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On thin residues and basis digraphs of nilpotent table algebras and applications to nilpotent groups

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ABSTRACT

In this paper we study characterizations of nilpotent table algebras in terms of thin residues; thin residue matrices; thin residue digraphs; and basis digraphs. We will also discuss applications to nilpotent groups. In particular, we give a conceptual explanation of the results in Takegahara and Yoshida (2008) [TY].

1. Introduction and main results

In the theory of finite groups, centers of group algebras over the complex numbers and rings of complex valued class functions on groups play an important role in the study of structures of finite groups. Recently, Takegahara and Yoshida [TY] obtained characterizations of a finite nilpotent group G in terms of the structure constants of the center of the group algebra $\mathbb{C}G$ and the ring of complex valued class functions on G. Table algebras abstract common features of centers of group algebras and rings of complex valued class functions on groups, and of the Bose–Mesner algebras of association schemes, etc. Table algebras have been studied in many papers, and applications to the theory of finite groups as well as association schemes are discussed (cf. [ACCHMX,AFM,B1,X1,X2,X4,X5]).

In this paper we study characterizations of nilpotent table algebras, and discuss applications to finite nilpotent groups. Since a nilpotent fusion ring or based ring (cf. [GN]) is a nilpotent table algebra, the results in this paper also hold for nilpotent fusion rings. Furthermore, an association scheme is nilpotent if its Bose–Mesner algebra is a nilpotent table algebra. So our results here are also true for nilpotent association schemes. However, we will not state our results for either nilpotent fusion rings or nilpotent association schemes.

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Let $(A, \mathbf{B})$ be a commutative table algebra. We will first prove that $(A, \mathbf{B})$ is a nilpotent table algebra if and only if for any $b \in \mathbf{B}$ such that $b \neq 1$, the thin residue $O^\vartheta(\mathbf{B}_b)$ is not equal to $\mathbf{B}_b$ (Theorem 1.1). An application of this result to finite groups is also presented (Corollary 1.2). Then we introduce the concept of the thin residue matrix $\mathrm{Mat}_{O^\vartheta}(\mathbf{B})$ of $(A, \mathbf{B})$, and establish the connections between the n-th power of $\mathrm{Mat}_{O^\vartheta}(\mathbf{B})$ and the n-th thin residue of $\mathbf{B}$ as well as between the n-th power of $\mathrm{Mat}_{O^\vartheta}(\mathbf{B})$ and the n-th thin radical of $\mathbf{B}$, for any positive integer n (Theorem 1.5). As a direct consequence, we obtain that $(A, \mathbf{B})$ is a nilpotent table algebra of class n if and only if $\mathrm{Mat}_{O^\vartheta}(\mathbf{B})$ is a nilpotent matrix of index of nilpotence n. For a finite group G, Takegahara and Yoshida [TY] introduced matrices S and T similar to $\mathrm{Mat}_{O^\vartheta}(\mathbf{B})$ for the center of the group algebra $\mathbb{C}G$ and the ring of complex valued class functions on G, respectively, and proved by lengthy but direct computations that G is a nilpotent group of class n if and only if S is a nilpotent matrix of index of nilpotence n if and only if T is a nilpotent matrix of index of nilpotence n (cf. [TY, Theorems 1.1 and 1.2]). Our approach in this paper is *not* to generalize the theorems in [TY] to table algebras, but to prove stronger properties about the structures of table algebras, and then obtain the characterizations of nilpotent table algebras and nilpotent groups as direct consequences. In particular, our Theorem 1.5 provides a conceptual explanation of [TY, Theorems 1.1 and 1.2]. We will also introduce the notion of the thin residue digraph $\Gamma_{O^\vartheta(\mathbf{B})}$, which is the digraph corresponding to the thin residue matrix $\mathrm{Mat}_{O^\vartheta}(\mathbf{B})$, and show that $(A, \mathbf{B})$ is a nilpotent table algebra of class n if and only if $\Gamma_{O^\vartheta(\mathbf{B})}$ does not contain any directed cycle and the length of longest directed path is $n-1$ (Theorem 1.10). Other properties for nilpotent table algebras and nilpotent finite groups can also be easily obtained from the thin residue digraphs (Corollaries 1.11, 1.12, and Proposition 3.7). In particular, our method developed in this paper yields a very simple and direct proof of [TY, Theorem 1.4]. Finally, we study the properties of the basis digraphs of table algebras, and prove that $(A, \mathbf{B})$ is a nilpotent table algebra if and only if for any $b \in \mathbf{B}$ such that $b \neq 1$, the principal component of the basis digraph Γ_b is cyclically partite (Theorem 1.13).

The rest of this section is devoted to explicit statements of definitions, notation, and the main results of the paper. Let us start with a very brief review of some known facts of table algebras, closed subsets, and quotient table algebras.

A *table algebra* $(A, \mathbf{B})$ is a finite dimensional associative algebra A over the complex numbers $\mathbb{C}$, and a distinguished basis $\mathbf{B} = \{b_0, b_1, b_2, \dots, b_k\}$ for A such that the following properties hold:

(i) $b_0 = 1_A$, the identity element of A. (We will also simply write 1_A as 1.)
(ii) The structure constants for $\mathbf{B}$ are nonnegative real numbers; that is, for all $b_i, b_j \in \mathbf{B}$, $b_i b_j = \sum_{m=0}^{k} \lambda_{ijm} b_m$, for some $\lambda_{ijm} \in \mathbb{R}_{\geqslant 0}$.
(iii) There is an algebra antiautomorphism (denoted by $*$) of A such that $(a^*)^* = a$ for all $a \in A$ and $b_i^* \in \mathbf{B}$ for all $b_i \in \mathbf{B}$. (Hence i^* is defined by $b_{i^*} = b_i^*$.)
(iv) For all $b_i, b_j \in \mathbf{B}$, $\lambda_{ij0} = 0$ if $j \neq i^*$; and $\lambda_{ii^*0} > 0$.

Let $(A, \mathbf{B})$ be a table algebra. Then $\mathbf{B}$ is called a *table basis*. An element $b_i \in \mathbf{B}$ is called a *thin* (or *linear*) *element* if $b_i b_i^* = \lambda_{ii^*0} 1_A$. If every element in a nonempty subset $\mathbf{N}$ of $\mathbf{B}$ is thin, then we say that $\mathbf{N}$ is *thin*. It is well known that there is a unique algebra homomorphism $\nu : A \to \mathbb{C}$ such that $\nu(b_i) = \nu(b_i^*) > 0$ for all $b_i \in \mathbf{B}$ (see Proposition 3.12 and Theorem 3.14 of [AFM]). The algebra homomorphism $\nu : A \to \mathbb{C}$ is called the *degree map* of $(A, \mathbf{B})$, and the values of $\nu(b_i)$, for all $b_i \in \mathbf{B}$, are called the *degrees* of $(A, \mathbf{B})$. For any $b_i \in \mathbf{B}$ and any nonempty subset $\mathbf{N}$ of $\mathbf{B}$, the *order* of b_i, $o(b_i)$, and the *order* of $\mathbf{N}$, $o(\mathbf{N})$, are defined by

$$o(b_i) := \frac{\nu(b_i)^2}{\lambda_{ii^*0}} \quad \text{and} \quad o(\mathbf{N}) := \sum_{b_i \in \mathbf{N}} o(b_i),$$

respectively. Furthermore, let

$$\mathbf{N}^{\tilde{+}} := \sum_{b_i \in \mathbf{N}} \frac{\nu(b_i)}{\lambda_{ii^*0}} b_i.$$

Let $(A, \mathbf{B})$ be a table algebra, with $\mathbf{B} = \{b_0 = 1_A, b_1, b_2, \ldots, b_k\}$. For any $a \in A$ with $a = \sum_{i=0}^{k} \alpha_i b_i$, define $\mathrm{Supp}_{\mathbf{B}}(a) := \{b_i \mid \alpha_i \neq 0\}$. For any nonempty subsets $\mathbf{R}$ and $\mathbf{L}$ of $\mathbf{B}$, define

$$\mathbf{RL} := \bigcup_{b \in \mathbf{R},\, c \in \mathbf{L}} \mathrm{Supp}_{\mathbf{B}}(bc) \quad \text{and} \quad \mathbf{R}^* = \{b^* \mid b \in \mathbf{R}\}.$$

Then for any nonempty subsets $\mathbf{R}$, $\mathbf{L}$, and $\mathbf{N}$, $(\mathbf{RL})\mathbf{N} = \mathbf{R}(\mathbf{LN})$; we will write both $(\mathbf{RL})\mathbf{N}$ and $\mathbf{R}(\mathbf{LN})$ as $\mathbf{RLN}$. A nonempty subset $\mathbf{N}$ of $\mathbf{B}$ is called a *closed subset* (or *table subset*) of $\mathbf{B}$ if $\mathbf{N}^*\mathbf{N} \subseteq \mathbf{N}$. It is well known that $\mathbf{N}$ is a closed subset of $\mathbf{B}$ if and only if $\mathbf{NN} \subseteq \mathbf{N}$, and if $\mathbf{N}$ is a closed subset of $\mathbf{B}$, then $1_A \in \mathbf{N}$, $\mathbf{N}^* = \mathbf{N}$, and $(\mathbb{C}\mathbf{N}, \mathbf{N})$ is also a table algebra, called a *table subalgebra* of $(A, \mathbf{B})$, where $\mathbb{C}\mathbf{N}$ is the $\mathbb{C}$-space with basis $\mathbf{N}$. For any $b_i \in \mathbf{B}$, we write $\mathbf{N}\{b_i\}$ as $\mathbf{N}b_i$, $\{b_i\}\mathbf{N}$ as $b_i\mathbf{N}$, and $\mathbf{N}\{b_i\}\mathbf{N}$ as $\mathbf{N}b_i\mathbf{N}$. Assume that $\mathbf{N}$ is a closed subset of $\mathbf{B}$. It is well known that $\{\mathbf{N}b_i\mathbf{N} \mid b_i \in \mathbf{B}\}$ forms a partition of $\mathbf{B}$. Let

$$b_i//\mathbf{N} := o(\mathbf{N})^{-1} (\mathbf{N}b_i\mathbf{N})^{\tilde{+}} = o(\mathbf{N})^{-1} \sum_{b_j \in \mathbf{N}b_i\mathbf{N}} \frac{\nu(b_j)}{\lambda_{jj^*0}} b_j, \quad \text{for any } b_i \in \mathbf{B}.$$

Let $\mathbf{B}//\mathbf{N} := \{b_i//\mathbf{N} \mid b_i \in \mathbf{B}\}$, and $A//\mathbf{N} := \mathbb{C}(\mathbf{B}//\mathbf{N})$. That is, $A//\mathbf{N}$ is the $\mathbb{C}$-space with basis $\mathbf{B}//\mathbf{N}$. Then $(A//\mathbf{N}, \mathbf{B}//\mathbf{N})$ is a table algebra such that for any $b_i \in \mathbf{B}$, $(b_i//\mathbf{N})^* = b_i^*//\mathbf{N}$, and the degree $\nu(b_i//\mathbf{N}) = o(\mathbf{N})^{-1} o(\mathbf{N}b_i\mathbf{N})$. (See [AFM, Theorem 4.9].) $(A//\mathbf{N}, \mathbf{B}//\mathbf{N})$ is called the *quotient table algebra* of $(A, \mathbf{B})$ with respect to $\mathbf{N}$. Note that in [AFM, Theorem 4.9], $(A, \mathbf{B})$ is assumed to be *standard*. Although we do not assume that $(A, \mathbf{B})$ is standard here, the quotient table algebra $(A//\mathbf{N}, \mathbf{B}//\mathbf{N})$ is actually defined over the *standard rescaling* of $\mathbf{B}$, and $\mathbf{B}//\mathbf{N}$ itself is standard.

The thin residues and the thin radicals of closed subsets of association schemes are studied in [Z]. These concepts can be defined similarly for closed subsets of table algebras. Let $(A, \mathbf{B})$ be a table algebra, and let $\mathbf{R}$ be a subset of $\mathbf{B}$. Then the intersection of all closed subsets of $\mathbf{B}$ that contain $\mathbf{R}$ is a closed subset of $\mathbf{B}$, called the closed subset *generated by* $\mathbf{R}$, and denoted by $\langle \mathbf{R} \rangle$. In particular, the closed subset $\langle \emptyset \rangle$ generated by the empty subset of $\mathbf{B}$ is $\{1\}$. For any $b \in \mathbf{B}$, the closed subset generated by $\{b\}$ is denoted by $\langle b \rangle$ or $\mathbf{B}_b$, and called the closed subset *generated by* b. Let $\mathbf{N}$ be a closed subset of $\mathbf{B}$. Then the *thin residue* of $\mathbf{N}$, $O^{\vartheta}(\mathbf{N})$, is defined by

$$O^{\vartheta}(\mathbf{N}) := \left\langle \bigcup_{b \in \mathbf{N}} \mathrm{Supp}_{\mathbf{B}}(bb^*) \right\rangle.$$

It is well known that the quotient $\mathbf{N}//O^{\vartheta}(\mathbf{N})$ is a thin closed subset. Furthermore, if $\mathbf{M}$ is a closed subset of $\mathbf{N}$ such that the quotient $\mathbf{N}//\mathbf{M}$ is thin, then $O^{\vartheta}(\mathbf{N}) \subseteq \mathbf{M}$. Set $(O^{\vartheta})^0(\mathbf{N}) := \mathbf{N}$, and for any integer $n \geqslant 1$, define the *n-th thin residue* of $\mathbf{N}$, $(O^{\vartheta})^n(\mathbf{N})$, inductively by

$$(O^{\vartheta})^n(\mathbf{N}) := O^{\vartheta}\big((O^{\vartheta})^{n-1}(\mathbf{N})\big).$$

On the other hand, the *thin radical* of $\mathbf{N}$, $O_{\vartheta}(\mathbf{N})$, is the set of all thin elements of $\mathbf{N}$. It is clear that the thin radical $O_{\vartheta}(\mathbf{N})$ is a closed subset of $\mathbf{B}$. Set $O_{\vartheta}^{(0)}(\mathbf{N}) := \{1\}$, and for any integer $n \geqslant 1$, define recursively the *n-th thin radical* of $\mathbf{N}$, $O_{\vartheta}^{(n)}(\mathbf{N})$, to be the closed subset of $\mathbf{B}$ such that

$$O_{\vartheta}^{(n)}(\mathbf{N})//O_{\vartheta}^{(n-1)}(\mathbf{N}) = O_{\vartheta}\big(\mathbf{N}//O_{\vartheta}^{(n-1)}(\mathbf{N})\big).$$

If $(A, \mathbf{B})$ is commutative, then

$$\mathbf{B} \supseteq O^{\vartheta}(\mathbf{B}) \supseteq (O^{\vartheta})^2(\mathbf{B}) \supseteq (O^{\vartheta})^3(\mathbf{B}) \supseteq \cdots$$

is called the *lower central series* of $(A, \mathbf{B})$, and

$$\{1\} \subseteq O_{\vartheta}(\mathbf{B}) \subseteq O_{\vartheta}^{(2)}(\mathbf{B}) \subseteq O_{\vartheta}^{(3)}(\mathbf{B}) \subseteq \cdots$$

is called the *upper central series* of $(A, \mathbf{B})$ (cf. [BX2]). Furthermore, if $(A, \mathbf{B})$ is commutative and $O_{\vartheta}^{(n)}(\mathbf{B}) = \mathbf{B}$ for some positive integer n, then $(A, \mathbf{B})$ is called a *nilpotent table algebra*, and the minimal n such that $O_{\vartheta}^{(n)}(\mathbf{B}) = \mathbf{B}$ is called the *nilpotent class* of $\mathbf{B}$ (cf. [BX2, Proposition 1.22 and Definition 1.23]). It is also known that $(A, \mathbf{B})$ is a nilpotent table algebra of class n if and only if n is the minimal positive integer such that $(O^{\vartheta})^n(\mathbf{B}) = \{1\}$ (cf. [B3,GN]).

Now we are ready to state the main results of the paper. The next theorem says that a nilpotent table algebra $(A, \mathbf{B})$ is characterized by the thin residues of its closed subsets $\mathbf{B}_b$, for all $b \in \mathbf{B}$ such that $b \neq 1$.

Theorem 1.1. *Let $(A, \mathbf{B})$ be a commutative table algebra. Then the following are equivalent.*

(i) *$(A, \mathbf{B})$ is a nilpotent table algebra.*
(ii) *For any $b \in \mathbf{B}$ such that $b \neq 1$, $O^{\vartheta}(\mathbf{B}_b) \neq \mathbf{B}_b$.*

For a finite group G, we have two commutative table algebras: $(Z(\mathbb{C}G), \mathrm{Cla}(G))$, the center of the group algebra $\mathbb{C}G$ with table basis $\mathrm{Cla}(G)$, the set of sums over various conjugacy classes of G; and $(\mathrm{Ch}(G), \mathrm{Irr}(G))$, the ring of complex valued class functions on G with pointwise addition and multiplication, and with table basis $\mathrm{Irr}(G)$ the set of irreducible characters of G. The (anti)automorphism of $(Z(\mathbb{C}G), \mathrm{Cla}(G))$ is induced from inversion on G. That is, for any conjugacy class C of G, let $C^+ = \sum_{x \in C} x$. Then $(C^+)^* = \sum_{x \in C} x^{-1}$. The (anti)automorphism of $(\mathrm{Ch}(G), \mathrm{Irr}(G))$ is given by complex conjugation on $\mathrm{Irr}(G)$, extended linearly to $\mathrm{Ch}(G)$. It is well known that G is a nilpotent group of class n if and only if $(Z(\mathbb{C}G), \mathrm{Cla}(G))$ is a nilpotent table algebra of class n if and only if $(\mathrm{Ch}(G), \mathrm{Irr}(G))$ is a nilpotent table algebra of class n (cf. [B1, Remarks 1.12 and 1.10]). Note that this result also follows from the more general relationship between a nilpotent table algebra and its dual (if the dual is also a table algebra); cf. [B2, Theorems 1 and 2].

As a direct consequence of Theorem 1.1, we have the following

Corollary 1.2. *Let G be a finite group. Then the following are equivalent.*

(i) *G is a nilpotent group.*
(ii) *For any conjugacy class C of G such that $C \neq \{1\}$, the commutator subgroup of G and $\langle C \rangle$, $[G, \langle C \rangle]$, is not equal to $\langle C \rangle$, where $\langle C \rangle$ is the (normal) subgroup of G generated by C.*

Theorem 1.1 and Corollary 1.2 will be proved in Section 2.

Now we introduce the concept of the thin residue matrix of a table algebra, which is closely related to the thin residue.

Definition 1.3. Let $(A, \mathbf{B})$ be a table algebra, with $\mathbf{B} = \{b_0 = 1_A, b_1, b_2, \ldots, b_k\}$, and structure constants λ_{ijm}, $0 \leqslant i, j, m \leqslant k$. Then the *thin residue matrix* of $(A, \mathbf{B})$, $\mathrm{Mat}_{O^{\vartheta}}(\mathbf{B})$, is the $k \times k$ nonnegative matrix whose rows and columns are indexed by $b_1, b_2, \ldots, b_k$ and whose (b_i, b_j)-entry is λ_{jj^*i}.

Let G be a finite group. Takegahara and Yoshida [TY] defined $S := \mathrm{Mat}_{O^{\vartheta}}(\mathrm{Cla}(G))$ and $T := \mathrm{Mat}_{O^{\vartheta}}(\mathrm{Irr}(G))$, and proved directly by calculations that G is nilpotent if and only if S is a nilpotent matrix if and only if T is a nilpotent matrix (cf. [TY, Theorems 1.1 and 1.2]). For a commutative table algebra $(A, \mathbf{B})$, our next theorem gives the intrinsic connections between the n-th power of $\mathrm{Mat}_{O^{\vartheta}}(\mathbf{B})$ and the n-th thin residue $(O^{\vartheta})^n(\mathbf{B})$ as well as between the n-th power of $\mathrm{Mat}_{O^{\vartheta}}(\mathbf{B})$ and the n-th thin radical $O_{\vartheta}^{(n)}(\mathbf{B})$, for any positive integer n, which in turn provide a conceptual explanation of the results in [TY].

Definition 1.4. Let $(A, \mathbf{B})$ be a table algebra, with $\mathbf{B} = \{b_0 = 1_A, b_1, b_2, \ldots, b_k\}$. For any $k \times k$ nonnegative matrix P with rows and columns indexed by $b_1, b_2, \ldots, b_k$, let

$$\mathfrak{C}(P) := \{b_i \mid \text{the } (b_i, b_j)\text{-entry of } P \text{ is not zero for some } b_j\}$$

and

$$\mathfrak{D}(P) := \{b_j \mid \text{the } (b_i, b_j)\text{-entry of } P \text{ is zero for all } b_i\}.$$

Then $\langle \mathfrak{C}(P) \rangle$ is called the closed subset of $\mathbf{B}$ *associated with* P, and $\langle \mathfrak{D}(P) \rangle$ is called the *null closed subset of* P.

Theorem 1.5. *Let* $(A, \mathbf{B})$ *be a commutative table algebra, and let* $S = \mathrm{Mat}_{O^\vartheta}(\mathbf{B})$*. Then for any positive integer* n*,*

$$\left(O^\vartheta\right)^n(\mathbf{B}) = \left\langle \mathfrak{C}\left(S^n\right) \right\rangle \tag{1.1}$$

and

$$O_\vartheta^{(n)}(\mathbf{B}) = \left\langle \mathfrak{D}\left(S^n\right) \right\rangle = \mathfrak{D}\left(S^n\right) \cup \{1\}. \tag{1.2}$$

In particular, $(O^\vartheta)^n(\mathbf{B}) = \{1\}$ *if and only if* $O_\vartheta^{(n)}(\mathbf{B}) = \mathbf{B}$ *if and only if* S^n *is a zero matrix.*

As mentioned above, for a commutative table algebra $(A, \mathbf{B})$ and a positive integer n, it is already known that $(O^\vartheta)^n(\mathbf{B}) = \{1\}$ if and only if $O_\vartheta^{(n)}(\mathbf{B}) = \mathbf{B}$. Here we obtain this result as a direct consequence of (1.1) and (1.2).

The next corollary is a direct consequence of Theorem 1.5.

Corollary 1.6. *Let* $(A, \mathbf{B})$ *be a commutative table algebra. Then the following are equivalent.*

(i) $(A, \mathbf{B})$ *is a nilpotent table algebra of class* n*.*
(ii) $\mathrm{Mat}_{O^\vartheta}(\mathbf{B})$ *is a nilpotent matrix of index of nilpotence* n*.*

Let G be a group. Set $Z_0(G) := \{1\}$, and for any positive integer i, define $Z_i(G)$ inductively to be the normal subgroup of G such that $Z_i(G)/Z_{i-1}(G) = Z(G/Z_{i-1}(G))$, the center of $G/Z_{i-1}(G)$. Then the upper central series of G is

$$Z_0(G) = \{1\} \subseteq Z_1(G) = Z(G) \subseteq Z_2(G) \subseteq \cdots.$$

On the other hand, set $G_1 := G$, and for any integer $i \geqslant 2$, define G_i inductively to be the commutator subgroup $[G, G_{i-1}]$ of G and G_{i-1}. Then the lower central series of G is

$$G = G_1 \supseteq G_2 = [G, G] \supseteq G_3 = \left[G, [G, G]\right] \supseteq \cdots.$$

Let G be a finite group. For a nonempty subset $\mathbf{R}$ of the table basis $\mathrm{Cla}(G)$, let

$$\mathbf{R}_G := \bigcup \{C \mid C \text{ is a conjugacy class of } G \text{ such that } C^+ \in \mathbf{R}\}. \tag{1.3}$$

It is clear that $\mathbf{R}$ is a closed subset of $\mathrm{Cla}(G)$ if and only if $\mathbf{R}_G$ is a normal subgroup of G. By applying Theorem 1.5 to the table algebra $(Z(\mathbb{C}G), \mathrm{Cla}(G))$, we get the next corollary immediately.

Corollary 1.7. *Let G be a finite group, and let $S = \mathrm{Mat}_{O^\vartheta}(\mathrm{Cla}(G))$. Then for any positive integer n,*

$$\langle \mathfrak{C}(S^n) \rangle_G = G_{n+1} \tag{1.4}$$

and

$$\langle \mathfrak{D}(S^n) \rangle_G = Z_n(G). \tag{1.5}$$

In particular, $Z_n(G) = G$ if and only if $G_{n+1} = \{1\}$ if and only if S^n is a zero matrix.

Note that [TY, Theorem 2.3] can be interpreted as a variation of (1.5). Since a finite group G is nilpotent of class n if and only if $(Z(\mathbb{C}G), \mathrm{Cla}(G))$ is a nilpotent table algebra of class n if and only if $(\mathrm{Ch}(G), \mathrm{Irr}(G))$ is a nilpotent table algebra of class n, as a direct consequence of Corollary 1.6, we have the following

Corollary 1.8. *(See [TY, Theorems 1.1 and 1.2].) Let G be a finite group, $S = \mathrm{Mat}_{O^\vartheta}(\mathrm{Cla}(G))$, and $T = \mathrm{Mat}_{O^\vartheta}(\mathrm{Irr}(G))$. Then the following are equivalent.*

(i) *G is a nilpotent group of class n.*
(ii) *S is a nilpotent matrix of index of nilpotence n.*
(iii) *T is a nilpotent matrix of index of nilpotence n.*

Theorem 1.5 and Corollary 1.7 will be proved in Section 2.

Let G be a finite group. Other matrices related to $\mathrm{Mat}_{O^\vartheta}(\mathrm{Cla}(G))$ and $\mathrm{Mat}_{O^\vartheta}(\mathrm{Irr}(G))$ are introduced in [TY], and some properties are proved by complicated computations. In Section 3 we will show that these properties can be easily obtained by applying the method developed in this paper.

Now let us turn to the characterizations of nilpotent table algebras in terms of the thin residue digraphs and the basis digraphs.

Definition 1.9. Let $(A, \mathbf{B})$ be a table algebra, with $\mathbf{B} = \{b_0 = 1_A, b_1, b_2, \ldots, b_k\}$, and structure constants λ_{ijm}, $0 \leqslant i, j, m \leqslant k$. The *thin residue digraph* of $(A, \mathbf{B})$, $\Gamma_{O^\vartheta(\mathbf{B})}$, is the directed graph (digraph) defined by

$$V(\Gamma_{O^\vartheta(\mathbf{B})}) := \mathbf{B} \setminus \{1\}, \qquad E(\Gamma_{O^\vartheta(\mathbf{B})}) := \{(b_i, b_j) \mid \lambda_{jj^*i} \neq 0\}.$$

That is, $\Gamma_{O^\vartheta(\mathbf{B})}$ is the digraph corresponding to the thin residue matrix $\mathrm{Mat}_{O^\vartheta}(\mathbf{B})$.

Let $(A, \mathbf{B})$ be a table algebra. Then $\Gamma_{O^\vartheta(\mathbf{B})}$ may not be a simple digraph. If $\Gamma_{O^\vartheta(\mathbf{B})}$ is not simple, then there exists $b_i \in \mathbf{B} \setminus \{1\}$ such that $(b_i, b_i) \in E(\Gamma_{O^\vartheta(\mathbf{B})})$, and $\Gamma_{O^\vartheta(\mathbf{B})}$ has a directed cycle with only one vertex b_i. The next theorem follows directly from Theorem 1.5. It provides a very easy way to determine whether a commutative table algebra is nilpotent or not. By applying this theorem, we will prove some properties for matrices related to the thin residue matrix (see Proposition 3.7 in Section 3), and obtain [TY, Theorem 1.4] as a direct consequence.

Theorem 1.10. *Let $(A, \mathbf{B})$ be a commutative table algebra. Then the following are equivalent.*

(i) *$(A, \mathbf{B})$ is a nilpotent table algebra of class n.*
(ii) *The digraph $\Gamma_{O^\vartheta}(\mathbf{B})$ contains no directed cycles, and the length of the longest directed path is $n - 1$.*

Let $(A, \mathbf{B})$ be a commutative table algebra. Then $(A, \mathbf{B})$ is not a nilpotent table algebra if and only if the digraph $\Gamma_{O^\vartheta}(\mathbf{B})$ contains a directed cycle by Theorem 1.10. Thus, we have the following corollary. Note that in Corollary 1.11(ii), it is possible that $r = 1$.

Corollary 1.11. *Let $(A, \mathbf{B})$ be a commutative table algebra. Then the following are equivalent.*

(i) *$(A, \mathbf{B})$ is not a nilpotent table algebra.*
(ii) *There are $b_{i_1}, b_{i_2}, \ldots, b_{i_r} \in \mathbf{B} \setminus \{1\}$ such that*

$$b_{i_m} \in \mathrm{Supp}_{\mathbf{B}}\big(b_{i_{m+1}} b^*_{i_{m+1}}\big), \quad \textit{for all } m = 1, 2, \ldots, r,$$

where $b_{i_{r+1}} = b_{i_1}$.

Let G be a group. For any character χ of G, let $\bar{\chi}$ denote the complex conjugate of χ. That is, for any $g \in G$, $\bar{\chi}(g) = \overline{\chi(g)}$, the complex conjugate of $\chi(g)$. For any two characters χ and ρ, let (χ, ρ) denote the inner product of χ and ρ. Let χ_0 be the principal irreducible character of G. The next corollary is a direct consequence of Theorem 1.10 (or Corollary 1.11). It provides a way to determine if a finite group is not a nilpotent group. Note that in Corollary 1.12(ii), it is possible that $r = 1$.

Corollary 1.12. *Let G be a finite group. Then the following are equivalent.*

(i) *G is not a nilpotent group.*
(ii) *There are irreducible characters $\chi_1, \ldots, \chi_r \in \mathrm{Irr}(G) \setminus \{\chi_0\}$ such that*

$$(\chi_i, \chi_{i+1} \overline{\chi_{i+1}}) \neq 0, \quad \textit{for all } i = 1, 2, \ldots, r,$$

where $\chi_{r+1} = \chi_1$.

A very short proof of Theorem 1.10 will be included in Section 3 for the convenience of the reader.

Let $(A, \mathbf{B})$ be a table algebra. Let us denote the structure constants of $(A, \mathbf{B})$ by λ_{bac}, for all $b, a, c \in \mathbf{B}$. For any $b \in \mathbf{B}$, we have a digraph Γ_b (cf. [AFM]) defined by

$$V(\Gamma_b) := \mathbf{B}, \qquad E(\Gamma_b) := \big\{(a, c) \bigm| \lambda_{bac} \neq 0\big\}.$$

Γ_b is called a *basis digraph* of $(A, \mathbf{B})$. Note that Γ_b may not be a simple digraph. If $b^* = b$, then Γ_b is an undirected graph. Similar digraphs for association schemes are defined in [BCN], and called *distribution diagrams*. For an association scheme $(X, \{R_i\}_{0 \leqslant i \leqslant d})$, the digraph G_{R_i}, defined by $V(G_{R_i}) := X$ and $E(G_{R_i}) := \{(x, y) \mid (x, y) \in R_i\}$, is called a *relation digraph*, for any $0 \leqslant i \leqslant d$. Note that the relation digraphs of association schemes are called the basis digraphs in [BP].

Let $(A, \mathbf{B})$ be a table algebra. For any $b \in \mathbf{B}$ such that $b \neq 1$, each weak component of Γ_b is also a strong component, and the vertex set of the (weak, strong) component of Γ_b that contains a vertex $a \in \mathbf{B}$ is $\mathbf{B}_b a$ (see Lemma 4.1 in Section 4 below). The component of Γ_b with vertex set $\mathbf{B}_b$ is called the *principal component* of Γ_b (Definition 4.2 in Section 4 below).

Let $\Gamma = (V, E)$ be a digraph. If there is an integer $h > 1$ and a partition $V_1, V_2, \ldots, V_h$ of the vertex set V such that for any $u \in V_i$ and $v \in V_j$ with $(u, v) \in E$, we have $j - i \equiv 1 \bmod h$, then Γ is called a *cyclically h-partite* digraph (cf. [CDS, p. 82]). In this paper a cyclically h-partite digraph is also simply called a *cyclically partite* digraph.

The next theorem says that a nilpotent table algebra can be characterized by the principal components of its nontrivial basis digraphs. Note that Barghi and Ponomarenko [BP] proved that an association scheme is a p-scheme if and only if each nontrivial relation digraph is cyclically p-partite.

Theorem 1.13. *Let $(A, \mathbf{B})$ be a commutative table algebra. Then the following are equivalent.*

(i) *$(A, \mathbf{B})$ is a nilpotent table algebra.*
(ii) *For any $b \in \mathbf{B}$ such that $b \neq 1$, the principal component of the digraph Γ_b is cyclically partite.*

Note that in Theorem 1.13(ii), it is possible that for some $b, c \in \mathbf{B} \setminus \{1\}$ such that $b \neq c$, Γ_b is cyclically h-partite and Γ_c is cyclically p-partite with $h \neq p$.

The rest of the paper is organized as follows. In Section 2 we prove Theorems 1.1 and 1.5. Then in Section 3 we prove Theorem 1.10 and, by using the thin residue digraphs, some other properties related to the thin residue matrices. We will study some basic properties of the basis digraphs and prove Theorem 1.13 in Section 4.

2. Proofs of Theorems 1.1 and 1.5

Let us first prove Theorem 1.1. Let $(A, \mathbf{B})$ be a table algebra, and let $\mathbf{R}$ be a nonempty subset of $\mathbf{B}$. Set $\mathbf{R}^0 = \{1\}$, and for any positive integer n, define $\mathbf{R}^n$ inductively by $\mathbf{R}^n = \mathbf{R}^{n-1}\mathbf{R}$. It is clear that the closed subset generated by $\mathbf{R}$, $\langle \mathbf{R} \rangle$, is given by

$$\langle \mathbf{R} \rangle = \bigcup_{n=1}^{\infty} \mathbf{R}^n. \tag{2.1}$$

In particular, for any $b \in \mathbf{B}$, $\mathbf{B}_b = \bigcup_{n=1}^{\infty} \mathrm{Supp}_{\mathbf{B}}(b^n)$. Furthermore, if $(A, \mathbf{B})$ is commutative, then for any nonempty subsets $\mathbf{R}$ and $\mathbf{L}$ of $\mathbf{B}$, $(\mathbf{RL})^n = \mathbf{R}^n\mathbf{L}^n$, for any positive integer n.

Let $(A, \mathbf{B})$ be a table algebra, and let $\mathbf{N}$ be a closed subset of $\mathbf{B}$. Then for any nonempty subset $\mathbf{R}$ of $\mathbf{B}$, let

$$\mathbf{R}/\!/\mathbf{N} := \{b/\!/\mathbf{N} \mid b \in \mathbf{R}\}.$$

Since the quotient table algebra $(A/\!/\mathbf{N}, \mathbf{B}/\!/\mathbf{N})$ is defined over the standard rescaling of $\mathbf{B}$ in this paper, all the known results for the quotient table algebras under the assumption that $(A, \mathbf{B})$ is standard can be applied in our discussions.

The next lemma is needed for the proof of Theorem 1.1.

Lemma 2.1. *Let $(A, \mathbf{B})$ be a commutative table algebra, and let $\mathbf{N}$ be a closed subset of $\mathbf{B}$. Then the following hold.*

(i) *For any nonempty subsets $\mathbf{R}$ and $\mathbf{L}$ of $\mathbf{B}$, $(\mathbf{R}/\!/\mathbf{N})(\mathbf{L}/\!/\mathbf{N}) = \mathbf{RL}/\!/\mathbf{N}$. In particular, for any positive integer n, $(\mathbf{R}/\!/\mathbf{N})^n = \mathbf{R}^n/\!/\mathbf{N}$.*
(ii) *For any nonempty subset $\mathbf{R}$ of $\mathbf{B}$, $\langle \mathbf{R}/\!/\mathbf{N} \rangle = \langle \mathbf{R} \rangle/\!/\mathbf{N} = \langle \mathbf{R} \rangle\mathbf{N}/\!/\mathbf{N}$.*
(iii) *For any $b \in \mathbf{B}$, $b/\!/\mathbf{N}$ is a thin element of $\mathbf{B}/\!/\mathbf{N}$ if and only if $\mathrm{Supp}_{\mathbf{B}}(bb^*) \subseteq \mathbf{N}$. In particular, if $\mathbf{B}$ is thin, then $\mathbf{B}/\!/\mathbf{N}$ is also thin.*

Proof. (i) By [X1, Lemma 2.5(iii)], $(\mathbf{R}/\!/\mathbf{N})(\mathbf{L}/\!/\mathbf{N}) = (\mathbf{RNL})/\!/\mathbf{N}$. Since $(A, \mathbf{B})$ is commutative, $\mathbf{RNL} = \mathbf{RLN}$. But $(\mathbf{RLN})/\!/\mathbf{N} = \mathbf{RL}/\!/\mathbf{N}$ by [X1, Lemma 2.5(i)]. So (i) holds.

(ii) From (2.1) and (i) we have that

$$\langle \mathbf{R}/\!/\mathbf{N} \rangle = \bigcup_{n=1}^{\infty} (\mathbf{R}/\!/\mathbf{N})^n = \bigcup_{n=1}^{\infty} \big(\mathbf{R}^n/\!/\mathbf{N}\big) = \left(\bigcup_{n=1}^{\infty} \mathbf{R}^n\right)/\!/\mathbf{N} = \langle \mathbf{R} \rangle/\!/\mathbf{N}.$$

Also $\langle \mathbf{R} \rangle\mathbf{N}/\!/\mathbf{N} = \langle \mathbf{R} \rangle/\!/\mathbf{N}$ by [X1, Lemma 2.5(i)]. So (ii) holds.

(iii) For any $b, c \in \mathbf{B}$, $\mathrm{Supp}_{\mathbf{B}/\!/\mathbf{N}}((b/\!/\mathbf{N})(c/\!/\mathbf{N})) = \mathrm{Supp}_{\mathbf{B}}(bc)/\!/\mathbf{N}$ by (i). So (iii) holds. □

Let $(A, \mathbf{B})$ and $(C, \mathbf{D})$ be table algebras. Let $\varphi : A \to C$ be an algebra homomorphism such that for any $b \in \mathbf{B}$, $\varphi(b)$ is a positive scalar multiple of an element in $\mathbf{D}$. Then φ is called a *table algebra homomorphism* from $(A, \mathbf{B})$ to $(C, \mathbf{D})$. If a table algebra homomorphism $\varphi : (A, \mathbf{B}) \to (C, \mathbf{D})$ is bijective, then φ is called a *table algebra isomorphism*, $(A, \mathbf{B})$ and $(C, \mathbf{D})$ are called *isomorphic table algebras*, and denoted by $(A, \mathbf{B}) \cong (C, \mathbf{D})$ or simply $\mathbf{B} \cong \mathbf{D}$. It is clear that if $\mathbf{B} \cong \mathbf{D}$ and $\mathbf{B}$ is thin, then $\mathbf{D}$ is also thin. Let $(A, \mathbf{B})$ be a commutative table algebra, and let $\mathbf{M}$ and $\mathbf{N}$ be closed subsets of $\mathbf{B}$. Then by [B2, Theorem 4], $\mathbf{MN}$ is a closed subset of $\mathbf{B}$, and

$$\mathbf{MN}//\mathbf{N} \cong \mathbf{M}//(\mathbf{M} \cap \mathbf{N}). \tag{2.2}$$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. (i) $\Rightarrow$ (ii) Let $b \in \mathbf{B}$ such that $b \neq 1$, and let m be the minimal positive integer such that $b \notin (O^\vartheta)^m(\mathbf{B})$. Then $b \in (O^\vartheta)^{m-1}(\mathbf{B})$, and it follows from $(O^\vartheta)^{m-1}(\mathbf{B})//(O^\vartheta)^m(\mathbf{B})$ a thin closed subset that $b//(O^\vartheta)^m(\mathbf{B})$ is thin. Hence, the closed subset $\langle b//(O^\vartheta)^m(\mathbf{B})\rangle$ generated by $b//(O^\vartheta)^m(\mathbf{B})$ is thin. Note that by Lemma 2.1(ii) and (2.2),

$$\langle b//\big(O^\vartheta\big)^m(\mathbf{B})\rangle = \big(\mathbf{B}_b \cdot \big(O^\vartheta\big)^m(\mathbf{B})\big)//\big(O^\vartheta\big)^m(\mathbf{B}) \cong \mathbf{B}_b//\big(\mathbf{B}_b \cap \big(O^\vartheta\big)^m(\mathbf{B})\big).$$

Thus, $\mathbf{B}_b//(\mathbf{B}_b \cap (O^\vartheta)^m(\mathbf{B}))$ is thin, and hence $O^\vartheta(\mathbf{B}_b) \subseteq \mathbf{B}_b \cap (O^\vartheta)^m(\mathbf{B})$. But $b \notin (O^\vartheta)^m(\mathbf{B})$. So $\mathbf{B}_b \cap (O^\vartheta)^m(\mathbf{B}) \neq \mathbf{B}_b$. Thus, $O^\vartheta(\mathbf{B}_b) \neq \mathbf{B}_b$, and (ii) holds.

(ii) $\Rightarrow$ (i) Since $\mathbf{B}$ is a finite set, it is enough to prove that

$$\text{for any closed subset } \mathbf{N} \text{ of } \mathbf{B} \text{ such that } \mathbf{N} \neq \{1\},\ O^\vartheta(\mathbf{N}) \neq \mathbf{N}. \tag{2.3}$$

Suppose toward a contradiction that (2.3) is false. Since the orders of closed subsets of $\mathbf{B}$ comprise a finite set of positive real numbers, we may choose a closed subset $\mathbf{N} \neq \{1\}$ with $o(\mathbf{N})$ minimal such that $O^\vartheta(\mathbf{N}) = \mathbf{N}$. Now $\mathbf{N} \neq \mathbf{B}_b$ for any $b \in \mathbf{N}$, by (ii). Hence, there exist proper closed subsets $\mathbf{C}$, $\mathbf{D}$ of $\mathbf{N}$ with $\mathbf{N} = \mathbf{CD}$. (For example, let $\mathbf{C}$ be a maximal closed subset of $\mathbf{N}$ and $\mathbf{D} = \mathbf{B}_b$ for any $b \in \mathbf{N} \setminus \mathbf{C}$.) Choose such $\mathbf{C}$, $\mathbf{D}$ with $o(\mathbf{C}) + o(\mathbf{D})$ minimal. Now $o(\mathbf{D}) < o(\mathbf{N})$ implies that (2.3) holds for $\mathbf{D}$. Hence, $\mathbf{Q} := O^\vartheta(\mathbf{D})$ is a proper closed subset of $\mathbf{D}$, and $\mathbf{D}//\mathbf{Q}$ is thin. By (2.2),

$$\mathbf{N}//\mathbf{CQ} = \mathbf{CD}//\mathbf{CQ} = (\mathbf{CQ})\mathbf{D}//\mathbf{CQ} \cong \mathbf{D}//(\mathbf{D} \cap \mathbf{CQ}).$$

But $\mathbf{Q} \subseteq \mathbf{D} \cap \mathbf{CQ}$ implies that $\mathbf{D}//(\mathbf{D} \cap \mathbf{CQ}) = (\mathbf{D}//\mathbf{Q})//((\mathbf{D} \cap \mathbf{CQ})//\mathbf{Q})$ by [X1, Theorem 4.4]. Hence, $\mathbf{N}//\mathbf{CQ}$ is thin by Lemma 2.1(iii). Also, $o(\mathbf{C}) + o(\mathbf{Q}) < o(\mathbf{C}) + o(\mathbf{D})$ implies that $\mathbf{CQ} \neq \mathbf{N}$. Therefore, $O^\vartheta(\mathbf{N}) \subseteq \mathbf{CQ} \subsetneq \mathbf{N}$, a contradiction. This proves (2.3), and (i) holds. $\square$

The next (well-known) corollary is immediate from Theorem 1.1.

Corollary 2.2. *Let $(A, \mathbf{B})$ be a nilpotent table algebra. Then for any closed subset $\mathbf{N}$ of $\mathbf{B}$, the table subalgebra $(\mathbb{C}\mathbf{N}, \mathbf{N})$ is also nilpotent.*

Note that Corollary 2.2 can be proved without using Theorem 1.1, and the proof of "(i) $\Rightarrow$ (ii)" in Theorem 1.1 is trivial by applying Corollary 2.2.

Let G be a finite group. For a nonempty subset $\mathbf{R}$ of the table basis $\mathrm{Cla}(G)$, let $\mathbf{R}_G$ be the same as in (1.3), and let $\langle \mathbf{R}_G\rangle$ be the (normal) subgroup of G generated by $\mathbf{R}_G$. The next lemma is immediate.

Lemma 2.3. *Let G be a finite group. Then the following hold.*

(i) *For any nonempty subset $\mathbf{R}$ of the table basis $\mathrm{Cla}(G)$, $\langle \mathbf{R}_G\rangle = \langle \mathbf{R}\rangle_G$.*
(ii) *For any closed subset $\mathbf{N}$ of $\mathrm{Cla}(G)$, $(O^\vartheta(\mathbf{N}))_G = [G, \mathbf{N}_G]$, the commutator subgroup of G and $\mathbf{N}_G$.*

Let G be a finite group. Then for any conjugacy class C of G, $\langle C^+\rangle_G = \langle C\rangle$ and $(O^\vartheta(\langle C^+\rangle))_G = [G, \langle C\rangle]$ by Lemma 2.3. Thus, $[G, \langle C\rangle] \neq \langle C\rangle$ if and only if $O^\vartheta(\langle C^+\rangle) \neq \langle C^+\rangle$. Since G is nilpotent if and only if the table algebra $(Z(\mathbb{C}G), \mathrm{Cla}(G))$ is nilpotent, Corollary 1.2 follows directly from Theorem 1.1.

Now we prove Theorem 1.5. We need the next lemma first.

Lemma 2.4. *Let $(A, \mathbf{B})$ be a commutative table algebra. Then for any nonempty subset $\mathbf{R}$ of $\mathbf{B}$,*

$$\Big\langle \bigcup_{b\in\mathbf{R}} \mathrm{Supp}_{\mathbf{B}}\big(bb^*\big)\Big\rangle = O^{\vartheta}\big(\langle\mathbf{R}\rangle\big). \tag{2.4}$$

Proof. Let $\mathbf{N} := \langle\bigcup_{b\in\mathbf{R}} \mathrm{Supp}_{\mathbf{B}}(bb^*)\rangle$. Then it follows from $\mathrm{Supp}_{\mathbf{B}}(bb^*) \subseteq O^{\vartheta}(\langle\mathbf{R}\rangle)$, for any $b \in \mathbf{R}$, that $\mathbf{N} \subseteq O^{\vartheta}(\langle\mathbf{R}\rangle)$. On the other hand, since $(A, \mathbf{B})$ is commutative, it follows from Lemma 2.1(iii) that for any $b \in \mathbf{R}$, $b/\!/\mathbf{N}$ is a thin element of $\mathbf{B}/\!/\mathbf{N}$. So $\langle\mathbf{R}/\!/\mathbf{N}\rangle$ is a thin closed subset of $\mathbf{B}/\!/\mathbf{N}$. Note that $\langle\mathbf{R}\rangle \supseteq \mathbf{N}$ and $\langle\mathbf{R}\rangle/\!/\mathbf{N} = \langle\mathbf{R}/\!/\mathbf{N}\rangle$ by Lemma 2.1(ii). So $\langle\mathbf{R}\rangle/\!/\mathbf{N}$ is a thin closed subset. Thus, $O^{\vartheta}(\langle\mathbf{R}\rangle) \subseteq \mathbf{N}$, and the lemma holds. $\square$

The next example says that Lemma 2.4 is not true if $(A, \mathbf{B})$ is not commutative.

Example 2.5. (The association scheme is adapted from [H].) Let $(X, \{R_i\}_{0\leqslant i\leqslant 9})$ be an association scheme such that

$$\sum_{i=0}^{9} iA_i = \begin{pmatrix}
0&1&2&2&3&3&4&4&5&5&6&6&7&7&8&8&9&9\\
1&0&3&3&2&2&5&5&4&4&8&8&9&9&6&6&7&7\\
2&3&0&2&1&3&6&7&8&9&4&7&4&6&5&9&5&8\\
2&3&2&0&3&1&7&6&9&8&7&4&6&4&9&5&8&5\\
3&2&1&3&0&2&8&9&6&7&5&9&5&8&4&7&4&6\\
3&2&3&1&2&0&9&8&7&6&9&5&8&5&7&4&6&4\\
4&5&6&7&9&8&0&4&5&1&7&2&2&6&3&9&8&3\\
4&5&7&6&8&9&4&0&1&5&2&7&6&2&9&3&3&8\\
5&4&9&8&6&7&5&1&0&4&3&9&8&3&7&2&2&6\\
5&4&8&9&7&6&1&5&4&0&9&3&3&8&2&7&6&2\\
6&9&4&7&5&8&7&2&3&8&0&6&4&2&5&3&1&9\\
6&9&7&4&8&5&2&7&8&3&6&0&2&4&3&5&9&1\\
7&8&4&6&5&9&2&6&9&3&4&2&0&7&1&8&5&3\\
7&8&6&4&9&5&6&2&3&9&2&4&7&0&8&1&3&5\\
9&6&5&8&4&7&3&8&7&2&5&3&1&9&0&6&4&2\\
9&6&8&5&7&4&8&3&2&7&3&5&9&1&6&0&2&4\\
8&7&5&9&4&6&9&3&2&6&1&8&5&3&4&2&0&7\\
8&7&9&5&6&4&3&9&6&2&8&1&3&5&2&4&7&0
\end{pmatrix},$$

where A_i is the adjacency matrix of R_i, $0 \leqslant i \leqslant 9$. Let $(\mathcal{A}, \mathbf{B})$ be the Bose–Mesner algebra of the association scheme $(X, \{R_i\}_{0\leqslant i\leqslant 9})$, where $\mathbf{B} = \{A_0, A_1, \ldots, A_9\}$. Then $(\mathcal{A}, \mathbf{B})$ is a noncommutative table algebra. It is easy to check that (2.4) does not hold for $\mathbf{R} = \{A_8\}$.

Proof of Theorem 1.5. Let us first prove (1.1) by induction on n. It is clear that (1.1) holds for $n = 1$ by the definition of the thin residue. Now assume that $n > 1$ and (1.1) holds for $n - 1$. Then we show that (1.1) holds for n. Let R be a $k \times k$ nonnegative matrix whose rows and columns are indexed by $b_1, b_2, \ldots, b_k$ and whose (b_i, b_j)-entry is $\rho_{ij} \geqslant 0$. Then by Lemma 2.4,

$$\begin{aligned}
O^{\vartheta}\big(\big\langle\mathfrak{C}(R)\big\rangle\big) &= \Big\langle \bigcup_{b_t\in\mathfrak{C}(R)} \mathrm{Supp}_{\mathbf{B}}\big(b_t b_t^*\big)\Big\rangle \\
&= \big\langle b_i \in \mathbf{B} \;\big|\; \lambda_{tt^*i} \neq 0 \text{ for some } b_t\ (t \geqslant 1) \text{ with } \rho_{tj} \neq 0 \text{ for some } b_j\big\rangle
\end{aligned}$$

$$= \left\langle b_i \in \mathbf{B} \;\middle|\; \sum_{t \geqslant 1} \lambda_{tt^*i}\rho_{tj} \neq 0 \text{ for some } b_j \right\rangle$$

$$= \left\langle b_i \in \mathbf{B} \;\middle|\; \text{the } (b_i, b_j)\text{-entry of } SR \text{ is nonzero for some } b_j \right\rangle$$

$$= \left\langle \mathfrak{C}(SR) \right\rangle.$$

In particular, let $R = S^{n-1}$. Then $O^{\vartheta}(\langle \mathfrak{C}(S^{n-1})\rangle) = \langle \mathfrak{C}(SS^{n-1})\rangle = \langle \mathfrak{C}(S^n)\rangle$. Thus, by the induction assumption,

$$\left(O^{\vartheta}\right)^n(\mathbf{B}) = O^{\vartheta}\left(\left(O^{\vartheta}\right)^{n-1}(\mathbf{B})\right) = O^{\vartheta}\left(\left\langle \mathfrak{C}\left(S^{n-1}\right)\right\rangle\right) = \left\langle \mathfrak{C}\left(S^n\right)\right\rangle.$$

So (1.1) holds for n.

Next we prove (1.2). For any positive integer n, define $\mathbf{T}^{(n)} := \{b_j \in \mathbf{B} \setminus \{1\} \mid \lambda_{jj^*i} = 0 \text{ for all } b_i \notin O_{\vartheta}^{(n-1)}(\mathbf{B})\}$, and prove by induction that for all $n \geqslant 1$,

$$O_{\vartheta}^{(n)}(\mathbf{B}) = \mathfrak{D}\left(S^n\right) \cup \{1\} = \mathbf{T}^{(n)} \cup \{1\}. \tag{2.5}$$

This is immediate for $n = 1$, by the definitions of $O_{\vartheta}^{(0)}(\mathbf{B})$, $O_{\vartheta}^{(1)}(\mathbf{B})$, and $\mathfrak{D}(S)$. Suppose that (2.5) holds for some $n \geqslant 1$. Let $\mathbf{N} := O_{\vartheta}^{(n)}(\mathbf{B})$. Then for $j \geqslant 1$,

$$\begin{aligned} b_j \in O_{\vartheta}^{(n+1)}(\mathbf{B}) \quad &\Longleftrightarrow \quad (b_j/\!/\mathbf{N})\left(b_j^*/\!/\mathbf{N}\right) = 1/\!/\mathbf{N} \quad \Longleftrightarrow \quad \mathrm{Supp}_{\mathbf{B}}\left(b_j b_j^*\right) \subseteq \mathbf{N} \\ &\Longleftrightarrow \quad \lambda_{jj^*t} = 0 \quad \text{for all } b_t \notin O_{\vartheta}^{(n)}(\mathbf{B}) \quad \Longleftrightarrow \quad b_j \in \mathbf{T}^{(n+1)}. \end{aligned}$$

By the induction hypothesis, for $t \geqslant 1$,

$$b_t \in O_{\vartheta}^{(n)}(\mathbf{B}) \quad \Longleftrightarrow \quad \text{the } (b_i, b_t)\text{-entry of } S^n \text{ is } 0 \quad \text{for all } 1 \leqslant i \leqslant k.$$

Let β_{it} be the (b_i, b_t)-entry of S^n, $1 \leqslant i, t \leqslant k$. If $j \geqslant 1$ and $b_j \in O_{\vartheta}^{(n+1)}(\mathbf{B})$, then for all $i \geqslant 1$, the (b_i, b_j)-entry of $S^{n+1} = S^n S$ is

$$\sum_{t \geqslant 1} \beta_{it}\lambda_{jj^*t} = \sum_{b_t \in O_{\vartheta}^{(n)}(\mathbf{B}) \setminus \{1\}} \beta_{it}\lambda_{jj^*t} + \sum_{b_t \notin O_{\vartheta}^{(n)}(\mathbf{B})} \beta_{it}\lambda_{jj^*t} = 0.$$

Thus, $O_{\vartheta}^{(n+1)}(\mathbf{B}) \subseteq \mathfrak{D}(S^{n+1}) \cup \{1\}$. If $b_j \notin O_{\vartheta}^{(n+1)}(\mathbf{B})$, then $\lambda_{jj^*u} \neq 0$ for some $b_u \notin O_{\vartheta}^{(n)}(\mathbf{B})$. Hence, by the induction assumption again, the (b_i, b_u)-entry of S^n is nonzero for some i. So the (b_i, b_j)-entry of $S^n S$ is nonzero, and $b_j \notin \mathfrak{D}(S^{n+1}) \cup \{1\}$. This proves that (2.5) holds for $n+1$, and establishes (1.2). □

Let G be a finite group. Then the map

$$\left\{\text{closed subsets of } \mathrm{Cla}(G)\right\} \to \{\text{normal subgroups of } G\}, \quad \mathbf{N} \mapsto \mathbf{N}_G$$

is bijective. Furthermore, for any closed subsets $\mathbf{M}$ and $\mathbf{N}$ of $\mathrm{Cla}(G)$ such that $\mathbf{N} \subseteq \mathbf{M}$,

$$\mathbf{M}/\!/\mathbf{N} \text{ is thin} \quad \text{if and only if} \quad \mathbf{M}_G/\mathbf{N}_G \subseteq Z(G/\mathbf{N}_G).$$

Therefore, for any positive integer n,

$$\left(O_{\vartheta}^{(n)}\left(\mathrm{Cla}(G)\right)\right)_G = Z_n(G).$$

It is also clear that for any positive integer n,

$$\left(\left(O^{\vartheta}\right)^n\left(\mathrm{Cla}(G)\right)\right)_G = G_{n+1}$$

by Lemma 2.3(ii). Hence, Corollary 1.7 follows directly from Theorem 1.5.

3. Thin residue digraphs

In this section we first study some basic properties of the thin residue digraph of a table algebra, and prove Theorem 1.10. Then we show that some properties proved by lengthy and complicated calculations in [TY, Theorem 1.4] can be easily obtained by the method developed in this paper.

Let $\Gamma = (V, E)$ be a digraph. For any vertices $u, v \in V$, if there is an ordered list of vertices $u_1 = u$, $u_2, \ldots, u_{n+1} = v$ such that $(u_i, u_{i+1}) \in E$ for $i = 1, 2, \ldots, n$, then we say that there is a *directed path* from u to v of length n. If $u = v$, then a directed path from u to u is a *directed cycle*. If Γ is not simple, and $(u, u) \in E$ for some $u \in V$, then Γ has a directed cycle with only one vertex u.

Definition 3.1. Let Γ be a digraph. A vertex v of Γ is called a *cycle-less* vertex if there is no directed path that begins with v and intersects a directed cycle.

Let Γ be a digraph. Then a vertex v of Γ is cycle-less if and only if the length of any directed path beginning with v (if it exists) is finite. If v is a cycle-less vertex of Γ, then all vertices on the directed paths beginning with v (if any) are cycle-less. Furthermore, every vertex of Γ is cycle-less if and only if Γ contains no directed cycles.

Let $(A, \mathbf{B})$ be a table algebra, with $\mathbf{B} = \{b_0 = 1_A, b_1, b_2, \ldots, b_k\}$. For any $k \times k$ nonnegative matrix P with rows and columns indexed by $b_1, b_2, \ldots, b_k$, let Γ_P be the digraph defined by

$$V(\Gamma_P) = \mathbf{B} \setminus \{1\} \quad \text{and} \quad E(\Gamma_P) = \left\{(b_i, b_j) \;\middle|\; \text{the } (b_i, b_j)\text{-entry of } P \text{ is not zero}\right\}.$$

That is, Γ_P is the digraph corresponding to P.

Lemma 3.2. *With the notation in the above paragraph, the following hold.*

- (i) *For any $b_i \in \mathbf{B} \setminus \{1\}$, $b_i \in \mathfrak{C}(P^n)$ for some positive integer $n > 1$ if and only if there exists $b_j \in \mathbf{B} \setminus \{1\}$ such that $(b_i, b_j) \in E(\Gamma_P)$ and $b_j \in \mathfrak{C}(P^{n-1})$.*
- (ii) *For any $b_i \in \mathbf{B} \setminus \{1\}$, $b_i \in \mathfrak{C}(P^n)$ for some positive integer n if and only if there is a directed path of length n that begins with b_i.*
- (iii) *If b_i is not a cycle-less vertex of Γ_P, then $b_i \in \mathfrak{C}(P^n)$ for any positive integer n.*
- (iv) *If b_i is a cycle-less vertex of Γ_P, and the length of the longest directed path that begins with b_i is $n \geqslant 1$, then $b_i \in \mathfrak{C}(P^r)$ for any positive integer $r \leqslant n$, but $b_i \notin \mathfrak{C}(P^m)$ for any positive integer $m > n$.*

Proof. (i) Assume that for any $1 \leqslant i, m \leqslant k$, the (b_i, b_m)-entries of P and P^{n-1} are p_{im} and β_{im}, respectively. Then the (b_i, b_m)-entry of P^n is $\sum_{l=1}^{k} p_{il}\beta_{lm}$. Since P and hence P^{n-1} are nonnegative matrices, the (b_i, b_m)-entry of P^n is not zero if and only if there is $1 \leqslant j \leqslant k$ such that $p_{ij} \neq 0$ and $\beta_{jm} \neq 0$. So (i) holds.

(ii) follows directly from (i) by induction on n.

(iii) Since b_i is not a cycle-less vertex, there is a directed path that begins with b_i and intersects a directed cycle. So for any positive integer n, there is a directed path of length n that begins with b_i. Hence, (iii) holds by (ii).

(iv) Since there is a directed path of length r that begins with b_i for any positive integer $r \leqslant n$, and there is no directed path of length m that begins with b_i for any positive integer $m > n$, (iv) follows directly from (ii). □

Definition 3.3. Let Γ be a digraph. If Γ has a directed path whose vertices are all cycle-less, then the length of the longest directed path with cycle-less vertices is called the *cycle-less length* of Γ, and denoted by $\ell(\Gamma)$. If Γ has no directed path with cycle-less vertices, then define $\ell(\Gamma) = 0$.

Let Γ be a digraph. If Γ has a directed path whose vertices are all cycle-less, then it is clear that the length of the path is finite. Thus, we have that $\ell(\Gamma) < \infty$.

Lemma 3.4. *Let $(A, \mathbf{B})$ be a table algebra, with $\mathbf{B} = \{b_0 = 1_A, b_1, b_2, \ldots, b_k\}$. Let P be a $k \times k$ nonnegative matrix whose rows and columns are indexed by $b_1, b_2, \ldots, b_k$. If $n = \ell(\Gamma_P)$, then*

$$\mathfrak{C}(P) \supsetneqq \mathfrak{C}(P^2) \supsetneqq \cdots \supsetneqq \mathfrak{C}(P^n) \supsetneqq \mathfrak{C}(P^{n+1}) = \mathfrak{C}(P^{n+2}) = \cdots.$$

Furthermore, $\mathfrak{C}(P^{n+1})$ contains no cycle-less vertices of Γ_P.

Proof. If $n = 0$, then Γ_P does not have any directed path whose vertices are all cycle-less. Let $b_i \in \mathfrak{C}(P)$. Then b_i is not a cycle-less vertex. Hence, $b_i \in \mathfrak{C}(P^m)$ for all positive integer m by Lemma 3.2(iii). This proves that $\mathfrak{C}(P) = \mathfrak{C}(P^2) = \mathfrak{C}(P^3) = \cdots$, and the lemma holds. If $n > 0$, then there is a directed path of length n, say $b_{i_{n+1}} \to b_{i_n} \to \cdots \to b_{i_2} \to b_{i_1}$, such that $b_{i_{n+1}}, b_{i_n}, \ldots, b_{i_2}, b_{i_1}$ are cycle-less vertices and $b_{i_1} \notin \mathfrak{C}(P)$. Hence for $m = 2, 3, \ldots, n+1$, $n = \ell(\Gamma_P)$ implies that the length of the longest directed path beginning with b_{i_m} is $m-1$. Thus, $b_{i_m} \in \mathfrak{C}(P^{m-1})$ but $b_{i_m} \notin \mathfrak{C}(P^m)$ by Lemma 3.2(iv), $m = 2, 3, \ldots, n+1$. Therefore, $\mathfrak{C}(P^r) \neq \mathfrak{C}(P^{r+1})$ for all $r = 1, 2, \ldots, n$. Furthermore, since for any cycle-less vertex b_j, the length of any directed path that begins with b_j is at most n, it follows from Lemma 3.2(iv) that $\mathfrak{C}(P^{n+1}), \mathfrak{C}(P^{n+2}), \ldots$ contain no cycle-less vertices of Γ_P. So $\mathfrak{C}(P^{n+1}) = \mathfrak{C}(P^{n+2}) = \cdots$ by Lemma 3.2(iii). It is clear that $\mathfrak{C}(P^r) \supseteq \mathfrak{C}(P^{r+1})$ for all positive integer r. Hence, the lemma holds. □

Now Theorem 1.10 is clear. We include a very short proof here for the convenience of the reader.

Proof of Theorem 1.10. By Lemma 3.2(iii) and (iv), $\mathrm{Mat}_{O^\vartheta}(\mathbf{B})$ is a nilpotent matrix if and only if every vertex of the digraph $\Gamma_{O^\vartheta(\mathbf{B})}$ is cycle-less. Hence, $\mathrm{Mat}_{O^\vartheta}(\mathbf{B})$ is a nilpotent matrix if and only if $\Gamma_{O^\vartheta(\mathbf{B})}$ has no directed cycles. Furthermore, if $\mathrm{Mat}_{O^\vartheta}(\mathbf{B})$ is a nilpotent matrix of index of nilpotence n, then $n = 1 + \ell(\Gamma_{O^\vartheta(\mathbf{B})})$ by Lemma 3.4. Thus, Theorem 1.10 holds by Corollary 1.6. □

The next corollary is immediate from Theorem 1.5 and Lemma 3.4.

Corollary 3.5. *Let $(A, \mathbf{B})$ be a commutative table algebra. Let $n = \ell(\Gamma_{O^\vartheta(\mathbf{B})})$. Then*

$$\left(O^\vartheta\right)^{n+1}(\mathbf{B}) = \left(O^\vartheta\right)^{n+2}(\mathbf{B}) = \cdots.$$

Let $(A, \mathbf{B})$ be a commutative table algebra. The next example says that for some positive integer $m < \ell(\Gamma_{O^\vartheta(\mathbf{B})})$, we may also have $(O^\vartheta)^{m+1}(\mathbf{B}) = (O^\vartheta)^{m+2}(\mathbf{B}) = \cdots$.

Let $(A, \mathbf{B})$ be a commutative table algebra. The nonzero entries of $\mathrm{Mat}_{O^\vartheta}(\mathbf{B})$ are not all ones in general. Since $\mathrm{Mat}_{O^\vartheta}(\mathbf{B})$ is a nonnegative matrix, changing its nonzero entries to 1 does not affect which entries of the powers of $\mathrm{Mat}_{O^\vartheta}(\mathbf{B})$ are nonzero. This fact is used implicitly in this section.

Example 3.6. Let $(U, \mathbf{V})$ be a commutative table algebra such that $\mathbf{V} := \{v_0 := 1, v_1, v_2, v_3\}$, $v_i^* = v_i$, $0 \leqslant i \leqslant 3$, and

$$v_1^2=3v_0+v_1+v_2,\qquad v_1v_2=v_1+2v_3,\qquad v_1v_3=v_3+2v_2,$$
$$v_2^2=3v_0+2v_2,\qquad v_2v_3=v_3+2v_1,\qquad v_3^2=3v_0+v_1+v_2.$$

(This table algebra is presented in [BX1, Example 1.14].) Let $(C,\mathbf{D})$ be a commutative table algebra such that $\mathbf{D}:=\{d_0:=1,d_1\}$ and $d_1^2=d_0$. Let $(A,\mathbf{B}):=(C\wr U,\mathbf{D}\wr\mathbf{V})$, the wreath product of $(C,\mathbf{D})$ and $(U,\mathbf{V})$. Then

$$\mathbf{B}:=\mathbf{D}\wr\mathbf{V}=\{d_0\otimes v_0,d_0\otimes v_1,d_0\otimes v_2,d_0\otimes v_3,d_1\otimes\mathbf{V}^+\},\quad\text{where }\mathbf{V}^+:=v_0+v_1+v_2+v_3.$$

(For the basic properties of wreath products of table algebras, the reader is referred to [X3].) Hence,

$$\mathrm{Mat}_{O^\vartheta}(\mathbf{B})=\begin{pmatrix}1&0&1&1\\1&1&1&1\\0&0&0&1\\0&0&0&0\end{pmatrix}.$$

Thus,

$$\mathfrak{C}\big(\mathrm{Mat}_{O^\vartheta}(\mathbf{B})\big)\supsetneqq\mathfrak{C}\big(\big(\mathrm{Mat}_{O^\vartheta}(\mathbf{B})\big)^2\big)=\mathfrak{C}\big(\big(\mathrm{Mat}_{O^\vartheta}(\mathbf{B})\big)^3\big)=\cdots.$$

So $\ell(\Gamma_{O^\vartheta(\mathbf{B})})=1$ by Lemma 3.4. (It is also very easy to get $\ell(\Gamma_{O^\vartheta(\mathbf{B})})=1$ from the digraph $\Gamma_{O^\vartheta(\mathbf{B})}$.) However, it is clear that

$$O^\vartheta(\mathbf{B})=\big(O^\vartheta\big)^2(\mathbf{B})=\big(O^\vartheta\big)^3(\mathbf{B})=\cdots=\{d_0\otimes v_0,d_0\otimes v_1,d_0\otimes v_2,d_0\otimes v_3\}.$$

That is, although $\mathfrak{C}(\mathrm{Mat}_{O^\vartheta}(\mathbf{B}))\supsetneqq\mathfrak{C}((\mathrm{Mat}_{O^\vartheta}(\mathbf{B}))^2)$, we have

$$\big\langle\mathfrak{C}\big(\mathrm{Mat}_{O^\vartheta}(\mathbf{B})\big)\big\rangle=\big\langle\mathfrak{C}\big(\big(\mathrm{Mat}_{O^\vartheta}(\mathbf{B})\big)^2\big)\big\rangle.$$

Let $(A,\mathbf{B})$ be a table algebra, with $\mathbf{B}=\{b_0=1_A,b_1,b_2,\dots,b_k\}$, and structure constants λ_{ijm}, $0\leqslant i,j,m\leqslant k$. Let ν be the degree map of $(A,\mathbf{B})$. Recall that $\nu(b_i)>0$ for all $0\leqslant i\leqslant k$. Now we introduce two matrices related to $\mathrm{Mat}_{O^\vartheta}(\mathbf{B})$. Let P and Q be $(k+1)\times(k+1)$ nonnegative matrices whose rows and columns are indexed by $b_0,b_1,\dots,b_k$. Assume that the (b_i,b_j)-entry of P is $\lambda_{jj^*i}/\nu(b_j)$, and the (b_i,b_j)-entry of Q is $\lambda_{jj^*i}/\nu(b_i)$, $0\leqslant i,j\leqslant k$. For a finite group G, two matrices similar to P and Q for $\mathrm{Cla}(G)$ and $\mathrm{Irr}(G)$ were introduced in [TY], and some properties of these two matrices were presented. In the following we show that the similar properties for P and Q can be easily obtained by means of the thin residue digraph $\Gamma_{O^\vartheta(\mathbf{B})}$. Let O_k be the $k\times k$ zero matrix.

Proposition 3.7. *With the notation in the above paragraph, for any positive integer $n>1$, the following are equivalent.*

(i) $P^n=\begin{pmatrix}1&\alpha\\0&O_k\end{pmatrix}$, *where* $\alpha=(\nu(b_1),\nu(b_2),\dots,\nu(b_k))$.
(ii) $Q^n=\begin{pmatrix}1&\gamma\\0&O_k\end{pmatrix}$, *where* $\gamma=(\nu(b_1)^2,\nu(b_2)^2,\dots,\nu(b_k)^2)$.
(iii) *The n-th power of* $\mathrm{Mat}_{O^\vartheta}(\mathbf{B})$ *is a zero matrix.*

Proof. Since $\lambda_{00^*0}=1$, $\lambda_{00^*i}=0$ for all $1\leqslant i\leqslant k$, and $\nu(b_0)=1$, we may write P and Q as

$$P=\begin{pmatrix}1&\alpha_1\\0&P_0\end{pmatrix}\quad\text{and}\quad Q=\begin{pmatrix}1&\gamma_1\\0&Q_0\end{pmatrix},$$

where the (b_i, b_j)-entry of P_0 is $\lambda_{jj^*i}/\nu(b_j)$, the (b_i, b_j)-entry of Q_0 is $\lambda_{jj^*i}/\nu(b_i)$, for all $1 \leqslant i, j \leqslant k$, and

$$\alpha_1 = \big(\lambda_{11^*0}/\nu(b_1), \lambda_{22^*0}/\nu(b_2), \ldots, \lambda_{kk^*0}/\nu(b_k)\big),$$
$$\gamma_1 = \big(\lambda_{11^*0}/\nu(b_0), \lambda_{22^*0}/\nu(b_0), \ldots, \lambda_{kk^*0}/\nu(b_0)\big) = (\lambda_{11^*0}, \lambda_{22^*0}, \ldots, \lambda_{kk^*0}).$$

Thus, the digraphs Γ_{P_0}, Γ_{Q_0}, and $\Gamma_{O^\vartheta(\mathbf{B})}$ are the same. Hence, the above discussions yield that for any positive integer $n > 1$, the n-th power of $\mathrm{Mat}_{O^\vartheta}(\mathbf{B})$ is equal to O_k if and only if $P_0^n = O_k$ if and only if $Q_0^n = O_k$. But

$$P^n = \begin{pmatrix} 1 & \alpha_1(I + P_0 + \cdots + P_0^{n-1}) \\ 0 & P_0^n \end{pmatrix} \quad \text{and} \quad Q^n = \begin{pmatrix} 1 & \gamma_1(I + Q_0 + \cdots + Q_0^{n-1}) \\ 0 & Q_0^n \end{pmatrix},$$

where I is the $k \times k$ identity matrix. Hence, the following are equivalent.

(a) $P^n = \left(\begin{smallmatrix} 1 & \alpha_1(I+P_0+\cdots+P_0^{n-1}) \\ 0 & O_k \end{smallmatrix}\right)$.
(b) $Q^n = \left(\begin{smallmatrix} 1 & \gamma_1(I+Q_0+\cdots+Q_0^{n-1}) \\ 0 & O_k \end{smallmatrix}\right)$.
(c) The n-th power of $\mathrm{Mat}_{O^\vartheta}(\mathbf{B})$ is O_k.

Note that if $P_0^n = Q_0^n = O_k$, then $I + P_0 + \cdots + P_0^{n-1} = (I - P_0)^{-1}$, and $I + Q_0 + \cdots + Q_0^{n-1} = (I - Q_0)^{-1}$. But for any $1 \leqslant j \leqslant k$, applying the degree map ν to both sides of $b_j b_j^* = \sum_{i=0}^k \lambda_{jj^*i} b_i$ yields that

$$\sum_{i=1}^k \lambda_{jj^*i}\nu(b_i) = \nu(b_j)^2 - \lambda_{jj^*0}.$$

Hence,

$$\alpha(I - P_0) = \alpha_1 \quad \text{and} \quad \gamma(I - Q_0) = \gamma_1.$$

Thus, if $P_0^n = Q_0^n = O_k$, then

$$\alpha_1\big(I + P_0 + \cdots + P_0^{n-1}\big) = \alpha_1(I - P_0)^{-1} = \alpha,$$

and

$$\gamma_1\big(I + Q_0 + \cdots + Q_0^{n-1}\big) = \gamma_1(I - Q_0)^{-1} = \gamma.$$

So the lemma follows from the equivalence of (a), (b), and (c). □

Let G be a finite group, and let $C_0 := \{1\}, C_1, \ldots, C_d$ be the conjugacy classes of G. Then $\mathrm{Cla}(G) = \{C_0^+, C_1^+, \ldots, C_d^+\}$. Let s_{ijm}, $0 \leqslant i, j, m \leqslant d$, denote the structure constants of $(Z(\mathbb{C}G), \mathrm{Cla}(G))$, and let $k_i := |C_i|$, the cardinality of C_i, $0 \leqslant i \leqslant d$. Then the degree map of $(Z(\mathbb{C}G), \mathrm{Cla}(G))$ is defined by $C_i^+ \mapsto k_i$. Let P_1 and Q_1 be two $(d+1) \times (d+1)$ matrices whose rows and columns are indexed by $C_0^+, C_1^+, \ldots, C_d^+$. Assume that the (C_i^+, C_j^+)-entry of P_1 is s_{jj^*i}/k_j, and the (C_i^+, C_j^+)-entry of Q_1 is s_{jj^*i}/k_i, $0 \leqslant i, j \leqslant d$. Furthermore, let $\chi_0, \chi_1, \ldots, \chi_d$ be the irreducible characters of G such that χ_0 is the principal character. The structure constants of $(\mathrm{Ch}(G), \mathrm{Irr}(G))$ are denoted by t_{ijm}, $0 \leqslant i, j, m \leqslant d$. Let $z_i := \chi_i(1)$, $0 \leqslant i \leqslant d$. Then the degree map of $(\mathrm{Ch}(G), \mathrm{Irr}(G))$ is induced by $\chi_i \mapsto z_i$. Let P_2 and Q_2 be two $(d+1) \times (d+1)$ matrices whose rows and columns are indexed by $\chi_0, \chi_1, \ldots, \chi_d$. Assume that the (χ_i, χ_j)-entry of P_2 is t_{jj^*i}/z_j, and the (χ_i, χ_j)-entry of Q_2 is t_{jj^*i}/z_i, $0 \leqslant i, j \leqslant d$. Let O_d denote the $d \times d$ zero matrix. As a direct consequence of Proposition 3.7 and Corollary 1.8, we have the following corollary.

Corollary 3.8. *With the notation in the above paragraph, for any positive integer $n > 1$, the following are equivalent.*

(i) $P_1^n = \begin{pmatrix} 1 & \alpha_1 \\ 0 & O_d \end{pmatrix}$, *where* $\alpha_1 = (k_1, k_2, \ldots, k_d)$.
(ii) $Q_1^n = \begin{pmatrix} 1 & \gamma_1 \\ 0 & O_d \end{pmatrix}$, *where* $\gamma_1 = (k_1^2, k_2^2, \ldots, k_d^2)$.
(iii) $P_2^n = \begin{pmatrix} 1 & \alpha_2 \\ 0 & O_d \end{pmatrix}$, *where* $\alpha_2 = (z_1, z_2, \ldots, z_d)$.
(iv) $Q_2^n = \begin{pmatrix} 1 & \gamma_2 \\ 0 & O_d \end{pmatrix}$, *where* $\gamma_2 = (z_1^2, z_2^2, \ldots, z_d^2)$.
(v) *The n-th power of* $\mathrm{Mat}_{O^\vartheta}(\mathrm{Cla}(G))$ *is a zero matrix.*
(vi) *The n-th power of* $\mathrm{Mat}_{O^\vartheta}(\mathrm{Irr}(G))$ *is a zero matrix.*

Note that P_1 and Q_2 in Corollary 3.8 are the same as A and B in [TY, Theorem 1.4], respectively, and the equivalence of (i), (iv), (v), and (vi) was proved in [TY, Theorem 1.4].

4. Basis digraphs

In this section we study basic properties of the basis digraphs of table algebras, and prove Theorem 1.13. Let $\Gamma = (V, E)$ be a digraph. Then Γ is *weakly* connected if replacing all of its directed edges with undirected edges produces a connected (undirected) graph. Γ is *strongly* connected if it contains a directed path from u to v for any distinct vertices $u, v \in V$. The *strong components* of Γ are the maximal strongly connected subgraphs, and the *weak components* are the maximal weakly connected subgraphs. For any vertices $u, v \in V$, if there is an ordered list of vertices $u_1 = u, u_2, \ldots, u_{n+1} = v$ such that either $(u_i, u_{i+1}) \in E$ or $(u_{i+1}, u_i) \in E$ for $i = 1, 2, \ldots, n$, then we say that there is a *semipath* between u and v of length n.

Lemma 4.1. *Let $(A, \mathbf{B})$ be a table algebra. Let $b \in \mathbf{B}$ such that $b \neq 1$. Then the following hold.*

(i) *For any $a, c \in \mathbf{B}$, there is a directed path from a to c of length n in the digraph Γ_b if and only if $c \in \mathrm{Supp}_{\mathbf{B}}(b^n a)$.*
(ii) *For any $a, c \in \mathbf{B}$, if there is a semipath between a and c in the digraph Γ_b, then there is also a directed path from a to c.*
(iii) *A weak component of Γ_b is also a strong component of Γ_b.*
(iv) *For any $a \in \mathbf{B}$, the vertex set of the (weak, strong) component of Γ_b that contains a is $\mathbf{B}_b a$.*
(v) *The digraph Γ_b is strongly connected if and only if $\mathbf{B}_b = \mathbf{B}$.*

Proof. (i) Let us prove the statement by induction on n. It is clear that the statement is true for $n = 1$. Now assume that $n > 1$ and the statement is true for $n - 1$. Then we prove the statement for n. If $c \in \mathrm{Supp}_{\mathbf{B}}(b^n a)$, then there exists $c_1 \in \mathrm{Supp}_{\mathbf{B}}(b^{n-1} a)$ such that $c \in \mathrm{Supp}_{\mathbf{B}}(bc_1)$. Hence, there is a directed path from a to c_1 of length $n - 1$ in the digraph Γ_b by induction assumption, and $(c_1, c) \in E(\Gamma_b)$. So there is a directed path from a to c of length n. On the other hand, if there is a directed path from a to c of length n, then there exists $c_1 \in \mathbf{B}$ such that there is a directed path from a to c_1 of length $n - 1$ and $(c_1, c) \in E(\Gamma_b)$. Hence, $c_1 \in \mathrm{Supp}_{\mathbf{B}}(b^{n-1} a)$ by induction assumption, and $c \in \mathrm{Supp}_{\mathbf{B}}(bc_1)$. Thus, $c \in \mathrm{Supp}_{\mathbf{B}}(b^n a)$, and the statement is true for n.

(ii) It is enough to prove that for any $a, c \in \mathbf{B}$, if $(c, a) \in E(\Gamma_b)$, then there is a directed path from a to c in the digraph Γ_b. Note that $(c, a) \in E(\Gamma_b)$ if and only if $a \in \mathrm{Supp}_{\mathbf{B}}(bc)$ if and only if $c \in \mathrm{Supp}_{\mathbf{B}}(b^* a)$. But $b^* \in \mathrm{Supp}_{\mathbf{B}}(b^n)$ for some positive integer n. So $c \in \mathrm{Supp}_{\mathbf{B}}(b^n a)$, and there is a directed path from a to c in the digraph Γ_b by (i). This proves (ii). Now (iii) follows directly from (ii), (iv) from (i) and (iii), and (v) from (iv). □

Lemma 4.1(v) was proved in [AFM, Proposition 4.3]. Since each weak component of Γ_b is also a strong component by Lemma 4.1(iii), a weak (or strong) component of Γ_b will simply be called a *component* of Γ_b.

Definition 4.2. Let $(A, \mathbf{B})$ be a table algebra. Let $b \in \mathbf{B}$ such that $b \neq 1$. Then the component of Γ_b with vertex set $\mathbf{B}_b$ is called the *principal component* of Γ_b.

Let Γ be a digraph such that any weak component of Γ is also a strong component. Then

$$\max\{d \mid d \text{ is the diameter of a weak (or strong) component of } \Gamma\}$$

is called the *maximal component diameter* of Γ, and denoted by $\mathrm{mcd}(\Gamma)$.

Lemma 4.3. *Let $(A, \mathbf{B})$ be a table algebra. Let $b \in \mathbf{B}$ such that $b \neq 1$. Then*

$$\mathrm{mcd}(\Gamma_b) = \textit{the diameter of the principal component}.$$

Proof. Let $a, c \in \mathbf{B}$ such that $a \neq c$ and the distance from a to c is n for some positive integer n. Then we show that n is less than or equal to the diameter of the principal component of Γ_b. If $n = 1$, there is nothing to prove. Now assume that $n > 1$. Then $c \in \mathrm{Supp}_{\mathbf{B}}(b^n a)$ but $c \notin \mathrm{Supp}_{\mathbf{B}}(b^m a)$ for any $1 \leqslant m \leqslant n-1$ by Lemma 4.1(i). Hence, $\mathrm{Supp}_{\mathbf{B}}(ca^*) \cap \mathrm{Supp}_{\mathbf{B}}(b^n) \neq \emptyset$ but $\mathrm{Supp}_{\mathbf{B}}(ca^*) \cap \mathrm{Supp}_{\mathbf{B}}(b^m) = \emptyset$ for any $1 \leqslant m \leqslant n-1$. Let $d \in \mathrm{Supp}_{\mathbf{B}}(ca^*) \cap \mathrm{Supp}_{\mathbf{B}}(b^n)$. Then $d \neq 1$, $d \in \mathrm{Supp}_{\mathbf{B}}(b^n)$, but $d \notin \mathrm{Supp}_{\mathbf{B}}(b^m)$ for any $1 \leqslant m \leqslant n-1$. Thus, $d \in \mathbf{B}_b$ and the distance from d to 1 is n by Lemma 4.1(i). So n is less than or equal to the diameter of the principal component. Hence, the lemma holds. □

The next lemma says that a thin element is characterized by its basis digraph.

Lemma 4.4. *Let $(A, \mathbf{B})$ be a table algebra. Let $b \in \mathbf{B}$ such that $b \neq 1$. Then the following are equivalent.*

(i) *b is a thin element of $\mathbf{B}$.*
(ii) *Each component of the digraph Γ_b is a directed cycle.*

Furthermore, if b is thin, then the number of vertices of any component of Γ_b is a factor of $|\mathbf{B}_b|$.

Proof. (i) ⇒ (ii) Assume that b is thin. Then $\mathbf{B}_b$ is also thin. For any $c \in \mathbf{B}_b$, let $c' := \nu(c)^{-1}c$, where ν is the degree map of $(A, \mathbf{B})$. Then $\mathbf{B}'_b := \{c' \mid c \in \mathbf{B}_b\}$ is a cyclic group generated by $b' := \nu(b)^{-1}b$. Let $a \in \mathbf{B}$, and let $\mathrm{sta}_{\mathbf{B}'_b}(a) := \{u' \in \mathbf{B}'_b \mid u'a = a\}$ be the stabilizer of a in $\mathbf{B}'_b$. Then $\mathrm{sta}_{\mathbf{B}'_b}(a)$ is a subgroup of $\mathbf{B}'_b$, and it follows from $\mathbf{B}$ a finite set that $(b')^n \in \mathrm{sta}_{\mathbf{B}'_b}(a)$ for some positive integer n. Assume that m is the smallest positive integer such that $(b')^m \in \mathrm{sta}_{\mathbf{B}'_b}(a)$. If $m = 1$, then $\mathbf{B}_b a = \{a\}$, and $(a, a) \in \Gamma_b$. So the component of Γ_b that has vertex a is a directed cycle. If $m > 1$, since $|\mathrm{Supp}_{\mathbf{B}}(b^i a)| = 1$ for any positive integer i, we may assume that $\mathrm{Supp}_{\mathbf{B}}(b^i a) = \{a_i\}$, $i = 1, 2, \ldots, m-1$. Then $a, a_1, a_2, \ldots, a_{m-1}$ are all distinct, and $\mathbf{B}_b a = \{a, a_1, a_2, \ldots, a_{m-1}\}$. Thus, the component of Γ_b that has vertex a is also a directed cycle. This proves (ii). Moreover, $|\mathbf{B}_b a| = |\mathrm{sta}_{\mathbf{B}'_b}(a)| = m$, and hence $|\mathbf{B}_b a|$ is a factor of $|\mathbf{B}_b|$.

(ii) ⇒ (i) Since the principal component of Γ_b is a directed cycle, we must have that $\mathrm{Supp}_{\mathbf{B}}(bb^*) = \{1\}$. So b is thin, and (i) holds. □

Lemma 4.5. *Let $(A, \mathbf{B})$ be a table algebra. Let $b \in \mathbf{B}$ such that $b \neq 1$. Assume that $\mathbf{B}_b$ has a proper closed subset $\mathbf{N}$. Then $\langle b/\!/\mathbf{N}\rangle = \mathbf{B}_b/\!/\mathbf{N}$.*

Proof. Since $\mathbf{B}_b//\mathbf{N}$ is a closed subset of $\mathbf{B}//\mathbf{N}$, and $b//\mathbf{N} \in \mathbf{B}_b//\mathbf{N}$, we see that $\langle b//\mathbf{N}\rangle \subseteq \mathbf{B}_b//\mathbf{N}$. On the other hand, (2.1) and [X1, Lemma 2.5(iii)] imply that

$$\langle b//\mathbf{N}\rangle = \bigcup_{n=1}^{\infty}(b//\mathbf{N})^n \supseteq \bigcup_{n=1}^{\infty}\big(\mathrm{Supp}_{\mathbf{B}}\big(b^n\big)//\mathbf{N}\big) = \left(\bigcup_{n=1}^{\infty}\mathrm{Supp}_{\mathbf{B}}\big(b^n\big)\right)//\mathbf{N} = \mathbf{B}_b//\mathbf{N}.$$

Thus, $\langle b//\mathbf{N}\rangle = \mathbf{B}_b//\mathbf{N}$. $\square$

Now we discuss the relations between the basis digraphs of a table algebra and its quotient table algebra.

Lemma 4.6. *Let $(A,\mathbf{B})$ be a table algebra. Let $b \in \mathbf{B}$ such that $b \neq 1$. Assume that $\mathbf{B}_b$ has a proper closed subset $\mathbf{N}$. Then for any $a, c \in \mathbf{B}$, the following hold.*

(i) *If $(a,c) \in E(\Gamma_b)$, then $(a//\mathbf{N}, c//\mathbf{N}) \in E(\Gamma_{b//\mathbf{N}})$.*
(ii) *If $(A,\mathbf{B})$ is commutative, then a and c are in the same component of Γ_b if and only if $a//\mathbf{N}$ and $c//\mathbf{N}$ are in the same component of $\Gamma_{b//\mathbf{N}}$.*

Proof. (i) If $c \in \mathrm{Supp}_{\mathbf{B}}(ba)$, then $c//\mathbf{N} \in \mathrm{Supp}_{\mathbf{B}//\mathbf{N}}((b//\mathbf{N})(a//\mathbf{N}))$ by [AFM, Theorem 4.9]. So (i) holds.

(ii) If a and c are in the same component of Γ_b, then $a//\mathbf{N}$ and $c//\mathbf{N}$ are in the same component of $\Gamma_{b//\mathbf{N}}$ by (i). Now assume that $a//\mathbf{N}$ and $c//\mathbf{N}$ are in the same component of $\Gamma_{b//\mathbf{N}}$. Then by Lemmas 4.1(iv) and 4.5, $c//\mathbf{N} \in (\mathbf{B}_b//\mathbf{N})(a//\mathbf{N})$. Since $(A,\mathbf{B})$ is commutative, $(\mathbf{B}_b//\mathbf{N})(a//\mathbf{N}) = \mathbf{B}_b a//\mathbf{N}$ by Lemma 2.1(i). Thus, $c//\mathbf{N} = c_1//\mathbf{N}$ for some $c_1 \in \mathbf{B}_b a$. Note that $(A,\mathbf{B})$ commutative and $c//\mathbf{N} = c_1//\mathbf{N}$ yield that $c \in \mathbf{N}c_1$. Hence, $c \in \mathbf{N}\mathbf{B}_b a$. But $\mathbf{N}$ is a proper closed subset of $\mathbf{B}_b$. So $\mathbf{N}\mathbf{B}_b = \mathbf{B}_b$, and $c \in \mathbf{B}_b a$. Therefore, a and c are in the same component of Γ_b by Lemma 4.1(iv). $\square$

Lemma 4.7. *Let $(A,\mathbf{B})$ be a table algebra. Let $b \in \mathbf{B}$ such that $b \neq 1$. Then the following hold.*

(i) *If there is a proper closed subset $\mathbf{N}$ of $\mathbf{B}_b$ such that the principal component of the digraph $\Gamma_{b//\mathbf{N}}$ is cyclically partite, then so is the principal component of the digraph Γ_b.*
(ii) *Let $(A,\mathbf{B})$ be commutative. Then the principal component of the digraph Γ_b is cyclically partite if and only if there is a proper closed subset $\mathbf{N}$ of $\mathbf{B}_b$ such that $b//\mathbf{N}$ is a thin element of the quotient table algebra $(A//\mathbf{N}, \mathbf{B}//\mathbf{N})$.*

Proof. (i) From Lemma 4.5, the vertex set of the principal component of the digraph $\Gamma_{b//\mathbf{N}}$ is $\mathbf{B}_b//\mathbf{N}$. If the principal component of the digraph $\Gamma_{b//\mathbf{N}}$ is cyclically partite, then there is an integer $h > 1$ and a partition $\widetilde{V_1}, \widetilde{V_2}, \ldots, \widetilde{V_h}$ of the vertex set $\mathbf{B}_b//\mathbf{N}$ such that for any $a//\mathbf{N} \in \widetilde{V_i}$ and $c//\mathbf{N} \in \widetilde{V_j}$ with $(a//\mathbf{N}, c//\mathbf{N}) \in E(\Gamma_{b//\mathbf{N}})$, $j - i \equiv 1 \bmod h$. Let $V_i := \{a \in \mathbf{B}_b \mid a//\mathbf{N} \in \widetilde{V_i}\}$, $i = 1, 2, \ldots, h$. Then $V_1, V_2, \ldots, V_h$ form a partition of $\mathbf{B}_b$. Let $a \in V_i$ and $c \in V_j$ such that $(a,c) \in E(\Gamma_b)$. Then $a//\mathbf{N} \in \widetilde{V_i}$, $c//\mathbf{N} \in \widetilde{V_j}$, and $(a//\mathbf{N}, c//\mathbf{N}) \in E(\Gamma_{b//\mathbf{N}})$ by Lemma 4.6(i). Thus, $j - i \equiv 1 \bmod h$. This proves that the principal component of the digraph Γ_b is also cyclically h-partite.

(ii) If there is a proper closed subset $\mathbf{N}$ of $\mathbf{B}_b$ such that $b//\mathbf{N}$ is a thin element, then by Lemma 4.4, the principal component of the digraph $\Gamma_{b//\mathbf{N}}$ is a directed cycle of length greater than 1. Thus, the principal component of the digraph $\Gamma_{b//\mathbf{N}}$ is cyclically partite, and hence the principal component of the digraph Γ_b is also cyclically partite by (i). On the other hand, assume that the principal component of the digraph Γ_b is cyclically h-partite for some positive integer $h > 1$. Let $V_1, V_2, \ldots, V_h$ be a partition of $\mathbf{B}_b$ such that for any $a \in V_i$ and $c \in V_j$, if $(a,c) \in E(\Gamma_b)$, then $j - i \equiv 1 \bmod h$. Let $a_i \in V_i$, $1 \leqslant i \leqslant h$. Then for any $1 \leqslant i < j \leqslant h$, since $a_i, a_j \in \mathbf{B}_b$ and $a_i \neq a_j$, there exists a directed path from a_i to a_j, and the length of any directed path from a_i to a_j is congruent to $j - i$ modulo h. Let

$$\mathbf{N} := \bigcup_{r=1}^{\infty}\mathrm{Supp}_{\mathbf{B}}\big(b^{hr}\big).$$

Then $\mathbf{N}$ is a closed subset of $\mathbf{B}$. For any $1 \leqslant i \leqslant h$, and any $c \in \mathbf{N}a_i$, it follows from $c \in \mathrm{Supp}_{\mathbf{B}}(b^{hr}a_i)$, for some positive integer r, that there is a directed path from a_i to c of length congruent to 0 modulo h by Lemma 4.1(i). Thus, $a_j \notin \mathbf{N}a_i$, for any $1 \leqslant i < j \leqslant h$, and hence $\mathbf{N}a_1, \mathbf{N}a_2, \dots, \mathbf{N}a_h$ are all distinct. So $\mathbf{N}$ is a proper closed subset of $\mathbf{B}_b$ (because $h > 1$). Furthermore, for any $c \in V_i$ such that $c \neq a_i$, the length of any directed path from a_i to c is a multiple of h. Thus, $c \in \mathbf{N}a_i$ by Lemma 4.1(i), and hence $V_i \subseteq \mathbf{N}a_i$, $1 \leqslant i \leqslant h$. But $\{V_1, V_2, \dots, V_h\}$ is a partition of $\mathbf{B}_b$. Therefore, we must have that $V_i = \mathbf{N}a_i$, $1 \leqslant i \leqslant h$. Hence, $\mathbf{B}_b = \bigcup_{i=1}^{h} \mathbf{N}a_i$, and it follows from $(A, \mathbf{B})$ commutative that

$$\mathbf{B}_b /\!/ \mathbf{N} = \{a_1 /\!/ \mathbf{N}, a_2 /\!/ \mathbf{N}, \dots, a_h /\!/ \mathbf{N}\}.$$

Note that $\mathrm{Supp}_{\mathbf{B}}(ba_i) \subseteq V_{i+1}$, $1 \leqslant i \leqslant h-1$, and $\mathrm{Supp}_{\mathbf{B}}(ba_h) \subseteq V_1$. So there exist $c_i \in V_i$, $1 \leqslant i \leqslant h$, such that $(a_i, c_{i+1}) \in E(\Gamma_b)$, $1 \leqslant i \leqslant h-1$, and $(a_h, c_1) \in E(\Gamma_b)$. Hence by Lemma 4.6(i), $(a_i /\!/ \mathbf{N}, c_{i+1} /\!/ \mathbf{N}) \in E(\Gamma_{b/\!/\mathbf{N}})$, $1 \leqslant i \leqslant h-1$, and $(a_h /\!/ \mathbf{N}, c_1 /\!/ \mathbf{N}) \in E(\Gamma_{b/\!/\mathbf{N}})$. But $c_i /\!/ \mathbf{N} = a_i /\!/ \mathbf{N}$, $1 \leqslant i \leqslant h$, and the vertex set of the principal component of the digraph $\Gamma_{b/\!/\mathbf{N}}$ is $\mathbf{B}_b /\!/ \mathbf{N}$ by Lemma 4.5. Thus, the principal component of the digraph $\Gamma_{b/\!/\mathbf{N}}$ is a directed cycle. Hence, $b /\!/ \mathbf{N}$ is thin by Lemma 4.4, and (ii) holds. $\square$

As a direct consequence of Lemma 4.7(ii), we have the following

Corollary 4.8. *Let $(A, \mathbf{B})$ be a commutative table algebra. Let $b \in \mathbf{B}$ such that $b \neq 1$. Then the principal component of the digraph Γ_b is cyclically partite if and only if $O^{\vartheta}(\mathbf{B}_b) \neq \mathbf{B}_b$.*

Now Theorem 1.13 follows directly from Theorem 1.1 and Corollary 4.8.

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