by

$$x = t^2, \qquad y = t^3 \qquad (t \in k).$$

Since $\mathbf{m} \supset P$, the point corresponding to $\mathbf{m}$ (via the Nullstellensatz) is on C, and we can write $\mathbf{m} = (X - a^2, Y - a^3)$, where $a \in k$.

Suppose (i) and (ii) fail, that is, $\operatorname{char}(k) = 0$ and $a \neq 0$. Suppose there is a height-one prime ideal Q such that $P^{\uparrow} \cap Q^{\uparrow} = \{\mathbf{m}\}$. Let g be a monic irreducible polynomial generating Q, and note that $g(t^2, t^3) = 0$ if and only if $t = a$. With $h(T) = g(T^2, T^3)$, we see that a is the only root of h. Since k is algebraically closed, $h(T) = (T - a)^n$ for some positive integer n. But then $h(T)$ has a non-zero linear term, contradicting the fact that $h(T) \in k[T^2, T^3]$.

For the converse, we note that if $\mathbf{m} = (X, Y)$, then $(X)^{\uparrow} \cap P^{\uparrow} = \mathrm{V}(X, Y^2) = \{\mathbf{m}\}$. Now assume that $\operatorname{char}(k) = p > 0$ and that $\mathbf{m} \neq (X, Y)$, that is, $a \neq 0$. Write $p = 2r + 3s$ with $r, s \geq 0$, and let $g = X^r Y^s - a^p$. Then g is irreducible (linear, if $p = 2$ or 3). Since $g(T^2, T^3) = (T - a)^p$, (a^2, a^3) is the only point on C where g vanishes. Thus $P^{\uparrow} \cap (g)^{\uparrow} = \{\mathbf{m}\}$. □

There is a slightly fancier way to verify the assertions in the example. Notice that there exists a height-one prime $Q = (g)$ of R with $P^{\uparrow} \cap Q^{\uparrow} = \{\mathbf{m}\}$ if and only if $\overline{\mathbf{m}} := \mathbf{m}/P$ is the radical of a principal ideal of R/P. The following lemma, from W. Krauter's 1981 Ph.D. dissertation [16] (cf. also [36, Lemma 3] and [31]) explains what's going on:

Lemma 2.2. *Let R be a one-dimensional Noetherian ring such that R_{red} has only finitely many singular maximal ideals. Then* $\operatorname{Pic}(R)$ *is a torsion group if and only if every maximal ideal of R is the radical of a principal ideal.*

Proof. Since nilpotents have no effect on either of the two conditions, we may assume that R is reduced. Suppose $\operatorname{Pic}(R)$ is torsion, and let $\mathbf{m}$ be a maximal ideal of R (possibly of height zero). Choose an element $f \in \mathbf{m}$ and outside every singular maximal ideal (except possibly $\mathbf{m}$) and outside every minimal prime (except possibly $\mathbf{m}$). Write $(f) = I \cap I_1 \cap \cdots \cap I_t$, an intersection of primary ideals with distinct radicals, and with $\sqrt{I} = \mathbf{m}$. Then $(f) = IJ$, where $J = I_1 \ldots I_t$. Each prime containing J is non-singular and of height one, so J is invertible (check locally). Then $J^n = (g)$ for some $n \geq 1$, and $I^n g = (f^n)$. Since g is a non-zerodivisor, it follows that I^n is principal.

Conversely, assume every maximal ideal is the radical of a principal ideal, and let I be an invertible ideal. Then I is isomorphic to an invertible ideal J outside the union of the singular maximal ideals, [28, Lemma 4.3]. Let $\mathbf{m}_1, \ldots, \mathbf{m}_s$ be the maximal ideals containing J. The rings $R_{\mathbf{m}_i}$ are discrete valuation rings, and we let $J_{\mathbf{m}_i} = \mathbf{m}_i^{e_i} R_{\mathbf{m}_i}$. By checking locally, we see that $J = \mathbf{m}_1^{e_1} \ldots \mathbf{m}_s^{e_s}$. Now let $\mathbf{m}_i = \sqrt{(x_i)}$, and write $x_i R_{\mathbf{m}_i} = \mathbf{m}_i^{f_i} R_{\mathbf{m}_i}$. Checking locally again, we have $(x_i) = \mathbf{m}_i^{f_i}$. Now let $g_i = f_1 \ldots \widehat{f_i} \ldots f_s$, and check that $J^{f_1 \cdots f_s} = (x_1^{e_1 g_1} \ldots x_s^{e_s g_s})$. □

We now state the axioms that characterize the posets U that are order-isomorphic to $\mathrm{Spec}(R)$ for an affine domain R over a field k that is algebraic over a finite field:

Axioms 2.3.
(P0) U is countable.
(P1) U has a unique minimal element.
(P2) U has dimension two.
(P3) For each element x of height one, $x^{\uparrow}$ is infinite.
(P4) For each two distinct elements x, y of height one, $x^{\uparrow} \cap y^{\uparrow}$ is finite.
(P5) Given a finite set S of height-one elements and a finite set T of height-two elements, there is a height-one element w such that

(1) $w < t$ for each $t \in T$; and
(2) if $x \in U, s \in S$ and $w < x > s$, then $x \in T$.

Axioms (P0) - (P4) are obviously satisfied for any two-dimensional domain that is finitely generated as an algebra over a countable Noetherian Hilbert ring. (A Hilbert ring is a ring in which each prime ideal is an intersection of maximal ideals, and any finitely generated algebra over a Hilbert ring is again a Hilbert ring.) It is Axiom (P5) that makes a difference. In the special case where $S := \{s\}$ and $T = \{t\}$ with $s < t$, (P5) provides a height-one element w such that $s^{\uparrow} \cap w^{\uparrow} = \{t\}$. Thus Example 2.1 shows that $\mathrm{Spec}(k[X, Y])$ has no such element if $\mathrm{char}(k) = 0$. In fact, much more is true:

Theorem 2.4. *Let k be a field, and let R be a two-dimensional affine domain over k. If $\mathrm{Spec}(R)$ satisfies (P5), then k is an algebraic extension of a finite field.*

Proof. Suppose first that $R = k[X, Y]$. Let $P = (X^3 + XY - Y^2)$, the kernel of the map $R \twoheadrightarrow S := k[T(T-1), T^2(T-1)]$ taking X to $T(T-1)$ and Y to $T^2(T-1)$. Let $\mathbf{m}$ be an arbitrary maximal ideal containing P. Since $\mathrm{Spec}(R)$ satisfies (P5), there is a height-one prime Q such that $P^{\uparrow} \cap Q^{\uparrow} = \{\mathbf{m}\}$. Writing $Q = (f)$, we see that $\mathbf{m}/P$ is the radical of the principal ideal $(f + P)$. This shows that every maximal ideal of R/P is the radical of a principal ideal. By Lemma 2.2, $\mathrm{Pic}(R/P)$ is torsion. We can easily compute $\mathrm{Pic}(R/P) = \mathrm{Pic}(S)$ from the Mayer-Vietoris sequence [27] associated to the conductor square for S:

(2.4.1)
$$\begin{array}{ccc} S & \hookrightarrow & k[T] \\ \downarrow & & \downarrow \\ k & \hookrightarrow & \frac{k[T]}{T(T-1)} \end{array}$$

By [27], $\mathrm{Pic}(S) \cong G/H$, where $G = (\frac{k[T]}{T(T-1)})^{\times}$, the group of units of $\frac{k[T]}{T(T-1)}$, and H is the join of the images of the horizontal and vertical maps on groups of units. Since $G = k^{\times} \times k^{\times}$ and H is the diagonal embedding of $k^{\times}$ in G, we see that $\mathrm{Pic}(S) \cong k^{\times}$. Thus $k^{\times}$ is a torsion group. Therefore $\mathrm{char}(k) = p > 0$ (else 2 has infinite order in $k^{\times}$), and every non-zero element is algebraic over the prime field. This shows that k is an algebraic extension of a finite field.

In the general case, we use the Noether Normalization Lemma to express R as an integral extension of a subring $T \cong k[X,Y]$ and apply the next lemma, with $A = A' = T$ and $A'' = R$. □

Lemma 2.5. [37, Lemma 3] *Let $A' \subseteq A \subseteq A''$ be integral extensions of Noetherian domains of dimension two, and assume that A' is integrally closed. If* $\mathrm{Spec}(A'')$ *satisfies (P5) of* (2.3)*, so does* $\mathrm{Spec}(A)$.

Proof. Let S be a finite set of height-one prime ideals of A and T a finite set of maximal ideals of A. Let T'' be the finite set of prime ideals, necessarily maximal, lying over primes in T, and let $S'' = \{Q'' \in \mathrm{Spec}(A'') \mid Q'' \cap A' = Q \cap A' \text{ for some } Q \in S\}$. Let P'' be a height-one prime ideal of A'' satisfying (1) and (2) of (P5) for the sets S'' and T'' (cf. Axioms 2.3). We claim that $P := P'' \cap A$ satisfies (1) and (2) for the sets S and T. For (1), let $\mathbf{m} \in T$, and choose any $\mathbf{m}'' \in \mathrm{Spec}(A'')$ lying over $\mathbf{m}$. Then $\mathbf{m}'' \in T''$, so $P'' \subset \mathbf{m}''$; hence $P \subset \mathbf{m}$. As for (2), suppose $P \subset \mathcal{M}$ and $Q \subset \mathcal{M}$, where $\mathcal{M} \in \mathrm{Spec}(A)$ and $Q \in S$. We must show that $\mathcal{M} \in T$. By "going up", there is a prime $\mathcal{M}''$ of A'' such that $P'' \subset \mathcal{M}''$ and $\mathcal{M}'' \cap A = \mathcal{M}$. Now apply "going down" to the extension $A' \subseteq A''$ to get a prime Q'' such that $\mathcal{M}'' \supset Q''$ and $Q'' \cap A' = Q \cap A'$. Since $Q'' \in S''$, (P5)(2) (for the prime P'' and the sets S'' and T'') implies that $\mathcal{M}'' \in T''$, whence $\mathcal{M} = \mathcal{M}'' \cap A \in T$. □

As we shall see, the converse of Theorem 2.4 is true, though the proof is more difficult. At this point, it is not even clear that there exist posets satisfying Axioms 2.3. (Try building one from scratch; it's not easy!) The next theorem shows that there is *at most one* such poset. Before stating the theorem, we define an operation $A \mapsto A^{\#}$ on subsets of a poset X satisfying Axioms 2.3. Given a subset A of X, let $A^{\#}$ be obtained by adjoining to A the unique minimal element of X and the sets $x^{\uparrow} \cap y^{\uparrow}$, where x and y range over distinct height-one elements in A. (*Clarification: Here and in the sequel "height" always refers to height in X, not the relative height in A.*) Clearly $A^{\#\#} = A^{\#}$. Moreover Axiom (P4) guarantees that $A^{\#}$ is finite if A is finite.

Theorem 2.6. [36, Theorem 1] *Let U and V be posets satisfying Axioms 2.3. Given finite subsets A and B of U and V, respectively, every height-preserving isomorphism from $A^\#$ onto $B^\#$ can be extended to an isomorphism from U onto V. (In particular, U and V are isomorphic: take $A = B = \emptyset$.)*

Proof. We may assume that $A = A^\#$ and $B = B^\#$. It suffices to prove the following: For each height-preserving isomorphism $\theta : A \xrightarrow{\cong} B$ and each $x \in U - A$, θ extends to a height-preserving isomorphism θ' from $A' := (A \cup \{x\})^\#$ onto some set $B' = (B')^\# \subset Y$. For then, by symmetry, we can extend the domain of $(\theta')^{-1}$ so that it includes an arbitrary $y \in V - B'$. Since U and V are countable, we will get the desired extension of θ by iterating this back-and-forth stepwise procedure. We refer the reader to the proof of [36, Theorem 1] for the details, which are elementary and boring. □

The proof of the converse of Theorem 2.4 has two main ingredients. The first is a variant of the finiteness theorem for the class number of an algebraic number field. We refer the reader to [37] for the technical shenanigans that reduce the following theorem to the classical result on the class number:

Theorem 2.7. *Let R be a finitely generated $\mathbb{Z}$-algebra of dimension one. Then* $\mathrm{Pic}(R)$ *is finite.* □

Corollary 2.8. *Let R be a one-dimensional Noetherian ring that is finitely generated as an algebra over $\mathbb{Z}$ or over a field k that is an algebraic extension of a finite field. Then every maximal ideal of R is the radical of a principal ideal.*

Proof. By Lemma 2.2 it is enough to prove that $\mathrm{Pic}(R)$ is torsion. In view of Theorem 2.7, it will suffice to show that $\mathrm{Pic}(R)$ is torsion when R is a finitely generated k-algebra and k is algebraic over a finite field. Write $R = k[X_1, \ldots, X_m]/(f_1, \ldots, f_n)$, and choose a finite field $\mathbb{F}$ such that each f_j is in $\mathbb{F}[X_1, \ldots, X_m]$. For each intermediate field F between $\mathbb{F}$ and k, let $R_F = F[X_1, \ldots, X_m]/(f_1, \ldots, f_m)$. Then $\mathrm{Pic}(R) = \mathrm{Pic}(\varinjlim R_F) = \varinjlim(\mathrm{Pic}(R_F))$. Since each $\mathrm{Pic}(R_F)$ is finite by Theorem 2.7, $\mathrm{Pic}(R)$ is torsion. □

The second main ingredient is the following Bertini-type theorem:

Theorem 2.9. [36, Lemma 4] *Let k be an algebraically closed field, let $A = k[x_1, \ldots, x_n]$ be a two-dimensional affine domain over k, and let (f, g) be an A-regular sequence. Then there is a non-empty Zariski-open subset U of $\mathbb{A}^{n+1}(k)$ such that $\sqrt{(f + (\alpha + \sum_{i=1}^n \beta_i x_i)g)}$ is a prime ideal whenever $(\alpha, \beta_1, \ldots \beta_n) \in U$.* □

Here is the main result of this section.

Theorem 2.10. [37, Theorem 2] *Let k be a field, and let R be a two-dimensional affine domain over k. These are equivalent:*

(1) $\operatorname{Spec}(R)$ *satisfies (P5).*
(2) $\operatorname{Spec}(R)$ *is order-isomorphic to* $\operatorname{Spec}(\mathbb{Z}[X])$
(3) k is an algebraic extension of a finite field.

Proof. In view of Theorem 2.4 and Theorem 2.6, it will suffice to show that $\operatorname{Spec}(R)$ satisfies (P5) whenever $R = \mathbb{Z}[X]$ or R is an affine domain over a field k that is algebraic over a finite field. Let S and $T := \{\mathbf{m}_1 \dots, \mathbf{m}_t\}$ be the finite sets of primes we are given in Axiom (P5). We may harmlessly assume that $T \neq \emptyset$ and, by enlarging S if necessary, that each $\mathbf{m}_j$ contains some prime in S. Put $I = \bigcap S$, and choose, by Corollary 2.8, $f_j \in \mathbf{m}_j$ such that $\mathbf{m}_j = \sqrt{I + (f_j)}$. Put $f = f_1 \cdots f_t$ and $J = \bigcap T$; then $\sqrt{I + (f)} = J$. We seek a height-one prime ideal P such that $\sqrt{I + P} = J$.

Suppose first that k is the algebraic closure of a finite field and that R is a two-dimensional Cohen-Macaulay domain, finitely generated as a k-algebra. Since $I + (f)$ has height two, there is an element $g \in I$ such that (f, g) is A-regular. By Theorem 2.9 there is an element $\lambda \in A$ such that $P := \sqrt{(f + \lambda g)}$ is a prime ideal. Then $\sqrt{I + P} = J$, and so P satisfies (1) and (2) of (P5).

Suppose, now, that k is an algebraic extension of a finite field and that R is a two-dimensional affine domain over k. By the Noether Normalization Lemma [24, §33, Lemma 2] there are elements $\xi, \eta \in A$, algebraically independent over k, such that A is an integral extension of $A' := k[\xi, \eta]$. Let $\bar{k}$ be the algebraic closure of k, and let $B = (A \otimes_k \bar{k})/Q$, where the prime ideal Q is chosen so that $\dim(B) = 2$. Finally, let A'' be the integral closure of B. Then A'' satisfies Serre's condition (S_2) [24, Theorem 23.8] and hence is Cohen-Macaulay. By what we have just shown, $\operatorname{Spec}(A'')$ satisfies (P5), and now Lemma 2.5 shows that A satisfies (P5) as well.

Finally, we suppose that $R = \mathbb{Z}[X]$. We seek a height-one prime ideal P of $\mathbb{Z}[X]$ such that $J = \sqrt{I + P}$. Since $I + (f)$ has height two, there is a polynomial $g \in I$ such that f and g are relatively prime. Then, for each $j \geq 1$ the polynomial $f^k + Yg$ is irreducible in $\mathbb{Z}[X, Y]$, and hence irreducible in $\mathbb{Q}[X, Y]$ (cf., e.g., [15, Exercise 2, p. 102]). Since Bertini's Theorem is not available, we use a version of Hilbert's Irreducibility Theorem, as formulated in Chapter VIII of [17]. Combining Corollary 3 of [17, §2, p. 148] with the corollary in [17, §3, p. 152], we find that there are infinitely many prime integers p for which each of the $t + 1$ polynomials $f^j + pg, 1 \leq j \leq t + 1$, is irreducible in $\mathbb{Q}[X]$. Choose such a prime p with the additional property that $p\mathbb{Z} \neq \mathbf{m}_j \cap \mathbb{Z}$ for $1 \leq j \leq t$. For each $j \leq t+1$, let c_j be the greatest common divisor of the coefficients of $f^j + pg$; then $h_j := \frac{1}{c_j}(f^j + pg)$ is irreducible in $\mathbb{Z}[X]$.

We claim that there exists $j \leq t+1$ such that $h_j \in J = \mathbf{m}_1 \cap \cdots \cap \mathbf{m}_t$. For suppose not; then there exist i, j, ℓ, with $1 \leq i < j \leq t+1$ and $1 \leq \ell \leq t$ such that $h_i \notin \mathbf{m}_\ell$ and $h_j \notin \mathbf{m}_\ell$. Let $\mathbf{m}_\ell \cap \mathbb{Z} = q\mathbb{Z}$. Since $c_i h_i$ and $c_j h_j$ are both in $J \subseteq \mathbf{m}_\ell$, we see that the prime q is a common divisor of both c_i and c_j. Therefore $q \mid (c_i h_i - c_j h_j)$. Now $c_i h_i - c_j h_j = f^i - f^j = f^i(1 - f^{j-i})$. Since q and f^{j-i} are in $\mathbf{m}_\ell$, it follows that $q \mid f$. But also $pg = c_i h_i - f^i$ is a multiple of q, and our choice of p now forces $q \mid g$. This contradicts the assumption that f and g are relatively prime, and the claim is proved.

To complete the proof, we choose j as in the claim and put $P = h_j\mathbb{Z}[X]$. Then $P \subset J$, and $f^j = -pg + h_j c_j \in I + P$. It follows that $J = \sqrt{I + P}$ as desired. □

Actually, Theorem 1 of [37] says a bit more:

Theorem 2.11. [37, Theorem1] *Let D be an order in an algebraic number field. Then* $\mathrm{Spec}(D[X])$ *satisfies (P5) and therefore is order-isomorphic to* $\mathrm{Spec}(\mathbb{Z}[X])$.

We have put in quite a bit of detail in this chapter in order to reawaken interest in the following conjecture from [37]:

Conjecture 2.12. Let R be a two-dimensional domain finitely generated as a $\mathbb{Z}$-algebra. Then $\mathrm{Spec}(R)$ satisfies (P5) and hence is order-isomorphic to $\mathrm{Spec}(\mathbb{Z}[X])$.

It is easy to see that if $\mathrm{Spec}(R)$ satisfies (P5) so does $\mathrm{Spec}(R[\frac{1}{f}])$ for each nonzero $f \in R$. Thus $\mathrm{Spec}(D[X, \frac{1}{f}])$ is order-isomorphic to $\mathrm{Spec}(\mathbb{Z}[X])$ whenever D is an order in an algebraic number field. A stronger result is proved in [33]:

Theorem 2.13. [33, Main Theorem 1.2] *Let D be an order in an algebraic number field, let X an indeterminate, let $g_1, \ldots, g_n$ be nonzero elements of the quotient field of $D[X]$, and let $R = D[X, g_1, \ldots, g_n]$. Then* $\mathrm{Spec}(R)$ *is order-isomorphic to* $\mathrm{Spec}(\mathbb{Z}[X])$.

(The somewhat simpler case where $D = \mathbb{Z}$ is worked out in [22].)

Suppose k is a field that is not algebraic over a finite field. By Theorem 2.10, $\mathrm{Spec}(k[X, Y])$ is not isomorphic to $\mathrm{Spec}(\mathbb{Z}[X])$. Still, one can ask whether or not all two-dimensional affine domains over k have order-isomorphic spectra. The answer, in general, is “No”:

Example 2.14. [37, Corollary 7] Let k be an algebraically closed field with infinite transcendence degree over $\mathbb{Q}$, and let V be the surface in $\mathbb{A}^3(k)$ defined by the equation $X^4 + Y^4 + Z^4 + 1 = 0$. Then not every point of V is the set-theoretic intersection of two curves on V. Therefore, in the two-dimensional affine domain $R = k[X, Y, Z]/(X^4 + Y^4 + Z^4 + 1)$, there is a maximal ideal $\mathbf{m}$ such that, for each pair P, Q of height-one prime ideals, $\{\mathbf{m}\} \neq P^{\uparrow} \cap Q^{\uparrow}$. On the other hand, in $k[X, Y]$ every maximal ideal is of the form $(X - a, Y - b)$, and $\{(X - a, Y - b)\} = (X - a)^{\uparrow} \cap (Y - b)^{\uparrow}$. Thus $\operatorname{Spec}(R)$ and $\operatorname{Spec}(k[X, Y]$ are not order-isomorphic.

We know very little about the order-isomorphism classes of two-dimensional affine domains over k if k is not algebraic over a finite field. The following questions indicate the depths of our ignorance:

Questions 2.15. (1) Let k be an algebraic extension of $\mathbb{Q}$, and let R be a two-dimensional affine domain over k. Is $\operatorname{Spec}(R)$ order-isomorphic to $\operatorname{Spec}(k[X, Y])$? (2) At the other extreme, if R and S are two-dimensional affine domains over k and $\operatorname{Spec}(R)$ and $\operatorname{Spec}(S)$ are order-isomorphic, are R and S necessarily isomorphic as k-algebras? (3) Let ℓ be another algebraic extension of $\mathbb{Q}$. If $\operatorname{Spec}(k[X, Y])$ and $\operatorname{Spec}(\ell[X, Y])$ are order-isomoprhic, must k and ℓ be isomorphic fields?

3. Polynomial rings over semilocal one-dimensional domains

Naively one might suppose, since $\operatorname{Spec}(\mathbb{Q}[X])$ is order-isomorphic to $\operatorname{Spec}(\mathbb{Z})$, that also $\operatorname{Spec}(\mathbb{Q}[X, Y])$ is order-isomorphic to $\operatorname{Spec}(\mathbb{Z}[Y])$. The surprising negation of that conclusion, as discussed in the previous section, as well as the mystery surrounding $\operatorname{Spec}(\mathbb{Q}[X, Y])$, led W. Heinzer and S. Wiegand to investigate spectra for "simpler" two-dimensional polynomial rings. What if you started with a one-dimensional ring with spectrum even simpler than $\operatorname{Spec}(\mathbb{Z})$? Would the spectrum of the ring of polynomials be easier to fathom?

In particular Heinzer and S. Wiegand considered the question: What partially ordered sets arise as $\operatorname{Spec}(R[X])$ for R a one-dimensional semilocal Noetherian domain? Just as $\mathbb{Z}[X]$ played a special role in Section 2, the rings $\mathbb{Z}_{(p_1)\cup\cdots\cup(p_n)}[X]$ play a special role in the current section. (Here $p_1, \ldots, p_n$ are distinct prime integers, and $\mathbb{Z}_{(p_1)\cup\cdots\cup(p_n)}$ consists of rational numbers whose denominators are prime to each p_i.) Their investigation led to the following theorem:

Theorem 3.1. [9] *Let R be a countable Noetherian one-dimensional domain with exactly n maximal ideals. Then there exist exactly two possibilities for* $\operatorname{Spec}(R[x])$*:*

(1) *If R is not Henselian, then* $\operatorname{Spec}(R[X]) \cong \operatorname{Spec}(\mathbb{Z}_{(p_1)\cup\cdots\cup(p_n)}[X])$*, for distinct prime integers* $p_1, \ldots, p_n$*.*

(2) *If R is Henselian, then $n = 1$ and* $\operatorname{Spec}(R[X]) \cong \operatorname{Spec}(H[X])$*, where H is the Henselization of* $\mathbb{Z}_{(2)}$*.*

Examples of each are shown in Figures 3.2.1 and 3.3.1 below.

Example 3.2. The spectrum of $\mathbb{Z}_{(2)}[X]$ (where $\mathbb{Z}_{(2)}$ consists of rationals with odd denominators) is crudely drawn in Figure 3.2.1 below.

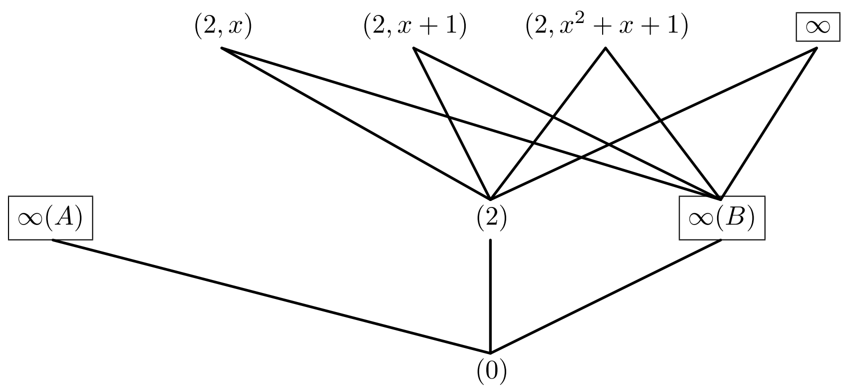


Figure 3.2.1. $\operatorname{Spec}(\mathbb{Z}_{(2)}[X])$

Diagram Notes: The "infinity box" symbol $\boxed{\infty}$ indicates that infinitely many points are in that spot. The relations between the prime ideals in $\boxed{\infty(B)}$ and the top primes are too complicated to draw accurately. Each of the primes in $\boxed{\infty(B)}$ is contained in just finitely many maximal ideals. For example, the two irreducible polynomials X and $X^2 + X + 2$ each generate height-one prime ideals in $\boxed{\infty(B)}$; (X) is contained in $(2, X)$ only, but $(X^2 + X + 2)$ is contained in both $(2, X)$ and $(2, X+1)$. The special height-one prime ideal (2) is in *all* of the height-two maximal primes. The box $\boxed{\infty(A)}$ represents the infinitely many height-one maximal ideals. Each height-two prime contains infinitely many height-one primes.

When $n = 1$, what distinguishes $\operatorname{Spec}(\mathbb{Z}_{(2)}[X])$ from $\operatorname{Spec}(H[X])$ is the following: In $\operatorname{Spec}(\mathbb{Z}_{(2)}[X])$, infinitely many height-one primes are contained in more than one maximal ideal. However, in $\operatorname{Spec}(H[X])$, $\mathbf{m}[X]$ is the *only* height-one prime contained in more than one maximal ideal (and it is contained in infinitely many).

Example 3.3. Although similar to the first picture, the illustration in Figure 3.3.1. of $\operatorname{Spec}(H[X])$, for H a countable Noetherian Henselian discrete rank-one valuation domain with maximal ideal $\mathbf{m}$, is cleaner:

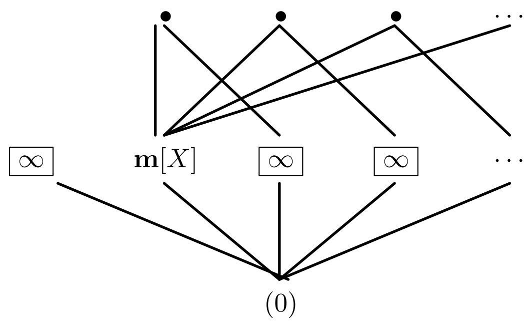


Figure 3.3.1. Spec($H[X]$)

Remarks 3.4. (1) Loosely speaking, *Henselian* rings are rings for which the conclusion of Hensel's Lemma holds. Complete local rings, e.g., power series rings over a field, are Henselian. (These will come up again in the next section.) Forming the *Henselization* of a local ring is less drastic than going to the completion. For example, the Henselization of a countable local ring is countable, whereas complete local rings of positive dimension are always uncountable. The precise definition of "Henselian" and the construction of the Henselization are given in [30, Section 33].

(2) In [35] C. Shah gave complete sets of invariants for Spec($R[X]$) for an arbitrary Noetherian semilocal domain R. If R is Henselian, with maximal ideal P, the two invariants are $r := |R|$ and $k := |(R/P)[X]|$. If R is not Henselian, with maximal ideals $P_1, \dots, P_n$, the invariants are $r := |R|$, n (the number of maximal ideals of R), and $k_1, \dots, k_n$, where $k_i := |(R/P_i)[X]|$. Shah gave examples to show that some combinations of invariants actually occur. There are some restrictions: Clearly $k_i \le r$ for each r. Also, $r \le k_i^{\aleph_0}$ for each i, by Lemma 4.2 below.

As pointed out by Kearnes and Oman [23], Shah assumed, incorrectly, that $a^{\aleph_0} = a$ for each cardinal $a \ge 2^{\aleph_0}$. In the proof of Theorem 3.1 of [8], Heinzer, Rotthaus and S. Wiegand refer to [35]; the arguments in Section 4 of this paper show that the statement of [8, Theorem 3.1] is correct.

(3) There are axioms similar to Axioms 2.3 that characterize the posets Spec($H[X]$) and Spec($\mathbb{Z}_{(p_1)\cup\cdots\cup(p_n)}[X]$) of Theorem 3.1 up to order-isomorphism. Of course (P3) is missing, and axioms analogous to (P5) distinguish the two cases (see Remark 3.5).

Remark 3.5. To state the distinguishing property between the two possibilities in Theorem 3.1 precisely, we use the "exactly less than" notation introduced in (0.1): In Spec($R[X]$), when R as above is *not* Henselian, we have:

(P5′) $L_e(T)$ is infinite for *every* finite set T of height-two maximal ideals.

If H is Henselian, however, we have:

(P5^H) Let T be a set of height-two maximal ideals. If $|T| \geq 2$, then $L_e(T) = \emptyset$; if $|T| = 1$, then $L_e(T)$ is infinite.

Remarks 3.6. Other Related Spectra

- Let R be a one-dimensional local Noetherian ring and let g, f be elements of $R[X]$ that either generate the unit ideal or form a regular sequence. W. Heinzer, D. Lantz and S. Wiegand characterized j-Spec$(R[X][\frac{g}{f}])$, and showed its relationship to the polynomials f and g. In some cases (for example, when R is a discrete valuation ring such as $\mathbb{Z}_{(2)}$, or when R is Henselian), they were able to characterize Spec$(R[X][\frac{g}{f}])$. Knowing j-Spec$(R[X][\frac{g}{f}])$ is not sufficient, however, to characterize Spec$(R[X][\frac{g}{f}])$ in general [5], [6], [7].
- In [38], R. Wiegand and W. Krauter found axioms that characterize the projective plane $\mathbb{P}^2(k)$ over the algebraic closure k of a finite field. The axioms are the same, regardless of the characteristic. A surprising consequence of the characterization is that a non-empty proper open subset U of $\mathbb{P}^2(k)$ is homeomorphic either to $\mathbb{P}^2(k) - \{\text{point}\}$ (the complement of a single point) or to $\mathbb{A}^2(k)$.

The projective line over $\mathbb{Z}$ has been studied too, in [1], [5], [21]. The poset structure is considerably more complex than that of the projective plane over the algebraic closure of a finite field. It is currently being investigated by S. Wiegand and her (current and former) students E. Celikbas and C. Eubanks-Turner.

4. Two-dimensional power series rings

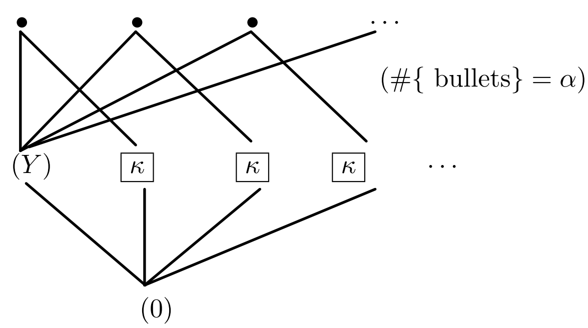
As part of an extensive project using power series rings to construct examples of rings with various properties, W. Heinzer, C. Rotthaus and S. Wiegand described the prime spectra of rings of the form $R[[Y]]$, for R a one-dimensional Noetherian domain and Y an indeterminate [8]. They completely characterized Spec$(R[[Y]])$ in the case where R is a countable domain. They did not, however, work through the cardinality arguments needed for the uncountable case. We review their results here and incidentally fill in the cardinality gap to obtain a characterization for the uncountable case as well.

First observe that, given variables X and Y, one can form "mixed polynomial/power series rings" over a field k in two ways—the second is of infinite transcendence degree over the first:

$$(1)\ k[[Y]][X] \quad \text{and} \quad (2)\ k[X][[Y]].$$

Since $k[[Y]]$ is a Henselian ring, Spec$(k[[Y]][X])$ is characterized in Theorem 3.1(2) of the previous section (actually this is Shah's extension from Remark 3.2(2), cf.

[35]). The interesting fact is that Spec($k[X][[Y]]$) is pretty similar to Spec($k[[Y]][X]$)—it just lacks the height-one maximals. Then Spec($k[X][[Y]]$) in turn is an example of Spec($R[[Y]]$), for R a general one-dimensional Noetherian domain. As we show in Theorem 4.3, the only variations in the partially ordered sets that occur as Spec($R[[Y]]$) for different one-dimensional Noetherian domains R of a given cardinality are the numbers of height-two maximal ideals of $R[[Y]]$. This number is the same as the number of maximal ideals of R (of $k[X]$ for that example), because each maximal ideal of $R[[Y]]$ has the form $(\mathbf{m}, Y)R[[Y]]$ where $\mathbf{m}$ is a maximal ideal of R, by [30, Theorem 15.1]. In particular, Spec $R[[Y]]$ has the following picture, by Theorem 4.3 below:



Spec($R[[Y]]$)

In the diagram α is the cardinality of the set of maximal ideals of R. The boxed cardinals κ (one for each maximal ideal of R) indicate that there are κ prime ideals in these positions; that is, $|L_e((\mathbf{m}, Y))| = \kappa$. The upshot of the cardinality addition to the result of [8], which is now included in Theorem 4.3, is that $\kappa = |R[[Y]]|$ and that $|L_e((\mathbf{m}, Y))| = |L_e((\mathbf{m}', Y))|$, for every pair $\mathbf{m}, \mathbf{m}'$ of maximal ideals of R.

We first use a remark from [8].

Remark 4.1. [8] Suppose that T is a commutative ring of cardinality δ, that $\mathbf{m}$ is a maximal ideal of T and that γ is the cardinality of $T/\mathbf{m}$. Then

(1) The cardinality of $T[[Y]]$ is $\delta^{\aleph_0}$, because the elements of $T[[Y]]$ are in one-to-one correspondence with $\aleph_0$-tuples having entries in T. If T is Noetherian, then $T[[Y]]$ is Noetherian, and so every prime ideal of $T[[Y]]$ is finitely generated. Since the cardinality of the set of finite subsets of $T[[Y]]$ is $\delta^{\aleph_0}$, it follows that $T[[Y]]$ has in all *at most* $\delta^{\aleph_0}$ prime ideals if T is Noetherian.

(2) If T is Noetherian, there are *at least* $\gamma^{\aleph_0}$ distinct height-one prime ideals of $T[[Y]]$ contained in $(\mathbf{m}, Y)T[[Y]]$. To see this, following the argument of [8], choose a subset $C = \{c_i \mid i \in I\}$ of T so that $\{c_i + \mathbf{m} \mid i \in I\}$ is a complete set of distinct coset representatives for $T/\mathbf{m}$. Then $|C| = \gamma$, and, for $c_i, c_j \in C$ with $c_i \neq c_j$, we have $c_i - c_j \notin \mathbf{m}$. Choose $a \in \mathbf{m}, a \neq 0$. Consider the set

$$G := \{a + \sum_{n\in\mathbb{N}} d_n Y^n \mid d_n \in C \text{ for each } n \in \mathbb{N}\}.$$

Each of the elements of G is in $(\mathbf{m}, Y)T[[Y]] \setminus YT[[Y]]$ and hence each element of G is contained in a height-one prime belonging to $L_e((\mathbf{m}, Y))$. Moreover, $|G| = |C^{\aleph_0}| = \gamma^{\aleph_0}$.

Let $P \in L_e((\mathbf{m}, Y))$. Suppose that two distinct elements of G are both in P, say $f = a + \sum_{n\in\mathbb{N}} d_n Y^n$ and $g = a + \sum_{n\in\mathbb{N}} e_n Y^n$ are in P, where each $d_n, e_n \in C$. Then we have

$$f - g = \sum_{n\in\mathbb{N}} d_n Y^n - \sum_{n\in\mathbb{N}} e_n Y^n = \sum_{n\in\mathbb{N}} (d_n - e_n) Y^n \in P.$$

Let t be the smallest power of Y so that $d_t \neq e_t$. Then $(f - g)/Y^t \in P$, since P is prime and $Y \notin P$. However, the constant term, $d_t - e_t$ is not in $\mathbf{m}$, contradicting the fact that $P \subseteq (\mathbf{m}, Y)T[[Y]]$. Thus there must be at least $|C|^{\aleph_0} = \gamma^{\aleph_0}$ distinct height-one primes contained in $L_e((\mathbf{m}, Y)T[[Y]])$, that is, $|L_e((\mathbf{m}, Y)T[[Y]])| \geq \gamma^{\aleph_0}$.

(3) Putting parts (1) and (2) together, we see that, for each maximal ideal $\mathbf{m}$ of T, $\gamma^{\aleph_0} \leq |L_e((\mathbf{m}, Y)T[[Y]])| \leq \delta^{\aleph_0}$, if T is Noetherian.

Lemma 4.2. *Let R be a Noetherian domain, Y an indeterminate and I a proper ideal of R. Let $\delta = |R|$ and $\gamma = |R/I|$. Then $\delta \leq \gamma^{\aleph_0}$, and $|R[[Y]]| = \delta^{\aleph_0} = \gamma^{\aleph_0}$.*

Proof. The first equality holds by Remark 4.1, and of course $\delta^{\aleph_0} \geq \gamma^{\aleph_0}$. For the reverse inequality, we note that the Krull Intersection Theorem [24, Theorem 8.10 (ii)] implies that $\bigcap_{n\geq 1} I^n = 0$. Therefore there is a monomorphism

$$R \hookrightarrow \prod_{n\geq 1} R/I^n. \tag{4.2.1}$$

Now R/I^n has a finite filtration with factors I^{r-1}/I^r for each r with $1 \leq r \leq n$. Since I^{r-1}/I^r is a finitely generated (R/I)-module, $|I^{r-1}/I^r| \leq \gamma^{\aleph_0}$. Therefore $|R/I^n| \leq (\gamma^{\aleph_0})^n = \gamma^{\aleph_0}$, for each n. Thus (4.2.1) implies $\delta \leq (\gamma^{\aleph_0})^{\aleph_0} = \gamma^{(\aleph_0^2)} = \gamma^{\aleph_0}$. Finally, $\delta^{\aleph_0} \leq (\gamma^{\aleph_0})^{\aleph_0} = \gamma^{\aleph_0}$, and so $\delta^{\aleph_0} = \gamma^{\aleph_0}$. □

Theorem 4.3. *Suppose that R is a one-dimensional Noetherian domain with cardinality $\delta := |R|$, and that the cardinality of the set of maximal ideals of R is α (α*

can be finite). Let $U = \operatorname{Spec} R[[Y]]$, where Y is an indeterminate over R. Then the poset U is characterized by the following axioms:

- ***(1):*** $|U| = \delta^{\aleph_0}$.
- ***(2):*** *U has a unique minimal element, namely* (0).
- ***(3):*** $\dim(U) = 2$ *and* $|\{$ *height-two elements of* $U\}| = \alpha$.
- ***(4):*** *There exists a unique height-one element $u_Y \in U$ (namely $u_Y = (Y)$) such that u is contained in every height-two element of U.*
- ***(5):*** *Every height-one element of U except for u_Y is in exactly one height-two element.*
- ***(6):*** *For every height-two element $t \in U$, $|L_e(t)| = |R[[Y]]| = \delta^{\aleph_0}$. If $t_1, t_2 \in U$ are distinct height-two elements, then the element u_Y from (4) is the unique height-one element less than both.*
- ***(7):*** *There are no height-one maximal elements in U. Every maximal element has height two. (This property, implicit in (5), is stated for emphasis.)*

Proof. Most of the proof is done in [8]. It remains to check the statement $|L_e(t)| = |R[[Y]]| = \delta^{\aleph_0}$ in item (6). This is immediate from Remark 4.1 and Lemma 4.2. □

Remarks 4.4. C. Eubanks-Turner, M. Luckas, and S. Saydam (former Ph.D. students of S. Wiegand) have characterized $\operatorname{Spec}(R[[X]][g/f])$, for R a one-dimensional Noetherian domain with infinitely many maximal ideals and g, f a generalized $R[[X]]$-sequence, in their recent work [4]. They specify various possibilities for the j-spectrum that depend upon f and g.

For example the following diagram shows the partially ordered set j-$\operatorname{Spec}(B)$, for $B := \mathbb{Z}[[X]][\frac{g}{f}]$, $f = 11880 + \sum_{i=1}^{\infty} X^i$ and $g = 9900 + \sum_{i=1}^{\infty} X^i$:

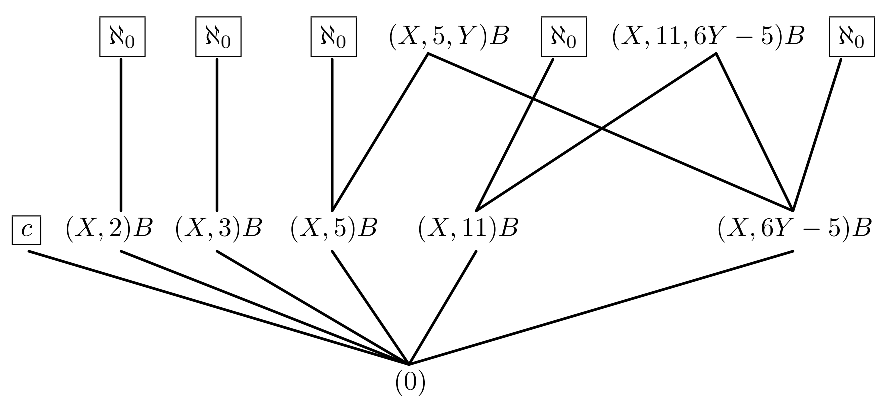


Figure 4.4.1: $\operatorname{Spec}(\mathbb{Z}[[x]][\frac{9900+\sum_{i=1}^{\infty} X^i}{\sum_{i=1}^{\infty} 11880+X^i}])$

Notes 4.5. [8, Remarks 3.6, Corollary 3.7] It is evident from Theorem 4.3 that

$$\mathrm{Spec}(\mathbb{Z}[[Y]]) \cong \mathrm{Spec}(\mathbb{Q}[X][[Y]]) \cong \mathrm{Spec}((\mathbb{Z}/2\mathbb{Z})[X][[Y]]) \ncong \mathrm{Spec}(\mathbb{R}[X][[Y]]).$$

(The last has uncountably many maximal ideals.) As Theorem 2.10 indicates,

$$\mathrm{Spec}(\mathbb{Z}[Y]) \cong \mathrm{Spec}((\mathbb{Z}/2\mathbb{Z})[X][Y]) \ncong \mathrm{Spec}(\mathbb{Q}[X][Y]).$$

Thus the situation for power series rings is different from the polynomial case.

References

[1] M. Arnavut,The projective line over the integers, *Commutative Algebra*, Arab. J. Sci. Eng., Sect. C, Theme Issues 26 (2001), 31–44.

[2] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge Stud. Adv. Math. 39, Cambridge, 1993.

[3] A. M. de Souza Doering and I. Lequain, The gluing of maximal ideals—applications to a problem in the spectrum of a Noetherian ring and to problems on the going up and going down in polynomial rings, Trans. Amer. Math. Soc. 260 (1980), 583–593.

[4] C. Eubanks-Turner, M. Luckas, and S. Saydam, Prime ideals in birational extensions of two-dimensional power series rings, preprint.

[5] W. Heinzer, D. Lantz and S. Wiegand, Projective lines over one-dimensional semilocal domains and spectra of birational extensions, in *Algebra Geometry and Applications* (C. Bajaj, ed.), Springer-Verlag, New York, 1994.

[6] W. Heinzer, D. Lantz and S. Wiegand, Prime ideals in birational extensions of polynomial rings, in *Commutative Algebra: Syzygies, Multiplicities and Birational Algebra* (W. Heinzer, C. Huneke and J. Sally, eds.), Contemp. Math. 159, 1994, 73–93.

[7] W. Heinzer, D. Lantz and S. Wiegand, Prime ideals in birational extensions of polynomial rings II, in *Zero-Dimensional Commutative Rings* (D. Anderson and D. Dobbs, eds.), Marcel Dekker, New York, 1995.

[8] W. Heinzer, C. Rotthaus and S. Wiegand, Mixed polynomial/power series rings and relations among their spectra, in *Multiplicative Ideal Theory in Commutative Algebra* (J. Brewer, S. Glaz, W. Heinzer, eds.), Springer, 2006, 227–242.

[9] W. Heinzer and S. Wiegand, Prime ideals in two-dimensional polynomial rings, Proc. Amer. Math. Soc. 107 (1989), 577–586.

[10] R. Heitmann, Prime ideal posets in Noetherian rings., Rocky Mountain J. Math. 7 (1977), 667–673.

[11] R. Heitmann, Examples of non-catenary rings, Trans. Amer. Math. Soc. 247 (1979), 125–136.

[12] M. Hochster, Prime ideal structure in commutative rings, Trans. Amer. Math. Soc. 137 (1969), 43–60.

[13] M. Hochster, Cohen-Macaulay modules, in *Conference on Commutative Algebra: Lawrence, Kansas 1972* (J. W. Brewer and Edgar A. Rutter, eds.), Lecture Notes in Math. 311, Springer-Verlag, Berlin, 1973, 120–152.

[14] M. Hochster, *Topics in the Homological Theory of Modules over Commutative Rings*, CBMS Regional Conference Ser. 24, Amer. Math. Soc., Providence, 1975.

[15] I. Kaplansky, *Commutative Rings*, Allyn and Bacon, Boston, 1970.

[16] W. Krauter, *Combinatorial Properties of Algebraic Sets*, Ph.D. Dissertation, University of Nebraska, 1981.

[17] S. Lang, *Diophantine Geometry*, Interscience, New York, 1959.

[18] G. Leuschke and R. Wiegand, Ascent of finite Cohen-Macaulay type, J. Algebra 228 (2000), 674–681.

[19] W. Lewis, The spectrum of a ring as a partially ordered set, J. Algebra 25 (1973), 419–434.

[20] W. Lewis and J. Ohm, The ordering of Spec R, Canadian J. Math 28 (1976), 820–835.

[21] A. Li and S. Wiegand, Polynomial behavior of prime ideals in polynomial rings and the projective line over the integers, Lecture Notes Pure Applied Math. Series 189: *Factorization in Integral Domains* (Daniel D. Anderson, ed.), Dekker, 1997, 383–399.

[22] A. Li and S. Wiegand, Prime ideals in two-dimensional domains over the integers, J. Pure Appl. Algebra 130 (1998), 313–324.
[23] K. Kearnes and G. Oman, Cardinalities of residue fields of Noetherian integral domains, Comm. Algebra, to appear.
[24] H. Matsumura, *Commutative Ring Theory*, Cambridge Stud. Adv. Math. 8, Cambridge Univ. Press, Cambridge, 1986.
[25] S. McAdam, Saturated chains in Noetherian rings, Indiana Univ. Math. J. 23 (1973/74), 719–728.
[26] S. McAdam, Intersections of height 2 primes, J. Algebra 40 (1977), 315–321.
[27] J. Milnor *Introduction to Algebraic K-Theory*, Princeton Univ. Press, Princeton, 1971.
[28] M. P. Murthy and R. G. Swan, Vector bundles over affine surfaces, Invent. Math. 36 (1976), 125–165.
[29] M. Nagata, On the chain problem of prime ideals, Nagoya Math. J. 10 (1956), 51–64.
[30] M. Nagata, *Local Rings*, John Wiley, New York, 1962.
[31] J. V. Pakala and T. S. Shores, On compactly packed rings, Pacific J. Math. 36 (1976), 197–201.
[32] A. Saydam and S. Wiegand, Noetherian domains with the same prime ideal structure as $Z_{(2)}[x]$, Arab. J. Sci. Eng., Sect. C, Theme Issues 26 (2001), 187–198.
[33] A. Saydam and S. Wiegand, Prime ideals in birational extensions of two-dimensional domains over orders, J. Pure Appl. Algebra 201 (2005) 142–153.
[34] F.-O. Schreyer, Finite and countable CM-representation type, in *Singularities, Representation of Algebras, and Vector Bundles: Proceedings Lambrecht 1985* (G.-M. Greuel and G. Trautmann, eds.), Lecture Notes in Math. 1273, Springer-Verlag, Berlin, 1987, 9–34.
[35] C. Shah, Affine and projective lines over one-dimensional semilocal domains, Proc. Amer. Math. Soc. 107 (1989), 577–586.
[36] R. Wiegand, Homeomorphisms of affine surfaces over a finite field, J. London Math. Soc. 18 (1978), 28–32.
[37] R. Wiegand, The prime spectrum of a two-dimensional affine domain, J. Pure Appl. Algebra 40 (1986), 209–214.
[38] R. Wiegand and W. Krauter, Projective surfaces over a finite field, Proc. Amer. Math. Soc. 83 (1981), 233–237.
[39] R. Wiegand and S. Wiegand, The maximal ideal space of a Noetherian ring, J. Pure Appl. Algebra 8 (1976), 129–141.
[40] R. Wiegand and S. Wiegand, Commutative rings whose finitely generated modules are direct sums of cyclics, Lecture Notes in Math. 616, Springer-Verlag, 1977, 406–423.
[41] R. Wiegand and S. Wiegand, Prime ideals and decompositions of modules, in *Non-Noetherian Commutative Ring Theory* (Scott Chapman and Sarah Glaz, eds.), Kluwer, 2000, 403–428.
[42] S. Wiegand, Locally maximal Bezout domains, Proc. Amer. Math. Soc. 47 (1975), 10–14.
[43] S. Wiegand, Intersections of prime ideals in Noetherian rings, Comm. Algebra 11 (1983), 1853–1876.

Department of Mathematics, University of Nebraska, Lincoln, NE 68588-0130
E-mail address: `rwiegand@math.unl.edu`

Department of Mathematics, University of Nebraska, Lincoln, NE 68588-0130
E-mail address: `swiegand@math.unl.edu`