# Edge-oblique polyhedral graphs 

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#### Abstract

Let $G=G(V, E, F)$ be a polyhedral graph with vertex set $V$, edge set $E$ and face set $F$. $e=(x, y ; \alpha, \beta) \in E(G)$ denotes an edge incident with the two vertices $x, y \in V(G), d(x) \leqslant d(y)$ and incident with the two faces $\alpha, \beta \in F(G), d(\alpha) \leqslant d(\beta)$. $K K=d(x), L=d(y) ; M=d(\alpha), N=d(\beta)]$ is the type of $e=(x, y ; \alpha, \beta)$. A graph which contains no two edges of a common edge-type is called edge-oblique and if it contains at most $z$ edges of each type it is called $z$-edge-oblique. In this work we shall prove, that there is only a finite number of edge-oblique and $z$-edge-oblique graphs. For the first case some bounds for the maximum degree and the number of edges are given.


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## 1. Introduction

By $\mathfrak{P}$ we denote the set of polyhedral graphs, and let $G=G(V, E, F) \in \mathfrak{P}$. The degree $d(x)$ of a vertex $x \in V(G)$ is the number of edges incident with $x$. The degree $d(\alpha)$ of a face $\alpha \in F(G)$ is the number of edges incident with $\alpha . e=(x, y ; \alpha, \beta) \in E(G)$ denotes an edge incident with the two vertices $x, y \in V(G), d(x) \leqslant d(y)$ and incident with the two faces $\alpha, \beta \in F(G), d(\alpha) \leqslant d(\beta)$. $[K=d(x), L=d(y) ; M=d(\alpha), N=d(\beta)]$ is the type of $e=(x, y ; \alpha, \beta)$. A face $\alpha \in F(G)$ is an $\left\langle a_{1}, a_{2}, \ldots, a_{l}\right\rangle$-face if $\alpha$ is an $l$-gon and the degrees of the vertices $x_{1}, x_{2}, \ldots, x_{l}$ incident with $\alpha$ in the cyclic order are $a_{1}, a_{2}, \ldots, a_{l}$. The lexicographic minimum $\left\langle b_{1}, b_{2}, \ldots, b_{l}\right\rangle: \alpha$ is a $\left\langle b_{1}, b_{2}, \ldots, b_{l}\right\rangle$-face is called the type of $\alpha$.
$\Delta(G):=\max \{d(a): a \in V \cup F\}$ is the maximum degree of $G$.


Fig. 1. Edge-oblique graph.


Fig. 2. Selfdual edge-oblique graph.

Because $G$ is a polyhedral graph there is no edge of type $\langle 3,3 ; 3,3\rangle$ in $G$ except if $G$ is the tetrahedron.

If $e \in E(G)$ is incident with an element $a \in V \cup F$ we write $a \sim e$ or $e \sim a$.
At the 9. High Tatras Conference on Cycles and Colourings in 2000 P.J. Owens asked the following question:

Let $k, l$ be two integers with $1 \leqslant l \leqslant k$. Does there exist a polyhedral graph $G$ with $k=|E(G)|$ edges and $l$ different types of edges?

The cases $l \in\{1,2\}$ are solved by Jendrol' and Tkáč in [2-4]. In this paper we are interested in the case $k=l$.
$G$ is called edge-oblique if for any type of edges there is at most one edge in $E(G)$ having this type.
$G$ is called face-oblique if for any type of faces there is at most one face in $F(G)$ having this type.

It has been proved that:
(1) The set of face-oblique graphs is not empty but finite $[5,6]$.
(2) The set of so called $z$-face-oblique graphs, which may contain up to $z$ faces of each type, is finite too [5].
(3) If $G$ is a face-oblique graph, then it contains a vertex of degree 3 [1].
(4) There are face-oblique graphs having only triangles as faces [6], but there is no face-oblique graph having vertices of degree 3 only.
(5) The set of self-dual face-oblique graphs is not empty [1].
(6) There are face-oblique graphs $G$ having no common face type with their faceoblique duals $G^{*}$ (so-called super-face-oblique graphs) [1].

Obviously, if $G$ is edge-oblique, its dual $G^{*}$ is edge-oblique, too.
Fig. 1 shows an edge-oblique graph $G_{1}$.
Fig. 2 shows an edge-oblique self-dual graph $G_{2}$.

## 2. Results and proofs

## Theorem.

(1) The set of edge-oblique graphs is finite. If $G$ is an edge-oblique graph then $\Delta(G) \leqslant 30$ and $|E(G)|<880$.
(2) The set of so called z-edge-oblique graphs, which may contain up to $z$ edges of each type is finite, too.
(3) If $G$ is an edge-oblique graph then $G$ contains a vertex of degree 3 as well as a triangle.
(4) There is neither an edge-oblique graph having only vertices of degree 3 nor one having only triangles as faces.
(5) There are self-dual edge-oblique graphs.
(6) There is no edge oblique graph $G$ such that $G$ and $G^{*}$ have no common edge-type (no super-edge-oblique graph).
(7) There is an edge oblique graph which is face-oblique, too.

Let $G=G(V, E, F) \in \mathfrak{P}$.
If $e=(x, y ; \alpha, \beta) \in E(G)$ is of type $[K, L ; M, N]$ we define charges:
$w(e):=w(K, L ; M, N):=1-\left(\frac{1}{K}+\frac{1}{L}+\frac{1}{M}+\frac{1}{N}\right)$.
Summing up the charges of all edges and using Euler's formula one gets

$$
\begin{aligned}
\sum_{e \in E(G)} w(e) & =|E(G)|-\sum_{x \in V(G)} d(x) \cdot \frac{1}{d(x)}-\sum_{\alpha \in F(G)} d(\alpha) \cdot \frac{1}{d(\alpha)} \\
& =|E(G)|-|V(G)|-|F(G)| \\
& =-2
\end{aligned}
$$

for any $G \in \mathfrak{P}$.
Proof of (1). In the following let $G=G(V, E, F)$ be an edge-oblique graph.
Let $E_{k}^{-}:=E_{k}^{-}(G):=\{e \in E(G): w(e) \leqslant 0 \wedge k=\max \{K, L, M, N\}\}$ be the set of edges with nonpositive charges and maximum degree $k$.

Let $E^{-}:=E^{-}(G):=\bigcup_{k=4}^{\Delta(G)} E_{k}^{-}$be the set of all edges with nonpositive charges.
$w\left(E^{-}\right):=\sum_{e \in E^{-}} w(e)=\sum_{k=4}^{\Delta(G)} \sum_{e \in E_{k}^{-}} w(e)$ sums up the charges of all these edges.

Apart from the two infinite sets $\{[3,3 ; 3, N]: N=13,14, \ldots\}$ with $w(3,3 ; 3, N)=-1 / N$ and $\{[3, L ; 3,3]: L=13,14, \ldots\}$ with $w(3, L ; 3,3)=-1 / L$ there are exactly 78 types of edges with nonpositive charges:

$$
\begin{array}{ll}
w(3,3 ; 3, N)_{N=4,5, \ldots, 12}=-\frac{1}{N}, & w(3, L ; 3,3)_{L=4,5, \ldots, 12}=-\frac{1}{L} \\
w(3,3 ; 4, N)_{N=4,5, \ldots, 12}=\frac{1}{12}-\frac{1}{N}, & w(4, L ; 3,3)_{L=4,5, \ldots, 12}=\frac{1}{12}-\frac{1}{L} \\
w(3,3 ; 5, N)_{N=5,6,7}=\frac{2}{15}-\frac{1}{N}, & w(5, L ; 3,3)_{L=5,6,7}=\frac{2}{15}-\frac{1}{L} \\
w(3,3 ; 6,6)=0, & w(6,6 ; 3,3)=0 \\
w(3,4 ; 3, N)_{N=4,5, \ldots 12}=\frac{1}{12}-\frac{1}{N}, & w(3, L ; 3,4)_{L=5,6, \ldots 12}=\frac{1}{12}-\frac{1}{L} \\
w(3,4 ; 4, N)_{N=4,5,6}=\frac{1}{6}-\frac{1}{N}, & w(4, L ; 3,4)_{L=4,5,6}=\frac{1}{6}-\frac{1}{L} \\
w(3,5 ; 3, N)_{N=5,6,7}=\frac{2}{15}-\frac{1}{N}, & w(3, L ; 3,5)_{L=6,7}=\frac{2}{15}-\frac{1}{L} \\
w(4,4 ; 3, N)_{N=5,6}=\frac{1}{6}-\frac{1}{N}, & w(3, L ; 4,4)_{L=5,6}=\frac{1}{6}-\frac{1}{L} \\
w(3,6 ; 3,6)=0, & w(4,4 ; 4,4)=0
\end{array}
$$

It is easy to calculate, that the sum of the charges of these 78 edges is

$$
\sum_{k=4}^{12} \sum_{e \in E_{k}^{-}} w(e)=-\frac{7891}{1540}=-5.1240 \ldots
$$

First we prove that $\Delta(G)=\max \{d(a): a \in V(G) \cup F(G)\} \leqslant 30$.

Proof. If there is an $a \in V(G) \cup F(G)$ with $d(a)=\Delta(G) \geqslant 31$ we first estimate the sum $w(a)$ of charges of all edges incident with $a$. For this purpose we may assume without loss of generality that $a$ is a vertex.

$$
w(a)=\sum_{e: a \sim e}\left(1-\frac{1}{K}-\frac{1}{\Delta}-\frac{1}{M}-\frac{1}{N}\right)=\sum_{e: a \sim e}\left(1-\frac{1}{K}-\frac{1}{M}-\frac{1}{N}\right)-1
$$

Inserting for $K, M, N$ the 31 degree combinations which yield the lowest possible charge and estimating all further addends by the next lowest possible value for $1-1 / K-1 / M$
$-1 / N$ one gets the following inequality:

$$
w(a)=\sum_{e: a \sim e} w(e) \geqslant \frac{470117}{90090}+(\Delta-31) \frac{11}{42} .
$$

Secondly, we estimate the sum of charges of all imaginable edges with nonpositive charge:

$$
\begin{aligned}
& w\left(E^{-}\right) \geqslant \sum_{k=4}^{12} \sum_{e \in E_{k}^{-}} w(e)+\sum_{k=13}^{\Delta} \sum_{e \in E_{k}^{-}} w(e) \geqslant-\frac{7891}{1540}-2 \sum_{k=13}^{\Delta} \frac{1}{k} \\
& \Rightarrow w(G)=\sum_{e \in E(G)} w(e) \geqslant w(a)+w\left(E^{-}\right) \\
& \quad \geqslant \frac{470117}{90090}+(\Delta-31) \frac{11}{42}-d \frac{7891}{1540}-2 \sum_{k=13}^{\Delta} \frac{1}{k} \\
& \quad \geqslant \frac{470117}{90090}-\frac{7891}{1540}+(\Delta-31) \frac{11}{42}-2 \sum_{k=13}^{31} \frac{1}{k}-2 \sum_{k=32}^{\Delta} \frac{1}{k} \\
& \quad \geqslant \frac{16987}{180180}+(\Delta-31)\left(\frac{11}{42}-\frac{2}{32}\right)-2(\ln (31)-\ln (12)) \\
& \quad \geqslant-\frac{4387217}{720720}+\left(\frac{67}{336} \Delta-2 \ln \left(\frac{31}{12}\right)\right) .
\end{aligned}
$$

This bound is a function of $\Delta$ which is monotonic increasing, and its value for $\Delta=31$ is greater than -2 which is a contradiction.

The set $\{[K, L ; M, N]: \max \{K, L, M, N\} \leqslant \Delta \leqslant 30\}$ of edge-types is finite. Because an edge-type occurs at most once in an edge-oblique graph, we have proved, that the set of edge-oblique graphs is finite.

Now let us prove: $|E(G)|<880$.
Because $\Delta(G) \leqslant 30$ there are at most 114 edges with a nonpositive charge namely the 78 edges listed above and the 36 edges of type $[3,3 ; 3, N], N=13,14, \ldots, 30$ and $[3, L ; 3,3], L=13,14, \ldots, 30$ with a charge

$$
w\left(E^{-}\right) \geqslant-\frac{7891}{1540}-2\left(\frac{1}{13}+\frac{1}{14}+\cdots+\frac{1}{30}\right)=-6.9075 \ldots
$$

A trivial estimation for the cardinality of $E(G)$ is the following:
The minimum value $\min \left\{w(e): e \in E(G) \backslash E^{-}(G)\right\}$ of the charge of an edge with positive charge is $w(e)=w([3,3 ; 4,13])=\frac{1}{156}$. If we choose $l$ in such a way, that $l \cdot \frac{1}{156}+w\left(E^{-}\right)>-2$, that means $l \geqslant 766$ it results:

If $|E(G)| \geqslant 766+114=880$ then the graph is not edge-oblique.

Remark. Refining our arguments by taking into consideration not only one element $a \in V \cup F$ with $d(a)=\Delta$ but two further elements $b, c$ with

$$
d(a) \geqslant d(b) \geqslant d(c) \geqslant d(t), \forall t \in(V \cup F) \backslash\{a, b\} ;
$$

it is possible to show, that $\Delta \leqslant 18$ and $|E(G)| \leqslant 138$. Since those bounds are also not sharp and the proofs are mainly done by computers and a computation by hand is rather long, we omit the longer proofs.

Proof of (2). We prove the finiteness of the family $\mathfrak{P}(z)$ of $z$-edge-oblique graphs by applying the discharging method:

The original charges $w(t), t \in V \cup E \cup F$ are

$$
w(t)= \begin{cases}1-\left(\frac{1}{K}+\frac{1}{L}+\frac{1}{M}+\frac{1}{N}\right) & \text { if } t=e=(x, y ; \alpha, \beta) \in E(G) \\ 0 & \text { is of type }[K, L ; M, N], \\ & \text { if } t \in V \cup F .\end{cases}
$$

The new charges are denoted by $w^{*}(t) . w^{*}(e)=w(e)$ if $e \in E(G)$ is of type $[K, L ; M, N]$ with $\max \{K, L, M, N\} \leqslant 2 z+13$. If $\max \{K, L, M, N\} \geqslant 2 z+14$ we shift charges from $e$ to
$x$ if $K \geqslant 2 z+14$,
$y$ if $L \geqslant 2 z+14$,
$\alpha$ if $M \geqslant 2 z+14$,
$\beta$ if $N \geqslant 2 z+14$
in accordance with the following rules:
Rule no. Type of $e=(x, y ; \alpha, \beta) \quad$ Discharging rule
1
$[3, L ; 3,3], L \geqslant 2 z+14$
Shift $-\frac{1}{13}-\frac{1}{L-1}$ from $e$ to $y$
2
$[3,3 ; 3, N], N \geqslant 2 z+14$
Shift $-\frac{1}{13}-\frac{1}{N-1}$ from $e$ to $\beta$

If rule 1 and 2 are not applicable
$[K, L ; M, N], K \geqslant 2 z+14$
(in this case $L \geqslant 2 z+14$, too)
4
$[K, L ; M, N], L \geqslant 2 z+14$
Shift $\frac{1}{13}-\frac{1}{K-1}$ from $e$ to $x$
Shift $\frac{1}{13}-\frac{1}{L-1}$ from $e$ to $y$
5

6
$[K, L ; M, N], M \geqslant 2 z+14$
(in this case $N \geqslant 2 z+14$, too)

$$
\text { Shift } \frac{1}{13}-\frac{1}{M-1} \text { from } e \text { to } \alpha
$$

$$
[K, L ; M, N], N \geqslant 2 z+14
$$

Shift $\frac{1}{13}-\frac{1}{N-1}$ from $e$ to $\beta$

In some cases more than one of rules $3-6$ must be used for one edge. After the discharging process, the charges $w(e)$ have changed into new charges $w^{*}(e)$. Furthermore, the vertices and faces have got charges $w^{*}(x)$ and $w^{*}(\alpha)$, too, which may be different from zero now.

Lemma. All vertices and faces have nonnegative charges.
Proof. (1) $w^{*}(x) \geqslant 0, \forall x \in V(G)$ : only if $x$ is a vertex of degree $d(x) \geqslant 2 z+14$ it has a charge different from 0 . In this case, $x$ may get at most $z$ times a charge of $-\frac{1}{13}-1 /(d(x)-1)$ as $x$ is incident with at most $z$ edges of type $[3, d(x) ; 3,3]$ and at least $d(x)-z$ times a charge of $\frac{1}{13}-1 /(d(x)-1)$.

$$
\begin{aligned}
& w^{*}(x) \geqslant z\left(-\frac{1}{13}-\frac{1}{d(x)-1}\right)+(d(x)-z)\left(\frac{1}{13}-\frac{1}{d(x)-1}\right), \\
& w^{*}(x) \geqslant \frac{d^{2}(x)-(14+2 z) d(x)+2 z}{13(d(x)-1)}, \\
& w^{*}(x)>0 \text { since } d(x) \geqslant 2 z+14 .
\end{aligned}
$$

(2) $w^{*}(\alpha) \geqslant 0, \forall \alpha \in F(G)$ : can be proved analogously.

$$
-2=\sum_{e \in E} w^{*}(e)+\sum_{x \in V} w^{*}(x)+\sum_{\alpha \in F} w^{*}(\alpha) .
$$

So the following inequality holds:

$$
\begin{aligned}
& -2 \geqslant \sum_{e \in E} w^{*}(e), \\
& -2 \geqslant \sum_{e \in E^{-*}} w^{*}(e)+\sum_{e \in E^{+*}} w^{*}(e),
\end{aligned}
$$

where

$$
\begin{aligned}
& E^{-*}:=\left\{e \in E, w^{*}(e) \leqslant 0\right\}, \\
& E^{+*}:=\left\{e \in E, w^{*}(e)>0\right\} .
\end{aligned}
$$

All edges without incident elements of degree $\geqslant 2 z+14$ keep their original charges, which are, as we have already seen either $\leqslant 0$ or $\geqslant \frac{1}{156}$. First we will prove, that all edges with incident elements of degree $\geqslant 2 z+14$ have charges $\geqslant \frac{1}{156}$.
(1) Exactly one of the values $K, L, M, N$ is $\geqslant 2 z+14$

| Applied rule | $w(e)$ | $w^{*}(e)$ |
| :--- | :--- | :--- |
| 1 | $-\frac{1}{L}$ | $-\frac{1}{L}+\frac{1}{13}+\frac{1}{L-1} \geqslant \frac{1}{13}>\frac{1}{156}$ |
| 2 | $-\frac{1}{N}$ | $-\frac{1}{N}+\frac{1}{13}+\frac{1}{N-1} \geqslant \frac{1}{13}>\frac{1}{156}$ |
| 4 | $\geqslant \frac{1}{12}-\frac{1}{L}$ | $\geqslant \frac{1}{12}-\frac{1}{L}-\frac{1}{13}+\frac{1}{L-1}>\frac{1}{156}$ |
| 6 | $\geqslant \frac{1}{12}-\frac{1}{N}$ | $\geqslant \frac{1}{12}-\frac{1}{N}-\frac{1}{13}+\frac{1}{N-1}>\frac{1}{156}$ |

(2) Exactly two of the values $K, L, M, N$ are $\geqslant 2 z+14$. Because of the definition of the edge-type these can be $K$ and $L, M$ and $N$ or $L$ and $N$ (w.l.o.g. $K, L \geqslant 2 z+14$ ) $w(e) \geqslant \frac{1}{3}-\frac{1}{K}-\frac{1}{L}$
$\Rightarrow w^{*}(e) \geqslant \frac{1}{3}-\frac{1}{K}-\frac{1}{L}-\frac{1}{13}+\frac{1}{K-1}-\frac{1}{13}+\frac{1}{L-1}>\frac{7}{39}>\frac{1}{156}$
(3) Exactly three of the values $K, L, M, N$ are $\geqslant 2 z+14$. Because of the definition of the edge-type these can be $K, L, N$ or $L, M, N$ (w.l.o.g. $K, L, N \geqslant 2 z+14$ )

$$
\begin{aligned}
& w(e) \geqslant \frac{2}{3}-\frac{1}{K}-\frac{1}{L}-\frac{1}{N} \\
& \Rightarrow w^{*}(e) \geqslant \frac{2}{3}-\frac{1}{K}-\frac{1}{L}-\frac{1}{N}-\frac{1}{13}+\frac{1}{K-1}-\frac{1}{13}+\frac{1}{L-1}-\frac{1}{13}+\frac{1}{N-1}>\frac{17}{39}>\frac{1}{156}
\end{aligned}
$$

(4) All of the values $K, L, M, N$ are $\geqslant 2 z+14$

$$
\begin{aligned}
& w(e)=1-\frac{1}{K}-\frac{1}{L}-\frac{1}{M}-\frac{1}{N} \\
& \Rightarrow w^{*}(e) \geqslant 1-\frac{1}{K}-\frac{1}{L}-\frac{1}{M}-\frac{1}{N}-\frac{1}{13}+\frac{1}{K-1}-\frac{1}{13}+\frac{1}{L-1}-\frac{1}{13}+\frac{1}{M-1}-\frac{1}{13}+ \\
& \frac{1}{N-1}>\frac{9}{13}>\frac{1}{156}
\end{aligned}
$$

Now an edge $e=(x, y ; \alpha, \beta)$ of type $[K, L ; M, N]$ with $w^{*}(e) \leqslant 0$ either belongs to the 78 types listed above with $\max \{K, L ; M, N\} \leqslant 12$ or is of type $[3, L ; 3,3]$ with $13 \leqslant L \leqslant 2 z+13$ or $[3,3 ; 3, N]$ with $13 \leqslant N \leqslant 2 z+13$. Each of them occurs at most $z$ times in $G$, therefore we have

$$
w^{*}\left(E^{-*}\right) \geqslant-z \cdot \frac{7891}{1540}-2 z \sum_{k=13}^{2 z+13} \frac{1}{k}
$$

and

$$
\begin{aligned}
& \left|E^{-*}\right| \leqslant z \cdot 78+2 z(2 z+13-13+1)=4 z^{2}+80 z . \\
& -2=w^{*}(G) \geqslant w^{*}\left(E^{+*}\right)+w^{*}\left(E^{-*}\right) \\
& -2 \geqslant \frac{1}{156}\left|E^{+*}\right|-z \cdot \frac{7891}{1540}-2 z \sum_{k=13}^{2 z+13} \frac{1}{k}
\end{aligned}
$$

> Selfdual edge-oblique graph


Fig. 3. Selfdual edge-oblique graph.

$$
\begin{aligned}
& -2 \geqslant \frac{1}{156}\left|E^{+*}\right|-z \cdot \frac{7891}{1540}-2 z\left[\ln \frac{2 z+13}{12}\right] \\
& \Rightarrow\left|E^{+*}\right| \leqslant 156\left[-2+z \cdot \frac{7891}{1540}-2 z \cdot \ln \frac{2 z+13}{12}\right]
\end{aligned}
$$

Since $\left|E^{+*}\right|$ and $\left|E^{-*}\right|$ are bounded, we have a limited number of edges for a $z$-edgeoblique graph. Therefore the set of such graphs must be finite.

Proof of (3). Suppose the edge-oblique graph $G$ has no vertex of degree 3. The only possible edge-types with negative charges are

$$
[4,4 ; 3,5],[4, L ; 3,3]_{L=4,5, \ldots, 11},[4, L ; 3,4]_{L=4,5},[5, L ; 3,3]_{L=5,6,7}
$$

The sum of the charges of these 14 edges is $-\frac{4321}{5544}=-0.77940 \ldots>-2$ Contradiction.
In an analogous way we can prove that an edge-oblique graph contains a triangle.
Remark 1. Since the charge-sum $-\frac{4321}{5544}$ is far away from -2 one can improve the result easily and show that there must occur at least 3 vertices of degree 3 and at least 3 triangles.

Proof of (4). Suppose all vertices of a graph $G$ have degree 3. Now consider a face $\alpha$ with maximum degree $d(\alpha)=\Delta$. There are $\Delta$ edges incident with $\alpha$ but only $\Delta-1$ possible edge-types for them namely $[3,3, M, \Delta]_{M=3, \ldots, \Delta}$. Therefore, one type must occur at least twice and the graph cannot be edge-oblique.

In a similar way it can be proved that an edge-oblique contains not only triangles.

Proof of (5). See Figs. 2 and 3.

Proof of (6). Suppose there is a graph $G \in \mathfrak{P}$ whose dual $G^{*}$ is edge-oblique, too and these graphs have no common edge-type. Therefore, $G$ contains at most one of the edge-types $[K, L ; M, N]$ and $[M, N ; K, L]$ and $G$ contains no edge-type of the form $[K, L ; K, L]$. To prove the nonexistence of such a graph we define a generalized edge-type in the following way:
$e=(x, y ; \alpha, \beta)$ is of the generalized edge-type $\langle K, L ; M, N\rangle$ if $[K, L ; M, N]$ is lexicographically smaller than $[M, N ; K, L]$ and $e$ is of one of these edge-types. Note that an edge has no generalized edge-type if it is of type $[K, L ; K, L]$, but such an edge does not occur in $G$.
$G$ contains at most one edge of each generalized edge-type. Similarly to before we define

$$
\begin{aligned}
& E^{-}:=\{e \in E(G): w(e) \leqslant 0\}, \\
& E_{k}:=\{e \in E(G): e \text { is of type }\langle K, L ; M, N\rangle \wedge \max \{K, L ; M, N\}=k\}, \\
& E_{k}^{-}:=E^{-} \cap E_{k} .
\end{aligned}
$$

Since $-2=\sum_{e \in E(G)} w(e)=\sum_{k=4}^{\Delta} \sum_{e \in E_{k}} w(e)$ the following two inequalities hold:
$-2 \geqslant \sum_{k=4}^{\Delta-1} \sum_{e \in E_{k}^{-}} w(e)+\sum_{e \in E_{\Delta}} w(e)$,
$-2 \geqslant \sum_{k=4}^{\Delta} \sum_{e \in E_{k}^{-}} w(e)$.
Now let us consider all possible generalized edge-types with negative charges:

| Generalized edge-type | $w(e)$ |  |
| :--- | :--- | :--- |
| $\langle 3,3 ; 3,4\rangle$ | $-\frac{1}{4}$ |  |
| $\langle 3,3 ; 4,4\rangle$ | $-\frac{1}{6}$ |  |
| $\langle 3,4 ; 4,4\rangle$ | $-\frac{1}{12} \quad \sum_{e \in E_{4}^{-}} w(e) \geqslant-\frac{1}{2} \quad \sum_{k=4}^{4} \sum_{e \in E_{k}^{-}} w(e) \geqslant-\frac{1}{2}$ |  |
| $\langle 3,3 ; 3,5\rangle$ | $-\frac{1}{5}$ |  |
| $\langle 3,3 ; 4,5\rangle,\langle 3,4 ; 3,5\rangle$ | $-\frac{7}{60}$ |  |

$\langle 3,3 ; 5,5$
$-\frac{1}{15}$
$\langle 3,4 ; 4,5\rangle,\langle 3,5 ; 4,4\rangle \quad-\frac{1}{30} \quad \sum_{e \in E_{5}^{-}} w(e) \geqslant-\frac{17}{30} \quad \sum_{k=4}^{5} \sum_{e \in E_{k}^{-}} w(e) \geqslant-\frac{16}{15}$
$\langle 3,3 ; 3,6\rangle$
$-\frac{1}{6}$
$\langle 3,3 ; 4,6\rangle,\langle 3,4 ; 3,6\rangle \quad-\frac{1}{12}$
$\langle 3,3 ; 5,6\rangle,\langle 3,5 ; 3,6\rangle \quad-\frac{1}{30} \quad \sum_{e \in E_{6}^{-}} w(e) \geqslant-\frac{2}{5} \quad \sum_{k=4}^{6} \sum_{e \in E_{k}^{-}} w(e) \geqslant-\frac{22}{15}$
$\langle 3,3 ; 3,7\rangle$
$-\frac{1}{7}$
$\langle 3,3 ; 4,7\rangle,\langle 3,4 ; 3,7\rangle \quad-\frac{5}{84}$
$\langle 3,3 ; 5,7\rangle,\langle 3,5 ; 3,7\rangle \quad-\frac{1}{105} \quad \sum_{e \in E_{-}^{-}} w(e) \geqslant-\frac{59}{210} \quad \sum_{k=4}^{7} \sum_{e \in E_{k}^{-}} w(e) \geqslant-\frac{367}{210}$
$\langle 3,3 ; 3,8\rangle$
$-\frac{1}{8}$
$\langle 3,3 ; 4,8\rangle,\langle 3,4 ; 3,8\rangle \quad-\frac{1}{24} \quad \sum_{e \in E_{8}^{-}} w(e) \geqslant-\frac{5}{24} \quad \sum_{k=4}^{8} \sum_{e \in E_{k}^{-}} w(e) \geqslant-\frac{1643}{840}$
$\langle 3,3 ; 3,9\rangle$
$-\frac{1}{9}$
$\langle 3,3 ; 4,9\rangle,\langle 3,4 ; 3,9\rangle \quad-\frac{1}{36} \quad \sum_{e \in E_{9}^{-}} w(e) \geqslant-\frac{1}{6} \quad \sum_{k=4}^{9} \sum_{e \in E_{k}^{-}} w(e) \geqslant-\frac{1783}{840}$
$\langle 3,3 ; 3,10\rangle$
$-\frac{1}{10}$
$\langle 3,3 ; 4,10\rangle,\langle 3,4 ; 3,10\rangle \quad-\frac{1}{60} \quad \sum_{e \in E_{10}^{-1}} w(e) \geqslant-\frac{2}{15} \quad \sum_{k=4}^{10} \sum_{e \in E_{k}^{-}} w(e) \geqslant-\frac{379}{168}$
$\langle 3,3 ; 3,11\rangle$

$$
-\frac{1}{11}
$$

$\langle 3,3 ; 4,11\rangle,\langle 3,4 ; 3,11\rangle \quad-\frac{1}{132} \quad \sum_{e \in E_{11}^{-1}} w(e) \geqslant-\frac{7}{66} \quad \sum_{k=4}^{11} \sum_{e \in E_{k}^{-}} w(e) \geqslant-\frac{1455}{616}$
$\langle 3,3 ; 3, N\rangle, N \geqslant 12 \quad-\frac{1}{N} \quad \sum_{e \in E_{N}^{-}} w(e) \geqslant-\frac{1}{N} \quad \sum_{k=4}^{N} \sum_{e \in E_{k}^{-}} w(e) \geqslant-\frac{1455}{616}$ $-\ln (N)+\ln (11)$


Fig. 4. Face- and edge-oblique graph.

For $\Delta \leqslant 8$ the fourth column yields a contradiction to inequality (2). For $\Delta \geqslant 9$ let us consider the edges of $E_{\Delta}$ with the lowest possible charges:

| Generalized edge-type | $w(e)$ |
| :--- | :---: |
| $\langle 3,3 ; 3, \Delta\rangle$ | $-\frac{1}{\Delta}$ |
| $\langle 3,3 ; 4, \Delta\rangle$ | $\frac{1}{12}-\frac{1}{\Delta}$ |
| $\langle 3,4 ; 3, \Delta\rangle$ | $\frac{1}{12}-\frac{1}{\Delta}$ |
| $\langle 3,3 ; 5, \Delta\rangle$ | $\frac{2}{15}-\frac{1}{\Delta}$ |
| $\langle 3,5 ; 3, \Delta\rangle$ | $\frac{2}{15}-\frac{1}{\Delta}$ |
| All other types | $\geqslant \frac{1}{6}-\frac{1}{\Delta}$ |

Since there are at least $\Delta$ edges belonging to $E_{\Delta}$, we have

$$
\begin{aligned}
& \sum_{e \in E_{\Delta}} w(e) \geqslant-\frac{1}{\Delta}+2\left(\frac{1}{12}-\frac{1}{\Delta}+\frac{2}{15}-\frac{1}{\Delta}\right)+(\Delta-5)\left(\frac{1}{6}-\frac{1}{\Delta}\right) \\
& \sum_{e \in E_{\Delta}} w(e) \geqslant \frac{\Delta}{6}-\frac{7}{5}
\end{aligned}
$$

In combination with column 4 this leads to a contradiction to inequality (1) for $9 \leqslant \Delta \leqslant 12$.

If $\Delta \geqslant 13$ we have

$$
\begin{aligned}
& -2 \geqslant \sum_{k=4}^{\Delta-1} \sum_{e \in E_{k}^{-}} w(e)+\sum_{e \in E_{\Delta}} w(e), \\
& -2 \geqslant-\frac{1455}{616}-\ln (\Delta)+\ln (11)+\frac{\Delta}{6}-\frac{7}{5}, \\
& -2 \geqslant-1.37+\frac{\Delta}{6}-\ln (\Delta)>-2
\end{aligned}
$$

a contradiction and (6) has been proved.
Proof of (7). See Fig. 4.

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