1. Hilbert Polynomials

Let $X \subset \mathbb{P}^n$ be a projective variety. We define the Hilbert Polynomial of X to be

$$\chi(\mathcal{O}_X(m)) = \sum_{i\geq 0}(-1)^i h^i(\mathcal{O}_X(m)) \tag{1.1}$$

where $\mathcal{O}_{\mathbb{P}^n}(1)$ is the hyperplane bundle on $\mathbb{P}^n$ and $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^n}(1) \otimes \mathcal{O}_X$ is its restriction to X.

We need a list of vanishing results:

Theorem 1.1 (Grothendieck). *For a scheme X,*

$$H^i(X, \mathcal{F}) = 0 \tag{1.2}$$

for $i > \dim X$ and all coherent sheaves $\mathcal{F}$ on X.

Note that this is not true for sheaves $\mathcal{F}$ that are not coherent. For example, if X is a smooth complex variety of dimension n, then

$$H^i(X, \mathbb{Z}) = H^i(X) \tag{1.3}$$

for $\mathcal{F} = \mathbb{Z}$, where $H^i(X)$ is the i-th singular cohomology. When X is projective, $H^{2n}(X, \mathbb{Z}) = \mathbb{C}$.

Theorem 1.2 (Serre). *For every projective scheme $X \subset \mathbb{P}^n$ and every coherent sheaf $\mathcal{F}$, there exists a number N, depending on X and $\mathcal{F}$ such that*

$$H^i(X, \mathcal{F}(m)) = H^i(X, \mathcal{F} \otimes \mathcal{O}_X(m)) = 0 \tag{1.4}$$

for all $i > 0$ and $m \geq N$.

Theorem 1.3 (Kodaira). *For every projective complex manifold $X \subset \mathbb{P}^n$,*

$$H^i(\mathcal{O}_X(-m)) = 0 \tag{1.5}$$

for all $0 \leq i < \dim X$ and $m > 0$.

For $P = \mathbb{P}^n$, $H^0(\mathcal{O}_P(m))$ can be identified with the space of homogeneous polynomials of degree m in $n+1$ variables

$$\begin{aligned} H^0(\mathbb{P}^n, \mathcal{O}(m)) &= \left\{ \sum_{d_0+d_1+\ldots+d_n=m} a_{d_0 d_1 \ldots d_n} X_0^{d_0} X_1^{d_1} \ldots X_n^{d_n} \right\} \\ &\cong \mathbb{C}^{\binom{m+n}{n}}. \end{aligned} \tag{1.6}$$

We have Kodiara-Serre duality

Theorem 1.4 (Kodiara-Serre). *Let X be a smooth projective variety of dimension n. Then*

$$H^i(E) \times H^{n-i}(E^\vee \otimes K_X) \longrightarrow H^n(K_X) = \mathbb{C} \tag{1.7}$$

is a perfect pairing for all vector bundles E on X, where $K_X = \wedge^n \Omega_X$ is the canonical bundle of X. Therefore,

$$H^i(E)^\vee \cong H^{n-i}(E^\vee \otimes K_X). \tag{1.8}$$

The canonical bundle of $P = \mathbb{P}^n$ can be computed from the Euler sequence

$$0 \longrightarrow \mathcal{O}_P \longrightarrow \mathcal{O}_P(1)^{\oplus(n+1)} \longrightarrow T_P \longrightarrow 0 \tag{1.9}$$

by which we have

$$(\wedge^n \Omega_P)^\vee = \wedge^n T_P = \mathcal{O}_P \otimes \wedge^n T_P \cong \wedge^{n+1} \mathcal{O}_P(1)^{\oplus(n+1)} = \mathcal{O}_P(n+1) \tag{1.10}$$

and hence

$$K_P = \mathcal{O}_P(-n-1). \tag{1.11}$$

Combining Kodaira-Serre duality and Kodaira vanishing, we can compute $h^i(\mathcal{O}_P(m))$ as follows:

$$h^i(\mathcal{O}_P(m)) = \begin{cases} 0 & \text{if } i \neq 0, n \\ \binom{m+n}{m} & \text{if } i = 0 \\ h^0(\mathcal{O}_P(-n-1-m)) & \text{if } i = n. \end{cases} \tag{1.12}$$

By Serre's vanishing theorem,

$$\chi(\mathcal{O}_X(m)) = h^0(\mathcal{O}_X(m)) \tag{1.13}$$

for m sufficiently large. So we may use the right hand side (RHS) of (1.13) as the definition of Hilbert polynomial since the left hand side (LHS) of (1.13) is indeed a polynomial of m:

Theorem 1.5. *Let $X \subset \mathbb{P}^n$ be a projective scheme. Then*

$$\begin{aligned} \chi(\mathcal{O}_X(m)) = d\binom{m+r}{r} + c_{r-1}\binom{m+r-1}{r-1} \\ + c_{r-2}\binom{m+r-2}{r-2} + \dots + c_0 \end{aligned} \tag{1.14}$$

for some constants $c_i \in \mathbb{Z}$, where $r = \dim X$ and $d = \deg X$.

Proof. We prove by induction on r. When $r = 0$, X is a 0-dimensional scheme of degree d. Namely, $\mathcal{O}_X$ is an artin ring of $\dim_{\mathbb{C}} \mathcal{O}_X = d$ over $\mathbb{C}$ as a vector space. So $h^0(\mathcal{O}_X(m)) = h^0(\mathcal{O}_X)$ and $\chi(\mathcal{O}_X(m)) = d$.

For $r > 0$, let Λ be a general hyperplane. Then $Y = X \cap \Lambda$ is a scheme of dimension $r - 1$ of degree d. We have the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-1) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0. \tag{1.15}$$

It follows that

$$\begin{aligned} \chi(\mathcal{O}_X(m)) - \chi(\mathcal{O}_X(m-1)) = \chi(\mathcal{O}_Y(m)) \\ = d\binom{m+r-1}{r-1} + \sum_{i=0}^{r-2} b_i \binom{m+i}{i} \end{aligned} \tag{1.16}$$

by induction hypothesis for some $b_i \in \mathbb{Z}$. Then

$$\begin{aligned}\chi(\mathcal{O}_X(m)) &= \chi(\mathcal{O}_X(0)) + \sum_{k=0}^{m} \chi(\mathcal{O}_Y(k)) \\ &= d\sum_{k=0}^{m} \binom{k+r-1}{r-1} + \sum_{i=0}^{r-2} b_i \left(\sum_{k=0}^{m} \binom{k+i}{i}\right) + c_0 \\ &= d\binom{m+r}{r} + \sum_{i=0}^{r-2} b_i \binom{m+i+1}{i+1} + c_0\end{aligned} \tag{1.17}$$

and (1.14) follows. □

For a projective scheme $X \subset \mathbb{P}^n$, if we have a free resolution of $\mathcal{O}_X$

$$0 \to E_l \to E_{l-1} \to ... \to E_0 = \mathcal{O}_P \to \mathcal{O}_X \to 0 \tag{1.18}$$

with each E_i being a direct sum $\oplus \mathcal{O}_P(e_{ij})$, then the Hilbert polynomial of X can be easily computed by

$$\chi(\mathcal{O}_X(m)) = \sum_{i=0}^{l} (-1)^i \chi(E_i(m)). \tag{1.19}$$

Such resolution always exists, which is the consequence of the famous Hilbert's Syzygy Theorem:

Theorem 1.6 (Hilbert's Syzygy)**.** *Every coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ has a free resolution, i.e., a long exact sequence*

$$0 \to E_l \to E_{l-1} \to ... \to E_0 \to \mathcal{O}_X \to 0 \tag{1.20}$$

with each E_i being a direct sum $\oplus \mathcal{O}_{\mathbb{P}^n}(e_{ij})$ and $l \le n$.

In particular, a complete intersection $X \subset \mathbb{P}^n$ has a well-known free resolution, called *Koszul Complex*. A complete intersection $X \subset \mathbb{P}^n$ of type $(d_1, d_2, ..., d_r)$ is a scheme $X = X_1 \cap X_2 \cap ... \cap X_r$ cut out by hypersurfaces $X_i \subset \mathbb{P}^n$ of degree d_i with $\dim X = n - r$. For such X, we have a free resolution

$$0 \longrightarrow \wedge^r E \longrightarrow \wedge^{r-1} E \longrightarrow ... \longrightarrow E \longrightarrow \mathcal{O}_P \longrightarrow \mathcal{O}_X \longrightarrow 0, \qquad E = \bigoplus_{i=1}^{r} \mathcal{O}_P(-d_i) \tag{1.21}$$

of $\mathcal{O}_X$, called the *Koszul complex* induced by the map

$$E \xrightarrow{\varphi} \mathcal{O}_P, \qquad E = \bigoplus_{i=1}^{r} \mathcal{O}_P(-d_i) \tag{1.22}$$

sending $(g_1, g_2, ..., g_r)$ to $g_1F_1 + g_2F_2 + ... + g_rF_r$ with F_i the defining equation of X_i. Then the Hilbert polynomial of X is very easy to compute:

$$\begin{aligned} \chi(\mathcal{O}_X(m)) &= \sum_{i=0}^{r} (-1)^i \chi(\wedge^i E \otimes \mathcal{O}_P(m)) \\ &= \sum_{I\subset\{1,2,...,r\}} (-1)^{|I|} \binom{m - d_I + n}{n} \end{aligned} \tag{1.23}$$

where we use the notations $|I| = i$ and $d_I = d_{a_1} + d_{a_2} + ... + d_{a_i}$ for an index set $I = \{a_1, a_2, ..., a_i\}$.

Of particular interest to us is the Hilbert polynomial of a plane curve $C \subset \mathbb{P}^2$ of degree d. By (1.23), it is given by

$$\chi(\mathcal{O}_C(m)) = \binom{m+2}{2} - \binom{m-d+2}{2} = md - \binom{d-1}{2} + 1. \tag{1.24}$$

As another example, let us consider the Hilbert polynomial of a rational normal curve. A rational normal curve C in $\mathbb{P}^n$ is the image of the embedding $f : \mathbb{P}^1 \to \mathbb{P}^n$ given by $H^0(\mathcal{O}_{\mathbb{P}^1}(n))$:

$$f(Z_0, Z_1) = \sigma(Z_0^n, Z_0^{n-1}Z_1, ..., Z_0Z_1^{n-1}, Z_1^n) \tag{1.25}$$

where σ is an automorphism of $\mathbb{P}^n$, i.e., a $\mathbb{P}GL(n + 1)$ action. When $n = 2$, it is a conic and when $n = 3$, it is a *twisted cubic*.

A rational normal curve is not linear degenerated, i.e., it is not contained in a hyperplane of $\mathbb{P}^n$; otherwise,

$$c_0Z_0^n + c_1Z_0^{n-1}Z_1 + ... + c_{n-1}Z_0Z_1^{n-1} + c_nZ_1^n = 0 \tag{1.26}$$

for all (Z_0, Z_1) and some constants $c_0, c_1, ..., c_n$, not all zero, which is impossible. It has degree n since (1.26) has n solutions in $\mathbb{P}^1$. When $n \geq 3$, it is NOT a complete intersection; otherwise, $C = X_1 \cap X_2 \cap ... \cap X_{n-1}$ for some hypersurfaces X_i of degree d_i; since C is linear non-degenerate, $d_i \geq 2$ and then $\deg C = n \geq 2^{n-1}$, which is impossible. The Hilbert polynomial of C is very easy to compute since $f^*\mathcal{O}_P(m) = \mathcal{O}_{\mathbb{P}^1}(mn)$ and hence

$$\chi(\mathcal{O}_C(m)) = \chi(\mathcal{O}_{\mathbb{P}^1}(mn)) = mn + 1. \tag{1.27}$$

Hilbert polynomials are deformational invariants.

Theorem 1.7. *Let $X \subset \mathbb{P}^n \times B$ be a flat family of closed subschemes of $\mathbb{P}^n$ over a variety B. Then there exists a polynomial $\phi(m)$ such that*

$$\chi(\mathcal{O}_{X_b}(m)) \equiv \phi(m) \tag{1.28}$$

for all $b \in B$, where X_b is the fiber of X over $b \in B$.

Question 1.8 (Flattened Twisted Cubics' Puzzle)**.** *Let*

$$\varphi : \mathbb{A}^2 \to \mathbb{P}^3 \times \mathbb{A}^1 \tag{1.29}$$

be the map given by

$$\varphi(s,t) = (1, st, s^2, s^3) \times (t). \tag{1.30}$$

Let X be the closure of $\varphi(\mathbb{A}^2)$ in $\mathbb{P}^3 \times \mathbb{A}^1$. Then X is irreducible and dominates $\mathbb{A}^1$ so it is flat over $\mathbb{A}^1$. The fiber X_1 of X over $t = 1$ is

$$X_1 = \overline{\{(1, s, s^2, s^3)\}} \tag{1.31}$$

which is a twisted cubic in $\mathbb{P}^3$. So its Hilbert polynomial is

$$\chi(\mathcal{O}_{X_1}(m)) = 3m + 1. \tag{1.32}$$

The fiber X_0 of X over $t = 0$ seems to be

$$X_0 = \overline{\{(1, 0, s^2, s^3)\}} = \Lambda \cap W \tag{1.33}$$

where Λ is the hyperplane $\{Z_1 = 0\}$ and W is the cubic surface $\{Z_2^3 = Z_3^2\}$. So its Hilbert polynomial is

$$\chi(\mathcal{O}_{X_0}(m)) = \binom{m+3}{3} - \binom{m+2}{3} - \binom{m}{3} + \binom{m-1}{3} = 3m. \tag{1.34}$$

So $\chi(\mathcal{O}_{X_1}(m)) \neq \chi(\mathcal{O}_{X_0}(m))$! What went wrong?

2. A Note On Koszul Complex

The idea of Koszul complex comes from a very simple observation in linear algebra. Let V be a vector space over a field k and $\phi : V \to k$ be a linear functional. We have a map $\partial_m : \wedge^m V \to \wedge^{m-1} V$ defined by

$$\partial_m(v_1 \wedge v_2 \wedge ... \wedge v_m) = \sum_{i=1}^{m} (-1)^i \phi(v_i) v_1 \wedge v_2 \wedge ... \wedge \widehat{v_i} \wedge ... \wedge v_m \tag{2.1}$$

It is easy to check that $\partial_{m-1} \circ \partial m = 0$. Therefore, we have a complex

$$... \to \wedge^m V \xrightarrow{\partial_m} \wedge^{m-1} V \xrightarrow{\partial_{m-1}} \wedge^{m-2} V \to ... \to V \xrightarrow{f} k \to 0 \tag{2.2}$$

which is called the Koszul complex associated to (or induced by) $\phi : V \to k$.

It is easy to check that (2.2) is actually exact as long as ϕ is surjective, i.e., $\phi \neq 0$. If $\dim V = r$, this sequence has length $r + 2$.

More generally, let R be a commutative ring, M be a module over R and $\phi : M \to R$ be a homomorphism of R-module. Then we can form a Koszul complex $\wedge^m M \to \wedge^{m-1} M$ just as in the case of vector spaces.

When $M = R^{\oplus r}$, we assume that $\phi : M \to R$ is given by

$$\phi(a_1, a_2, ..., a_r) = a_1 f_1 + a_2 f_2 + ... + a_r f_r. \tag{2.3}$$

We say that $f_1, f_2, ..., f_r$ is a *regular sequence* of R if f_i is not a zero divisor of $R/(f_1, f_2, ..., f_{i-1})$ for $i = 1, 2, ..., r$.

Then the Koszul complex $\wedge^\bullet M$ is exact if $f_1, f_2, ..., f_r$ is a regular sequence of R. This can be proved by induction on r and the observation that the sequence

$$\begin{aligned} 0 \to R/(f_1, f_2, ..., f_{i-1}) &\xrightarrow{\times f_i} R/(f_1, f_2, ..., f_{i-1}) \\ &\to R/(f_1, f_2, ..., f_{i-1}, f_i) \to 0 \end{aligned} \tag{2.4}$$

is exact.

Translating this geometrically, $f_1, f_2, ..., f_r$ being a regular sequence means $\operatorname{Spec} R/(f_1, f_2, ..., f_r)$ is a complete intersection in $\operatorname{Spec} R$. Hence the Koszul complex $\wedge^\bullet M$ gives a free resolution of $R/(f_1, f_2, ..., f_r)$. This explains (1.21).

One special case is that $R = (f_1, f_2, ..., f_r)$, where ϕ is surjective and $\wedge^\bullet M$ is exact. It follows that if we have a surjective map $E \to \mathcal{O}_X$ between vector bundles on X, then the corresponding Koszul complex $\wedge^\bullet E$ is exact.

Koszul complex has several generalizations. One generalization is that $\phi : M \to R$ is replaced by $\phi : M \to N$ for two R-modules M and N. In this case, we have a complex

$$\begin{aligned} \wedge^l M \to \wedge^{l-1} M \otimes N \to \wedge^{l-2} M \otimes \operatorname{Sym}^2 N \\ \to ... \to M \otimes \operatorname{Sym}^{l-1} N \to \operatorname{Sym}^l N \to 0 \end{aligned} \tag{2.5}$$

where the differential maps are defined in the obvious way.

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