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Risk–reward models for on-line leasing of depreciable equipment

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ABSTRACT

In this paper, we generalize the traditional on-line leasing problem to the case where the equipment is depreciable and the investor can always sell the used equipment for a positive price, which is an essential feature of many practical leasing problems. The traditional risk–reward model for the case with a certain forecast has been discussed for the on-line leasing of depreciable equipment. On the basis of this, an improved risk–reward model with a probability forecast is obtained here and presented as the main result. A relationship between the two risk–reward models, in which the latter includes the former as a special case, is proved. Numerical analysis shows that the competitive performance is significantly improved in the risk–reward models.

1. Introduction

On-line algorithms have been used in solving financial optimization problems [1–3]. When Sleator and Tarjan [4] advanced to comparing an on-line algorithm to the optimal off-line algorithm and Karlin et al. [5] coined the term competitive analysis, an extensive and systematic study started. In particular, on-line leasing has been widely studied [6–11]. Competitive analysis compares the performances of on-line algorithms to that of the optimal off-line algorithm and uses the competitive ratio to evaluate the on-line algorithms. However, competitive analysis is a worst-case analysis, and has been judged to be too conservative. Hence, probabilistic analysis is introduced in [10,12] to handle the problem, but this assumes that the inputs are subject to a known distribution. Therefore, probabilistic analysis has been criticized for making distributional assumptions that are too strong. As a result, al-Binali [13] advanced a risk–reward framework to blend competitive analysis and probabilistic analysis. The risk–reward model not only allows the investor to benefit from a correct forecast but also allows him/her to control the risk of performing too poorly with respect to the optimal off-line algorithm when the forecast is incorrect. The forecast in al-Binali's risk–reward model is usually certain. On the basis of an uncertain forecast, Dong et al. [11] put forward a more flexible risk–reward model, which depends on the risk tolerance level, the different forecasts and the probability of each forecast's correctness. The new risk–reward model not only includes al-Binali's risk–reward model, but also can achieve better performance in some situations which are decided by the probability of each forecast's correctness. So, in this paper we called this the improved risk–reward model. Generally, there are two actions in the risk–reward models: a riskless action that leads to a certain outcome and a risky action that leads to either a gain or a loss. The outcome is the competitive ratio achieved, and it is uncertain for the risky action. However, traditional competitive analysis does not give the on-line investor a risky choice, but simply selects the riskless action and achieves the optimal competitive ratio. al-Binali has given a detailed explanation of the operation of the risk–reward framework [13].

We used al-Binali's risk–reward model to discuss the on-line leasing of depreciable equipment, and obtained the optimal restricted ratios with and without an interest rate [14]. The problem of on-line leasing of depreciable equipment is described

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again as follows. An on-line investor needs depreciable equipment (e.g., a plane, a house) for finite periods (e.g., days, months), but he/she does not know the quantity in advance. Denote the rental cost in every period by L and the initial price of the equipment by P. Generally, $P \gg L$. Let N be the number of actual usage periods. At the beginning of each period, the on-line investor can determine whether the equipment will be needed for the current period; and he/she must decide between the following two choices: to lease the equipment with a leasing fee L, or to buy it with a depreciation D, which is the loss of the equipment's value in one time period, and a transaction cost C. After using the equipment for finite periods, the on-line investor can sell it in the secondary market. This leasing problem is obviously different from that considered in previous studies since it assumes that the equipment is depreciable and a secondary market exists, which is an essential feature of many practical leasing problems.

The risk–reward model obtained with a certain forecast is provided under certain assumptions [14]. The assumptions are that the life expectancy of the equipment is finite and long enough, the rental periods are contiguous and finite, and the number of rental periods is less than the equipment's life expectancy. Hence, one unit of new equipment can meet the need of the on-line investor. That is to say, we only consider the case where the net value of depreciable equipment after depreciation is larger than 0, i.e., $N < P/D$, which means that the equipment still has some value after N periods and the investor can always sell it for a positive price.

On the basis of the risk–reward model (we call this the general risk–reward model in the following exposition) obtained with a certain forecast, this paper mainly uses the improved risk–reward model to discuss the on-line leasing of depreciable equipment. With the probability forecast, we have obtained the optimal risk algorithm. Our analysis shows that the improved risk–reward model has extended the general risk–reward model. The rest of this paper is organized as follows. In Section 2, we first give some basic definitions and notation for al-Binali's risk–reward model [13]; then we restate our results for the general risk–reward model for on-line leasing of depreciable equipment, which was presented in [14]. In Section 3, we use the improved risk–reward model to discuss the on-line leasing of depreciable equipment and obtain the optimal risk algorithm; and we show that the improved risk–reward model has extended the general risk–reward model. In Section 4, we show numerically that the optimal restricted ratios dramatically decrease compared with the optimal competitive ratio. In Section 5, we conclude the paper and discuss future research topics.

2. The general risk–reward model for on-line leasing of depreciable equipment

2.1. al-Binali's risk–reward model with a certain forecast

In this subsection, we give definitions and notation for al-Binali's risk–reward model [13]. Consider a cost minimization problem $\wp$ consisting of a set I of inputs. Let $Cost_{ALG}(\sigma)$ be the cost of the on-line algorithm ALG on input σ, where $\sigma \in I$. The cost of an optimal off-line algorithm OPT on input σ is $Cost_{OPT}(\sigma) = \min_{ALG} Cost_{ALG}(\sigma)$. Then the competitive ratio of an algorithm ALG on problem $\wp$ is

$$R_{ALG} = \sup_{\sigma \in I} \frac{Cost_{ALG}(\sigma)}{Cost_{OPT}(\sigma)}.$$

The optimal competitive ratio for problem $\wp$ is

$$R^* = \inf_{ALG} R_{ALG}.$$

The risk of an algorithm ALG is defined as R_{ALG}/R^*. If r ($r \geq 1$) is the risk tolerance of the investor, then denote the set of all algorithms that satisfy the investor's risk tolerance by $J_r = \{ALG : R_{ALG} \leq rR^*\}$.

A forecast is assumed to be a subset of the input set I. Denote the forecast by F and $F \subset I$. We define $R_{F(ALG)}$ to be the competitive ratio of an algorithm ALG restricted to cases where the forecast is correct, i.e.,

$$R_{F(ALG)} = \sup_{\sigma \in F} \frac{Cost_{ALG}(\sigma)}{Cost_{OPT}(\sigma)},$$

and denote by $R_F = \inf_{ALG \in J_r} R_{F(ALG)}$ the optimal restricted ratio, which can also be seen as the best possible ratio for algorithms in J_r when the forecast is correct. Then, we need to measure the reward of ALG as an improvement over the optimal on-line algorithm. Hence, we define

$$f_{F(ALG)} = \frac{R^*}{R_{F(ALG)}}$$

as the reward of ALG. Given a problem $\wp$ consisting of input set I, a forecast $F \subset I$ and a risk tolerance r, there exists an optimal risk tolerant algorithm $ALG^* \in J_r$ such that

$$f_{F(ALG^*)} = \sup_{ALG^* \in J_r} f_{F(ALG)}.$$

Therefore, when the forecast is correct, the optimal restricted ratio is R_F and the reward of the optimal risk tolerant algorithm is $f_{F(ALG^*)} = R^*/R_F$. Otherwise it is not meaningful to talk about the reward. In other words, the risk–reward model uses a forecast to develop an algorithm which maximizes the reward should the forecast come true. However, it does not exceed the investor's risk tolerance for any input sequence.

2.2. Traditional competitive analysis for on-line leasing of depreciable equipment

Let L and D be the rental cost and the depreciation per period, respectively. The transaction costs of buying and selling are both $C(C < D)$. Then, we have: (1) the equipment leasing company must make profits; thus $L > D$ holds; (2) in the first period, the rental cost is less than the purchasing cost, as otherwise, the on-line investor would buy the equipment at the beginning; thus $L < 2C + D$ holds. Here, the assumptions are that the price of new equipment does not change, and the price of used equipment equals that of new equipment minus the total depreciation. Suppose that the on-line investor needs the equipment throughout N contiguous periods and he/she has no knowledge about N. Therefore, the cost of the optimal off-line algorithm is

$$\begin{aligned} Cost_{OPT}(N) &= \min\{NL, 2C + ND\} \\ &= \begin{cases} NL, & N < T_0; \\ ND + 2C, & N \geq T_0, \end{cases} \end{aligned} \tag{1}$$

where $T_0 = 2C/(L - D)$. For simplicity we assume that T_0 is an integer. Suppose the set of on-line strategies is $\{S(T)\}_{T \geq 1}$, where $S(T)$ is the strategy where the investor buys the equipment after leasing it for the first $T - 1$ periods and then uses it continuously for the following $N - T + 1$ periods, and then sells it in the secondary market. Therefore, the cost of the on-line strategy $S(T)$ is

$$Cost_{ON}(N) = \begin{cases} NL, & N < T; \\ (T-1)L + (N - T + 1)D + 2C, & N \geq T. \end{cases} \tag{2}$$

On the basis of (1) and (2), using traditional competitive analysis, we can obtain an optimal deterministic on-line leasing strategy for depreciable equipment and its optimal competitive ratio. The conclusions are given in Theorem 2.1. The proof is presented in [14], and is similar to that in [6], and is therefore omitted.

Theorem 2.1. *The optimal deterministic strategy for on-line leasing of depreciable equipment is: buy after leasing for $T_0 - 1$ periods and then use it continuously during the following $N - T_0 + 1$ periods, and then sell it in the secondary market. Moreover, the optimal competitive ratio of this strategy is*

$$R^* = 1 + \frac{(L - D)(2C + D - L)}{2LC},$$

where $T_0 = 2C/(L - D)$.

As can be seen, the ratio R^* is obtained by the optimal strategy S^*: if $N \leq T_0 - 1$, then the investor always leases the equipment; otherwise, the investor buys the equipment after leasing it for $T_0 - 1$ periods and then uses it continuously during the following $N - T_0 + 1$ periods, and then sells it in the secondary market.

2.3. The general risk–reward model for on-line leasing of depreciable equipment

In this subsection we restate our result for the general risk–reward model for on-line leasing of depreciable equipment. It was obtained by using al-Binali's risk–reward framework which is based on the deterministic strategy achieved above and two forecasts of $N < T_0$ and $N \geq T_0$. When forecast $N < T_0$ is correct, the algorithms where the investor always leases in the set $J_r = \{ALG : R_{ALG} \leq rR^*\}$ will be used by the on-line investor and the optimal restricted ratio is $R_F = 1$. For the forecast case of $N \geq T_0$, the risk algorithm was obtained in [14]. The improved risk–reward model presented in Section 3 is based on this case. Hence, for convenience we restate the conclusions of Theorem 2.2 and present a simple proof.

Theorem 2.2. *If the forecast $N \geq T_0$ is correct and the risk tolerance satisfies $r \geq \max(1, \Delta)$, then the optimal restricted ratio is*

$$R_F = 1 + \frac{(L - D)(D + 2C - LrR^*)}{D(D + 2C - L) + 2CL(rR^* - 1)}, \tag{3}$$

where

$$\Delta = \frac{(L - D)(2C + D) + 2CL}{R^*((L - D)(2C + D) + 2CD)}.$$

Proof. Our hypothesis for the strategy that the on-line investor adopts is $A(S)$, which is to lease for the first S periods and then buy. Hence, the adversary can make the competitive ratio be

$$\frac{SL + D + 2C}{\min\{(S + 1)D + 2C, (S + 1)L\}}.$$

On the basis of the risk tolerance, there are two cases as follows.

Case 1. $S < T_0$. According to the definition of risk tolerance, it holds that

$$\frac{SL + D + 2C}{(S+1)L} \leq rR^*.$$

If $D + 2C - rLR^* \geq 0$, we have

$$S \geq \frac{D + 2C - rLR^*}{(rR^* - 1)L} \equiv S_1. \tag{4}$$

Case 2. $S \geq T_0$. The restricted ratio in the false forecast satisfies

$$\frac{SL + D + 2C}{(S+1)D + 2C} \leq rR^*.$$

If $L - rDR^* > 0$, we have

$$S \leq \frac{(D + 2C)(rR^* - 1)}{L - rDR^*} \equiv S_2. \tag{5}$$

It is easy to check that $S_1 \leq S_2$. In fact, we have

$$\begin{aligned} S_2 - S_1 &= \frac{(D+2C)L(rR^*-1)^2 - (D+2C-rLR^*)(L-rDR^*)}{(rR^*-1)(L-rDR^*)L} \\ &= \frac{(D+2C)[L(rR^*)^2 - 2LrR^* + rDR^*] + rLR^*(L-rDR^*)}{(rR^*-1)(L-rDR^*)L} \\ &\geq \frac{rLR^*[L(rR^*)^2 - 2LrR^* + rDR^* + L - rDR^*]}{(rR^*-1)(L-rDR^*)L} \\ &= \frac{L^2 rR^*(rR^*-1)^2}{(rR^*-1)(l-rR^*D)L} \geq 0, \end{aligned}$$

where the inequality follows from the condition $D + 2C - LrR^* \geq 0$. As can be seen, S_1 and S_2 change with r. When $T_0 \geq S_1$, we have

$$r \geq \frac{(L-D)(2C+D) + 2CL}{R^*((L-D)L + 2CL)} \tag{6}$$

and when $T_0 \leq S_2$, we have

$$r \geq \frac{(L-D)(2C+D) + 2CL}{R^*((L-D)(2C+D) + 2CD)}. \tag{7}$$

It is easy to check that the right hand side of (7) is larger than that of (6). Therefore, $T_0 \in [S_1, S_2]$ implies

$$r \geq \frac{(L-D)(2C+D) + 2CL}{R^*((L-D)(2C+D) + 2CD)} \equiv \Delta. \tag{8}$$

The risk tolerance r means that S can only change in the interval $[S_1, S_2]$. If the forecast $N \geq T_0$ is correct, the optimal off-line strategy is to buy the equipment at the beginning. Hence, the restricted ratio in this forecast is

$$R_F(S) = \frac{SL + D + 2C}{(S+1)D + 2C}. \tag{9}$$

Since $\frac{\partial}{\partial S} R_F(S) > 0$, $R_F(S)$ attains its minimum given by (3) at $S = S_1$. $\square$

3. The improved risk–reward model for on-line leasing of depreciable equipment

3.1. The improved risk–reward model with a probability forecast

In al-Binali's risk–reward model, the forecast is usually certain. Under the assumption that the forecast is uncertain, Dong et al. [11] extended the certain forecast to the probability forecast. The probability forecast is established as follows. Divide the input I by a group of subsets, which are denoted by $F_1, F_2, \ldots, F_m$, where $\cup F_i = I$ and $F_i \cap F_j = \emptyset$ for $i \neq j$. Let P_i be the probability that the investor anticipates that the input $\sigma \in F_i$, where $\sum_{i=1}^{m} P_i = 1$. The set of $\{(F_i, P_i)|\ i = 1, 2, \ldots, m\}$ is called a probability forecast. Let

$$R_{PF_i(ALG)} = \sup_{\sigma \in F_i} \frac{Cost_{ALG}(\sigma)}{Cost_{OPT}(\sigma)}$$

be the restricted competitive ratio of algorithm *ALG* with the correct forecast F_i. Denote the reward of the correct forecast F_i by

$$f_{PF_i(ALG)} = \frac{R^*}{R_{PF_i(ALG)}},$$

where R^* is the optimal competitive ratio of algorithm *ALG* with no forecast. On the basis of this, with the probability forecast $\{(F_i, P_i)|i = 1, 2, \ldots, m\}$, Dong et al. defined $R_{PF(ALG)} = \sum_{i=1}^{m} P_i R_{PF_i(ALG)}$ and $f_{PF(ALG)} = R^*/R_{PF(ALG)}$ as the restricted competitive ratio and the reward, respectively. They also proved that the reward of the probability forecast has two desired properties [11].

Property 1. *For any on-line algorithm ALG*, $\min_i\{f_{PF_i(ALG)}\} \le f_{PF(ALG)} \le \max_i\{f_{PF_i(ALG)}\}$.

Let $\{(F_i, P_i)|i = 1, 2, \ldots, m\}$ be a probability forecast. Divide F_i into $F_{i,1}$ and $F_{i,2}$, where $F_{i,1} \cup F_{i,2} = F_i$ and $F_{i,1} \cap F_{i,2} = \emptyset$. And divide P_i into $P_{i,1}$ and $P_{i,2}$, where $P_{i,1} + P_{i,2} = P_i$. In this way, a more detailed probability forecast based on $\{(F_i, P_i)|i = 1, 2, \ldots, m\}$ is constructed. It can be denoted by $\{(F_1, P_1), (F_2, P_2), \ldots, (F_{i-1}, P_{i-1}), (F_{i,1}, P_{i,1}), (F_{i,2}, P_{i,2}), (F_{i+1}, P_{i+1}), \ldots, (F_m, P_m)\}$. Let $\hat{f}_{PF(ALG)}$ be the reward obtained with the newly constructed probability forecast. Then the following Property 2 is obtained.

Property 2. *For any on-line algorithm $ALG \in \{S(T)\}_{T\ge 1}$, $f_{PF(ALG)} \le \hat{f}_{PF(ALG)}$.*

Properties 1 and 2 show that the more detailed the probability forecast is, the greater the reward is. With the probability forecast, there exists a more generalized risk–reward model. Following previous notation, denote the risk tolerance level by r, and denote the set of all on-line algorithms with risk tolerance level r by $J_r = \{ALG|R_{ALG} \le rR^*\}$. The main purpose of the improved risk–reward framework is to look for an algorithm $ALG^* \in J_r$ that maximizes the reward with the probability forecast. Therefore, the mathematical model for obtaining the optimal risk algorithm is presented as follows:

$$\max_{ALG} f_{PF(ALG)} = \frac{R^*}{R_{PF(ALG)}} \tag{10}$$

$$s.t.\ R_{ALG} \le rR^*. \tag{11}$$

Steps for obtaining the optimal risk algorithm with the probability forecast:

1. With respect to the specific on-line problem, determine the optimal competitive ratio R^* that is obtained with no forecast.
2. Divide the total inputs I into $F_1, F_2, \ldots, F_m$, where $\cup F_i = I$ and $F_i \cap F_j = \emptyset$ for $i \ne j$.
3. Denote the probability that the investor anticipates that the input $\sigma \in F_i$ by P_i, where $\sum_{i=1}^{m} P_i = 1$.
4. Compute the restricted competitive ratio $R_{PF_i(ALG)}$ and $R_{PF(ALG)}$ with the probability forecast $\{(F_i, P_i)|i = 1, 2, \ldots, m\}$.
5. Set the risk tolerance level to r.
6. Solve the model (10)–(11) to obtain the optimal risk algorithm.

3.2. The improved risk–reward model for on-line leasing of depreciable equipment

In this subsection, we pursue the risk algorithm for the deterministic strategies $\{S(T)\}_{T\ge 1}$ by using a probability forecast. From Section 2 we know that the optimal competitive ratio of algorithm $S(T)$ obtained with no forecast is R^*.

As can be seen, $T_0 = 2C/(L - D)$ is the key point of the optimal off-line leasing algorithm. On the basis of this, we construct the probability forecast by using two forecasts: $F_1 = \{N : N < T_0\}$ with probability P_1 and $F_2 = \{N : N \ge T_0\}$ with probability P_2, where $P_1 + P_2 = 1$. In each period, the ratio of leasing cost to buying cost for an on-line investor is $L/(2C + D)$; and the ratio of selling profit to leasing profit for the owner is D/L. When the two ratios are equal, i.e., $L/(2C + D) = D/L$, using the steps for obtaining the optimal risk algorithm described in Section 3.1, we obtain the following Theorem 3.1, which gives optimal risk algorithm with strategies $\{S(T)\}_{T\ge 1}$ with a probability forecast $\{(F_1, P_1), (F_2, P_2)\}$.

Theorem 3.1. *For on-line leasing of depreciable equipment, with deterministic strategy $S(t)$ with a probability forecast $\{(F_1, P_1), (F_2, P_2)\}$, when setting the risk tolerance level $r \ge \max(1, \Delta)$, the optimal risk algorithm is $S(T^*)$ and T^* is*

$$T^* = \begin{cases} \dfrac{2C}{(L - D)}, & P_1 \ge 1/2; \\[2ex] \dfrac{2C\sqrt{\frac{P_1}{1-P_1}}}{L - D\sqrt{\frac{P_1}{1-P_1}}}, & \delta \le P_1 < 1/2; \\[2ex] \dfrac{D + 2C - L}{(R^* r - 1)L}, & P_1 < \delta, \end{cases} \tag{12}$$

where

$$\delta = \frac{(S_1L/(2C+S_1D))^2}{1+(S_1L/(2C+S_1D))^2}.$$

Proof. According to Section 2, the algorithm set with risk level r can be denoted by $S_{rl} = \{S(T) : T \in [S_1, S_2]\}$, where $r \geq \Delta$. With the probability forecast $\{(F_1, P_1), (F_2, P_2)\}$, we have

$$R_{PF(S(T))} = \sum_{i=1}^{2} P_i R_{PF_i(S(T))}$$

and

$$R_{PF_i(S(T))} = \sup_{\sigma \in F_i} \frac{Cost_{ON}(\sigma)}{Cost_{OPT}(\sigma)}, \quad i = 1, 2.$$

By computing, we obtain

$$R_{PF_1(S(T))} = \sup_{N<T_0} \frac{Cost_{ON}(N)}{Cost_{OPT}(N)} = \begin{cases} 1, & T \geq T_0; \\ \dfrac{L(T-1)+D+2C}{LT}, & T < T_0; \end{cases} \tag{13}$$

$$R_{PF_2(S(T))} = \sup_{N\geq T_0} \frac{Cost_{ON}(N)}{Cost_{OPT}(N)} = \begin{cases} \dfrac{L(T-1)+D+2C}{DT+2C}, & T \geq T_0; \\ \dfrac{L(T-1)+D+2C}{DT+2C}, & T < T_0. \end{cases} \tag{14}$$

Consequently, we have

$$R_{PF(S(T))} = \begin{cases} P_1 + (1-P_1)\dfrac{L(T-1)+D+2C}{DT+2C}, & T \geq T_0; \\ P_1\dfrac{L(T-1)+D+2C}{LT} + (1-P_1)\dfrac{L(T-1)+D+2C}{DT+2C}, & T < T_0, \end{cases} \tag{15}$$

and the derivative of $R_{PF(S(T))}$ with respect to T is

$$\frac{\partial}{\partial T} R_{PF(S(T))} = \begin{cases} (1-P_1)\dfrac{(L-D)(2C+D)}{(DT+2C)^2}, & T \geq T_0; \\ -P_1\dfrac{2C+D-L}{LT^2} + (1-P_1)\dfrac{(L-D)(2C+D)}{(DT+2C)^2}, & T < T_0. \end{cases} \tag{16}$$

On the basis of (16) and equality $D/L = L/(2C+D)$, we have $\frac{\partial}{\partial T} R_{PF(T)} < 0$ for $T < T_0$ and $P_1 \geq 1/2$. In fact, letting

$$G(T) = -\frac{2C+D-L}{LT^2} + \frac{(L-D)(2C+D)}{(DT+2C)^2},$$

when $P_1 \geq 1/2$, we obtain

$$\frac{\partial}{\partial T} R_{PF(S(T))} \leq \frac{1}{2} G(T).$$

On the other hand, we have

$$\frac{\partial}{\partial T} G(T) = \frac{2L^2(2C+D-L)}{(LT)^3} - \frac{2(L-D)D(2C+D)}{(DT+2C)^3} > 0,$$

which is obtained on the basis of $LT < DT + 2C$ for $T < T_0$. Therefore, we have

$$\frac{\partial}{\partial T} R_{PF(S(T))} \leq \frac{1}{2} G(T) < \frac{1}{2} G(T)|_{T=T_0} = 0.$$

Thus, when $P_1 \geq 1/2$, $R_{PF(S(T))}$ is monotonically decreasing at $T < T_0$, and monotonically increasing at $T \geq T_0$; and in this case the optimal risk algorithm is $S(T_0)$.

When $P_1 < 1/2$, $R_{PF(S(T))}$ is monotonically decreasing at $T < N^*$, and monotonically increasing at $T \geq N^*$, where

$$N^* = \frac{2C\sqrt{\frac{P_1}{(1-P_1)}}}{L - D\sqrt{\frac{P_1}{(1-P_1)}}}.$$

Table 1
Comparison between R^* and R_F.

L	D	C	r	R^*	R_F	$\frac{R^*-R_F}{R^*-1}$ (%)
1000	800	200	1.05	1.100	1.041	59
1200	1000	250	1.05	1.100	1.058	42
1600	1200	300	1.03	1.083	1.018	78.3

Table 2
Comparison between optimal restricted ratios and the optimal competitive ratio with $L = 80$, $D = 64$, $C = 18$.

r	R^*	R_F	$[S_1, S_2]$	P_1	R_{PF}
1.05	1.11	1.060	[1.51, 4.06]	1.00	1.000
				0.80	1.022
				0.50	1.055
				0.45	1.108
				0.30	1.093
				0.00	1.060

In addition, $N^* \geq S_1$ demands

$$P_1 \geq \frac{(S_1 L/(2C + S_1 D))^2}{1 + (S_1 L/(2C + S_1 D))^2} \triangleq \delta.$$

Hence, when $\delta \leq P_1 < 1/2$, the optimal risk leasing algorithm is $S(N^*)$.

Furthermore, when $P_1 \leq \delta$, the optimal risk algorithm is $S(S_1)$. This is identical to that obtained under al-Binali's risk–reward model since in this case the probability P_2 that forecasts F_2's correctness is very large and the probability forecast $\{(F_1, P_1), (F_2, P_2)\}$ is almost equivalent to the certain forecast F_2.

Therefore, from the above analysis we obtain the optimal risk leasing algorithm $S(T^*)$ that makes $R_{PF(S(T))}$ reach its minimum and T^* is given by Eq. (12). □

Corollary 3.2. *When $P_1 = 1$, $T^* = 2C/(L - D)$, and the optimal restricted ratio is $R_{PF} = 1$ which is also the optimal restricted ratio R_F obtained with the certain forecast $N < 2C/(L - D)$; when $P_1 = 0$, $T^* = (D + 2C - L)/(R^* r - 1)L$, and the optimal restricted ratio is*

$$1 + (L - D)(D + 2C - LrR^*)/(D(D + 2C - L) + 2CL(rR^* - 1)),$$

which is also the optimal restricted ratio R_F obtained with a the certain forecast $N \geq 2C/(L - D)$.

Corollary 3.2 shows that the improved risk–reward model has generalized al-Binali's risk–reward model.

4. Numerical analysis

Numerical examples are presented in this section to illustrate the improved performance of the two risk–reward models and the relationship between them. The results are presented in Tables 1 and 2.

As regards al-Binali's risk–reward model whose optimal restricted ratio is R_F, we take $(R^* - R_F)/(R^* - 1)$ as the improvement measurement over the traditional competitive ratio R^* since the largest improvement is $R^* - 1$. From Table 1, it is clear that with the correct forecast, the average improvement is 59.8%, which means that the investor can improve his/her performance significantly by taking the risk of achieving a competitive ratio larger than the optimal competitive ratio.

Given the risk tolerance, Table 2 presents the optimal competitive ratio R^*, the optimal restricted ratio R_F, the interval $[S_1, S_2]$, and the optimal restricted ratio R_{PF} with a different P_1. From Table 2, we can clearly conclude the relationships between R^*, R_F and R_{PF}. That is, R_F is a special case of R_{PF} ($P_1 = 0$) and the optimal restricted ratio R_{PF} is almost always less than the optimal competitive ratio R^* at every P_1.

5. Conclusions

The case of on-line leasing of depreciable equipment is discussed in this paper. Using the competitive ratio for the evaluation of on-line algorithms, based on the general risk–reward model obtained with al-Binali's certain forecast, we mainly use the improved risk–reward model to discuss the on-line leasing of depreciable equipment. Numerical analysis shows that the competitive performance is significantly improved in the two risk–reward models. There are many aspects of this problem meriting future research. It might be interesting to introduce factors of inflation and salvage to the models. It is also interesting to consider randomized competitive strategies for on-line leasing of depreciable equipment.

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