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Dual solutions of slip flow past a nonlinearly shrinking permeable sheet



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Abstract The aim of the paper is to investigate the flow of an incompressible viscous fluid past a nonlinearly shrinking permeable sheet. Partial slip condition is considered instead of no slip condition at the boundary. The self similar equations are obtained and then solved numerically by a shooting technique. Dual solutions are obtained for the flow past a nonlinearly shrinking sheet with slip condition in the presence of suction. It is found that for the first solution the momentum boundary layer thickness decreases with slip and suction parameters; but it increases with the power-law index of the shrinking velocity. The dual solutions for the velocity field are obtained for the positive values of power-law index n and for certain values of the other parameters in the study. Velocity slip controls the boundary layer separation. However, the power-law index acts to accelerate the boundary layer separation.

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1. Introduction

Viscous boundary layer flow due to a shrinking sheet is very important for its applications in various types of manufacturing processes in metallurgy and polymer industry which involve packaging process, for example, shrink wrapping. Recently, the flow induced by a shrinking sheet has drawn attention of several researchers not only for its increasing technological applications but also for its interesting physical characters, like rising shrinking balloon. Flow over a shrinking

sheet is quite different from the flow over a stretching sheet as the flow over a shrinking sheet would give rise to a velocity away from the sheet and vorticity generated at the shrinking sheet is not confined within the boundary layer. As a result, a steady flow is not possible unless adequate suction is applied at the surface. Wang [1] first investigated the flow due to shrinking sheet. Miklavcic and Wang [2] established the criterion for the existence and uniqueness of the similarity solution of the governing equation due to flow past a shrinking sheet. Also, they observed that the flow depends on externally imposed mass suction. Hayat et al. [3] obtained an analytical solution of magnetohydrodynamic (MHD) flow of a second grade fluid over a shrinking sheet. Later on, using homotopy analysis method (HAM), Hayat et al. [4] obtained an analytical solution for MHD rotating flow of a second grade fluid past a porous shrinking sheet. In another paper, Hayat et al. [5] discussed the mass transfer in case of a steady two

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dimensional MHD boundary layer flow of an upper-convected Maxwell fluid past a porous shrinking sheet in the presence of chemical reaction. Using HAM, Hayat et al. [6] obtained a series of solution of three-dimensional MHD and rotating flow over a porous shrinking sheet. Fang et al. [7] also investigated unsteady viscous flow over a shrinking surface with mass suction. Nadeem et al. [8] obtained the series solution for stagnation point flow of a second grade fluid over a shrinking sheet. Noor et al. [9] established a series solution for MHD viscous flow due to a shrinking sheet using Adomian decomposition method (ADM). Fang et al. [11] solved analytically the viscous flow over a porous shrinking sheet using second order slip flow. Also, Fang and Zhang [10] found a closed form exact solution for the thermal boundary layers over a shrinking sheet subjected to wall mass transfer. Nadeem and Faraz [14] investigated the thin film flow of a second grade fluid over a stretching/shrinking surface with variable fluid properties.

Over the last few decades many investigations were carried out for flow past a stretching/shrinking surface by considering linear stretching/shrinking velocity of the flat sheet [12,13]. However, the boundary layer flow may occur due to a nonlinear stretching/shrinking sheet. Akyildiz et al. [15] investigated the flow due to a nonlinear stretching sheet by considering the velocity $u = cx^n$ at $y = 0$, which was used for positive odd integer values of n . But such profile would fail for even integer values of n , since the flow at $y = 0$ would be in the wrong direction for $-\infty < x < 0$ (see [16]). Nadeem et al. [17] analysed the heat transfer characteristics for water based nanofluid flow over an exponentially stretching surface. Nandy et al. [18] investigated the combined effects of magnetic field and thermal radiation on unsteady flow and heat transfer of nanofluid over a porous shrinking sheet. They considered the problem using boundary layer approximations. Flow induced by a nonlinear shrinking sheet is not studied very much though it is very important and realistic, and appears frequently in many engineering processes. Using the homotopy analysis method, Nadeem and Hussain [19] solved analytically the problem of magnetohydrodynamic (MHD) flow of a viscous fluid over a nonlinear porous shrinking sheet. Ali et al. [20] extended the problem considered by Nadeem and Hussain [19] by considering the magnetic effects and reported the existence of dual solutions. Prasad et al. [21] investigated the problem of flow and heat transfer past a nonlinear porous shrinking sheet and obtained numerical solutions for asymptotically large shrinking rates. But they reported only the single solution which is one of the branches of the dual solutions.

Generally the first branch of the dual solutions is physically stable which can be verified by a stability analysis. Keeping this in mind, boundary layer flow past a shrinking sheet with a more general nonlinear power-law shrinking velocity has been considered. Dual solutions are obtained and analysed in detail. Of late, Bhattacharyya et al. [22] analysed the Soret and Dufour effects on stagnation point flow past a shrinking sheet and double crossing over is found in dual dimensionless temperature profiles for increasing Soret number and in dual dimensionless concentration profiles for the increase in Dufour number.

In all the studies mentioned above no-slip boundary condition was assumed. But fluids such as emulsions, suspensions, foams, and polymer solutions exhibit slip at the boundary and have important applications such as in the polishing of artificial heart valves and internal cavities. Many researchers

such as Andersson [23], Bhattacharyya et al. [24], and Mukhopadhyay [25] investigated boundary layer flow past a stretching/shrinking sheet with slip at the boundary. Moreover, due to the micro-scale dimensions of micro-electro-mechanical systems (MEMS), fluid flow does not obey the traditional no-slip flow. Navier [26] proposed a slip condition which is linearly proportional to the shear stress. Recently, Uddin et al. [27] analysed the magnetic effect on stretching/shrinking nonlinear nanomaterial sheet in the presence of Navier slip and convective heating. Devakar et al. [28] obtained the analytic solution of couple stress fluid. They also considered the slip boundary condition. In this paper the slip model proposed by Navier [26] has been used.

In this paper, the boundary layer slip flow over a nonlinear shrinking sheet with a more general shrinking velocity has been investigated. Using similarity transformation, the governing equations are transformed to a self similar ordinary differential equation and then the equation is solved numerically by a shooting method. Dual solutions of the flow problem for the governing parameters are obtained and the flow characteristics are discussed through graphs.

2. Flow analysis

Let us consider the flow of an incompressible viscous fluid past a flat shrinking sheet coinciding with the plane $y = 0$. The shrinking velocity is $U_w(x)$. The flow is confined to $y > 0$. The x -axis runs along the shrinking surface in the direction opposite to the sheet motion, and the y -axis is perpendicular to it (see, Fig. 1). The governing boundary layer equations for the steady two-dimensional flow for the problem are as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \quad (2)$$

where u and v are the velocity components of the fluid, and ν denotes the kinematic viscosity.

The appropriate boundary conditions are [16,25]

$$\left. \begin{aligned} u &= -c \operatorname{sgn}(x) |x|^n + N \nu \frac{\partial u}{\partial y}, -\infty < x < \infty, v = -v_w(x) \text{ at } y = 0, \\ u &\rightarrow 0 \text{ as } y \rightarrow \infty \end{aligned} \right\} \quad (3)$$

where c (> 0) is a constant, n (> 0) is a nonlinear shrinking parameter, $N = N_1 |x|^{-(\frac{n-1}{2})}$ is the velocity slip factor which changes with x , and N_1 is the positive slip constant. The no-slip case is recovered for $N = 0$. $v_w(x) > 0$ is the suction velocity and $v_w(x) < 0$ is the velocity of blowing. Here $v_w(x) = v_0 x^{(n-1)/2}$, v_0 is a constant.

It is to be noted that if $n > 1$ then N become singular. As the boundary layer does not start at $x = 0$, but it starts in the vicinity of $x = 0$, the solution for $n > 1$ is possible.

Now we define the similarity variables as

$$\eta = y \sqrt{c(n+1)/2\nu} |x|^{(n-1)/2}, \quad (4)$$

$$u = c \operatorname{sgn}(x) |x|^n f'(\eta), \quad (5)$$

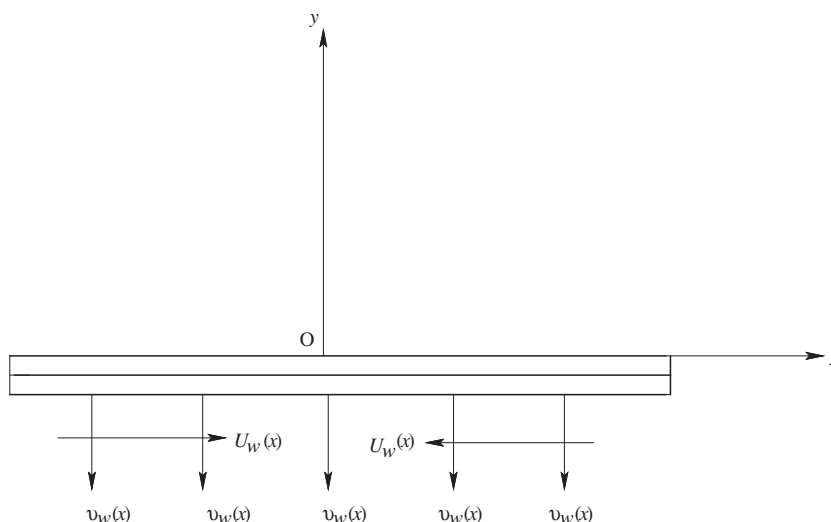


Figure 1 Sketch of the physical flow problem.

$$v = -\text{sgn}(x)\sqrt{c(n+1)v/2}|x|^{(n-1)/2}\left[f(\eta) + \frac{(n-1)}{(n+1)}\eta f'(\eta)\right]. \quad (6)$$

Upon the substitution of (4)–(6) into (1)–(3), the governing equations and the boundary conditions reduce to

$$f'''(\eta) + f(\eta)f''(\eta) - \left(\frac{2n}{n+1}\right)[f'(\eta)]^2 = 0, \quad (7)$$

$$f(0) = S, \quad f'(0) = -1 + Bf''(0), \quad f'(\eta) \rightarrow 0 \text{ as } \eta \rightarrow \infty, \quad (8)$$

where the prime denotes differentiation with respect to η , $S = v_0\sqrt{\frac{2}{c(n+1)v}}$ is the mass transfer parameter, $S > 0$ corresponds to mass suction and $S < 0$ corresponds to the mass injection. $B = N_1\sqrt{\frac{cv(n+1)}{2}}$ is the slip parameter. It is to be noted that here we shall consider only the case of suction ($S > 0$). Because injection will not permit the existence of solutions to the self-similar problem, it destroys the similarity flow over a shrinking sheet [1,7].

3. Results and discussion

The self-similar Eq. (7) along with the boundary conditions (8) is solved numerically by shooting method. In this method an initial guess for the solution is needed which must satisfy the boundary conditions of the problem. The problem may have dual solutions and determining the initial guess for the first solution is easier than the second solution. Numerical solutions for several sets of the governing parameters are obtained and presented through Figs. 2–6. To assess the accuracy of the method, a comparison corresponding to the values of $[f''(0)]$ for linear shrinking sheet in case of no-slip boundary condition is made with the results of Prasad et al. [21] and is presented in Table 1. From this table, it is clear that our results agree with their results up to second decimal place.

Fig. 2 depicts the nature of velocity profiles for different values of power-law index n of the shrinking velocity of the sheet. Dual velocity profiles exist. Fluid velocity decreases with increasing values of n in case of first branch while it increases

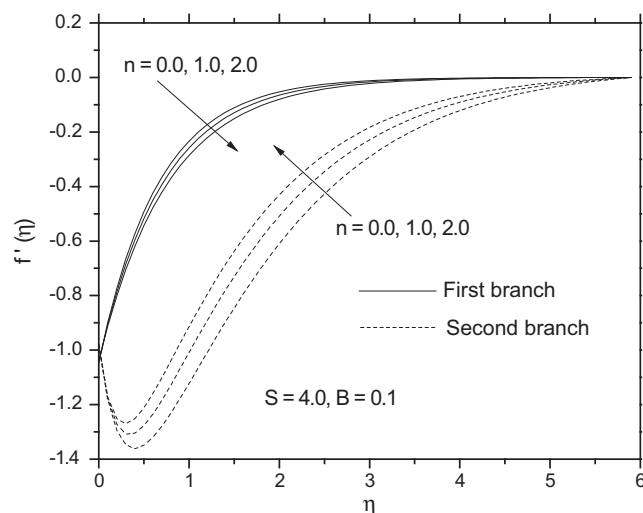


Figure 2 Velocity profiles for different values of power-law index n of the shrinking velocity.

in the second branch of the solution. That is, the boundary layer thickness increases with n in case of the first branch of the solution.

Fig. 3(a) exhibits the effects of mass transfer parameter S on the velocity field for linear shrinking sheet case while Fig. 3(b) exhibits similar results for nonlinear shrinking sheet case. In both cases the dual nature of velocity profiles is noted. From these figures, we notice that the velocity increases with increasing values of the mass transfer parameter S for first branch solution while fluid velocity decreases with increasing mass transfer parameter S in case of second branch solution. Boundary layer thickness decreases with increasing suction in case of first branch of solution. For larger value of S significant change in velocity profile is noted in case of second branch solution. One can notice that an increase in the value of the power-law index n helps to smoothen the velocity profile. This can be seen from the second branch solutions of Fig. 3(b) compared to that of Fig. 3(a). This conforms to the findings of Prasad et al. [21].

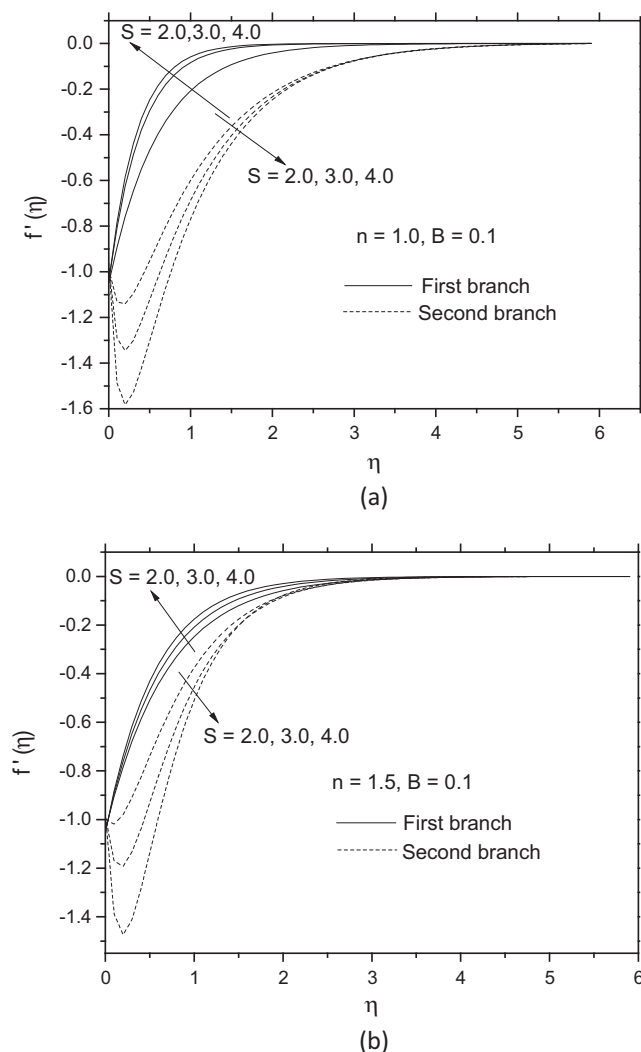


Figure 3 Velocity profiles for different values of mass transfer parameter S for (a) linear (b) nonlinear shrinking sheet.

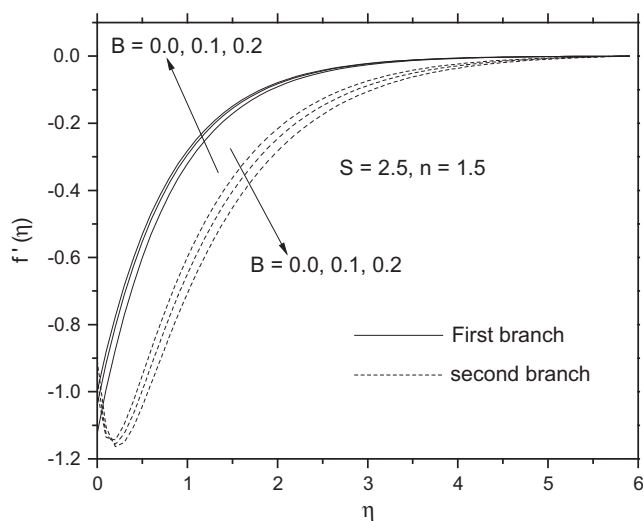


Figure 4 Velocity profiles for different values of slip parameter B .

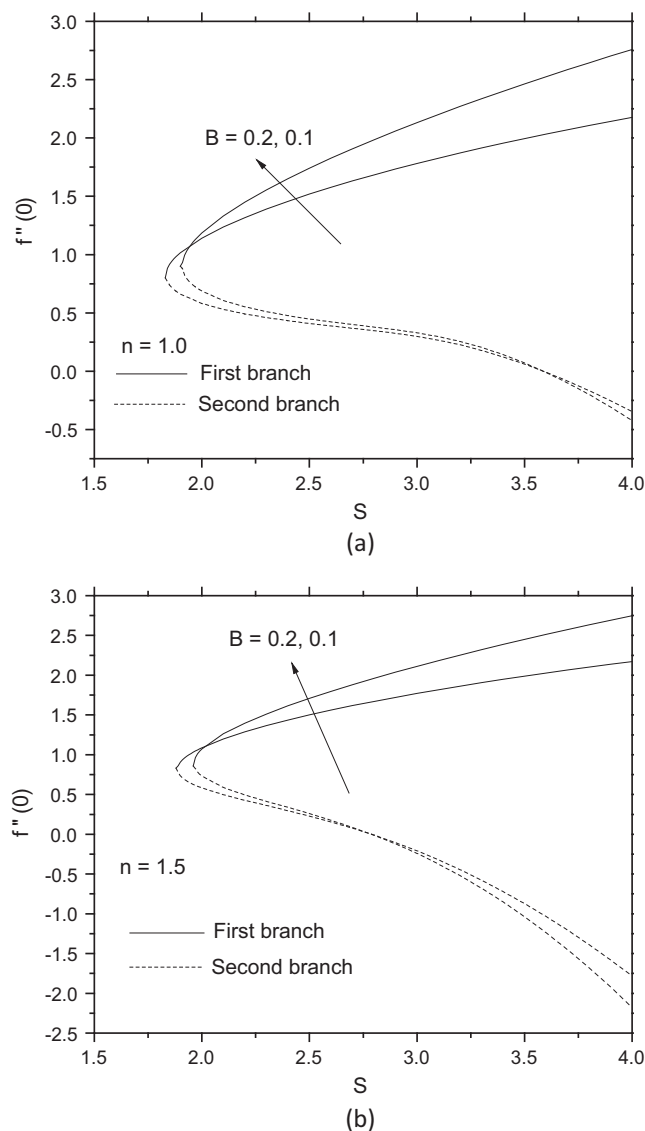


Figure 5 Variation of skin-friction coefficient with mass transfer parameter S for different values of slip parameter B for (a) linear (b) nonlinear shrinking sheet.

The effects of slip on velocity profiles can be found from Fig. 4. Fluid velocity increases with increasing slip parameter B in the first branch solution but the opposite behaviour is noted in the case of second branch solution. That is, the boundary layer thickness is less in the case of first branch solution compared to that in the second branch solution.

Fig. 5 displays the behaviour of the skin friction coefficient $f''(0)$ with mass transfer parameter S for different values of slip parameter B for linear [Fig. 5(a)] and nonlinear [Fig. 5(b)] shrinking sheet. The skin friction coefficient $f''(0)$ increases with increasing values of the suction parameter. The boundary layer separates from the surface at a critical value (S_c) beyond which boundary layer approximation is not valid. For increasing slip at the boundary i.e. at the sheet, the generation of vorticity due to shrinking velocity is slightly reduced and hence, with the imposed suction, that vorticity remains confined to the boundary layer region for larger shrinking velocity. That is to say that due to increasing velocity slip, steady solution

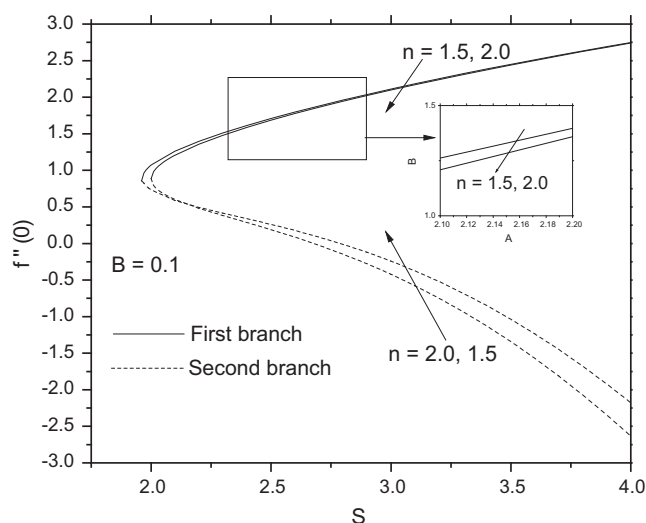


Figure 6 Variation of skin-friction coefficient with mass transfer parameter S for different values of power index n of the shrinking velocity.

Table 1 Values of Skin-friction $f''(0)$ for different values of mass transfer parameter S in case of linear shrinking sheet in the absence of slip.

	$S = 4.0$	$S = 4.5$	$S = 5.0$
Prasad et al. [21] with $m = 1$	-1.038378	-2.228009	-3.798063
Present study with $n = 1$, $B = 0$	-1.037634	-2.226580	-3.796187

is possible even for larger shrinking velocity. It is noted that the value of S_c decreases with slip parameter B for both linear and nonlinear shrinking sheet cases [Fig. 5]. That is, the velocity slip parameter controls the boundary layer separation. From Fig. 6, it is found that $f''(0)$ decreases with increasing n . The critical value of the suction parameter S_c decreases as n increases; that is, the power-law index n accelerates the boundary layer separation.

4. Concluding remarks

Steady boundary layer flow past a nonlinear shrinking permeable sheet is investigated. A more general power-law shrinking velocity along with the slip at the boundary is considered in this study. Dual solutions are obtained and the effects of the pertaining parameters are analysed with the help of the graphical representations. The range of shrinking velocity parameter for which the solution exists increases as the velocity slip parameter increases. Velocity slip parameter acts to control the boundary layer flow separation whereas the power-law index n of shrinking velocity acts to accelerate the boundary layer separation.

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