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Numerical characterization of n-cube subset partitioning[☆]

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ABSTRACT

A general quantitative description of vertex subsets of the $\mathbf{n}$-dimensional unit cube $\mathbf{E^n}$ through their partitions (direct problem) is given and the existence and composition problems for vertex subsets with given quantitative characteristics of partitions (inverse problem) are considered. Each of these subproblems is of significant theoretical and practical importance. Finding an efficient algorithmic solution to the inverse problem remains open. A complete and simple structural description of the numerical parameters of the unit cube partitions is presented.

1. General problem and related combinatorial counterparts

Let $\mathbf{E^n}$ be the set of vertices of the $\mathbf{n}$-dimensional unit cube $\mathbf{E^n} = \{(\mathbf{x_1}, \ldots, \mathbf{x_n})/\mathbf{x_i} \in \{0, 1\}, \mathbf{i} = 1, \ldots, \mathbf{n}\}$. For an arbitrary variable $\mathbf{x_i}$, $\mathbf{i} = \mathbf{1}, \ldots, \mathbf{n}$, we consider the partition of the cube into two subcubes according to the value of $\mathbf{x_i}$. Denote these subcubes by $\mathbf{E^{n-1}_{x_i=1}}$ and $\mathbf{E^{n-1}_{x_i=0}}$. Similarly, each subset of vertices $\mathbf{M} \subset \mathbf{E^n}$ can be partitioned into $\mathbf{M_{x_i=1}}$ and $\mathbf{M_{x_i=0}}$.

For a given $\mathbf{m}$, $\mathbf{0} \le \mathbf{m} \le \mathbf{2^n}$, let $\mathbf{M}$ be an $\mathbf{m}$-vertex subset of $\mathbf{E^n}$. The vector $\mathbf{S} = (\mathbf{s_1}, \ldots, \mathbf{s_n})$ is called **associated vector of partitions** for the set $\mathbf{M}$ if $\mathbf{s_i} = |\mathbf{M_{x_i=1}}|$ for all $\mathbf{1} \le \mathbf{i} \le \mathbf{n}$.

The main aim of the problem is to find simple conditions for integer-valued vectors to be associated vectors.

In particular, this problem arises out of the discrete isoperimetric problem for $\mathbf{E^n}$ [1,2,6], where some estimations have been proved for quantitative characteristics of partitions of arbitrary $\mathbf{n}$-cube subsets. By using these estimates, an almost complete description of the set of all solutions of the discrete isoperimetric problem is achieved.

We now address several combinatorial problems closely related to this subject.

Consider the (0, 1)-matrix representation of an $\mathbf{m}$-subset $\mathbf{M}$, $\mathbf{M} \subseteq \mathbf{E^n}$. This is an $\mathbf{m} \times \mathbf{n}$ matrix, whose rows correspond to vertices/elements of $\mathbf{M}$ and columns are variables of $\mathbf{E^n}$. All the rows are different and the number of 1s in columns correspond to the sizes of partition-subsets of $\mathbf{M}$ in appropriate directions. The problem of the existence of vertex subsets of $\mathbf{E^n}$ with given associated vectors is formulated in terms of (0, 1)-matrices, as the problem of existence of (0, 1)-matrices with given number of 1s in columns and with different rows. The class of (0, 1)-matrices, under similar conditions (given number of 1s in rows and in columns), has been studied by Ryser, who obtained simple necessary and sufficient conditions for the existence of such matrices, in particular [9].

Another class of related problems is known as the discrete tomography problem. The main problem of discrete tomography lies in reconstructing a finite set of elements from its discrete X-rays. The problem of existence of (0, 1)-matrices with given number of 1s in rows and in columns, is a simple case of reconstructing a set of elements in the two-dimensional

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grid from its discrete X-rays along horizontal and vertical directions. Recall that the complexity of particular tomography problems strictly depends on conditions and restrictions imposed [10,11].

Another problem is the characterization of degree sequences of hypergraphs.

Let $\mathbf{H} = (\mathbf{H(V)}, \mathbf{H(E)})$ be a hypergraph with vertex set $\mathbf{H(V)} = \{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$, and edge set $\mathbf{H(E)} = \{\mathbf{E_1}, \ldots, \mathbf{E_m}\}$. The degree of a vertex $\mathbf{v_i} \in \mathbf{H(V)}$ is $\mathbf{d_i}$, the number of edges of $\mathbf{H}$ which contain $\mathbf{v_i}$. The degree sequence of $\mathbf{H}$ is $\mathbf{d(H)} = \{\mathbf{d_1}, \ldots, \mathbf{d_n}\}$, where $\mathbf{d_1} \geq \cdots \geq \mathbf{d_n}$. Suppose we are given a sequence of nonnegative integers $\mathbf{d_1}, \ldots, \mathbf{d_n}$, such that $\mathbf{d_1} \geq \cdots \geq \mathbf{d_n}$. The hypergraph degree sequence problem is: does there exist a hypergraph with $\mathbf{d_1}, \ldots, \mathbf{d_n}$ as its degree sequence? The incidence matrix of $\mathbf{H}$ is a (0, 1)-matrix with $\mathbf{m}$ rows and $\mathbf{n}$ columns with $\mathbf{d_1}, \ldots, \mathbf{d_n}$ 1s in the columns. If the hypergraph is simple, the rows of the corresponding matrix will be different. Therefore, the problem of the existence of a simple hypergraph with a given degree sequence is equivalent to the problem of existence of a (0, 1)-matrix with the given parameters. Some complementary issues have been addressed in recent publications, but the main problem known as the hypergraph degree sequence problem is still open [3–5,7,8,12–14].

Current research considers the problem in combinatorial terms of multidimensional multivalued cubes.

2. Decomposition by equivalence

For a given $\mathbf{m}$, $\mathbf{0} \leq \mathbf{m} \leq \mathbf{2^n}$, let $\psi_\mathbf{m}$ denote the set of all associated vectors of partitions of $\mathbf{m}$-subsets of $\mathbf{E^n}$. Let $\Xi^\mathbf{n}_\mathbf{m+1}$ denote the $\mathbf{n}$-dimensional $(\mathbf{m}+\mathbf{1})$- valued grid, i.e., the set of all integer-valued vectors $\mathbf{S} = (\mathbf{s_1}, \ldots, \mathbf{s_n})$ with $\mathbf{0} \leq \mathbf{s_i} \leq \mathbf{m}$, $\mathbf{i} = \mathbf{1}, \ldots, \mathbf{n}$. It is evident that $\psi_\mathbf{m} \subseteq \Xi^\mathbf{n}_\mathbf{m+1}$. We distinguish the boundary cases: if $\mathbf{m} = \mathbf{0}$ or $\mathbf{m} = \mathbf{2^n}$, $|\psi_\mathbf{m}| = \mathbf{1}$ and the associated vectors are all $\mathbf{0}$ or all $\mathbf{2^{n-1}}$, respectively; for $\mathbf{m} = \mathbf{1}$, $\psi_\mathbf{m} = \Xi^\mathbf{n}_\mathbf{m+1} = \mathbf{E^n}$ and $|\psi_\mathbf{m}| = \mathbf{2^n}$.

In general, a large number of different subsets may have the same associated vector. Some understanding on this might be reached by comparing the number of all the different $\mathbf{m}$-subsets of $\mathbf{E^n}$ with the number of all $(\mathbf{0}, \mathbf{1})$-matrices; and with the number of all associated vectors. The main result below concerns the complete description of $\psi_\mathbf{m}$ given in terms of the geometry of $\Xi^\mathbf{n}_\mathbf{m+1}$:

the vectors of $\Xi^\mathbf{n}_\mathbf{m+1}$ are distributed among $\mathbf{m} \cdot \mathbf{n} + \mathbf{1}$ layers according to their weight, the sum of all coordinates. The $\tilde{\mathbf{0}} = (\mathbf{0}, \ldots, \mathbf{0})$ vector is located on the 0-th layer; the $\mathbf{i}$-th layer consists of all vectors with weight $\mathbf{i}$. Two vectors on consecutive layers which differ in only one component are called neighbors. The $\tilde{\mathbf{m}} = (\mathbf{m}, \ldots, \mathbf{m})$ vector is located on the $(\mathbf{m} \cdot \mathbf{n})$-th layer.

For $\mathbf{S} = (\mathbf{s_1}, \ldots, \mathbf{s_n})$, $\mathbf{S}' = (\mathbf{s}'_\mathbf{1}, \ldots, \mathbf{s}'_\mathbf{n})$, we say $\mathbf{S} \geq \mathbf{S}'$ if $\mathbf{s_i} \geq \mathbf{s}'_\mathbf{i}$ for $\mathbf{1} \leq \mathbf{i} \leq \mathbf{n}$. In $\Xi^\mathbf{n}_\mathbf{m+1}$ we distinguish several classes of specific vectors:

- **middle vectors $\tilde{\mathbf{m}}_{\mathbf{mid}+}$, $\tilde{\mathbf{m}}_{\mathbf{mid}-}$**
  $\tilde{\mathbf{m}}_{\mathbf{mid}+} = (\frac{\mathbf{m+1}}{\mathbf{2}}, \ldots, \frac{\mathbf{m+1}}{\mathbf{2}})$ and $\tilde{\mathbf{m}}_{\mathbf{mid}-} = (\frac{\mathbf{m-1}}{\mathbf{2}}, \ldots, \frac{\mathbf{m-1}}{\mathbf{2}})$ for odd $\mathbf{m}$ and $\tilde{\mathbf{m}}_{\mathbf{mid}+} = \tilde{\mathbf{m}}_{\mathbf{mid}-} = (\frac{\mathbf{m}}{\mathbf{2}}, \ldots, \frac{\mathbf{m}}{\mathbf{2}})$ for even $\mathbf{m}$.
  $\tilde{\mathbf{m}}_{\mathbf{mid}+}$ is located on the $\mathbf{n} \cdot \frac{\mathbf{m+1}}{\mathbf{2}}$-th layer of $\Xi^\mathbf{n}_\mathbf{m+1}$ and $\tilde{\mathbf{m}}_{\mathbf{mid}-}$ is situated on the $\mathbf{n} \cdot \frac{\mathbf{m-1}}{2}$-th layer of $\Xi^\mathbf{n}_\mathbf{m+1}$ for odd $\mathbf{m}$; for even $\mathbf{m}$ vector $\tilde{\mathbf{m}}_{\mathbf{mid}+} = \tilde{\mathbf{m}}_{\mathbf{mid}-}$ is located on the $\mathbf{n} \cdot \frac{\mathbf{m}}{2}$-th layer.
- **upper than half (upper h) and lower than half (lower h) vectors**

  Definition 1. A vector $\mathbf{S}' \in \Xi^\mathbf{n}_\mathbf{m+1}$ is called an upper $\mathbf{h}$-vector if $\mathbf{S}' \geq \tilde{\mathbf{m}}_{\mathbf{mid}+}$ and $\mathbf{S}'' \in \Xi^\mathbf{n}_\mathbf{m+1}$ is called a lower $\mathbf{h}$-vector if $\mathbf{S}'' \leq \tilde{\mathbf{m}}_{\mathbf{mid}-}$.

  We denote by $\hat{\mathbf{H}}$ the set of all upper $\mathbf{h}$-vectors and by $\check{\mathbf{H}}$ the set of all lower $\mathbf{h}$-vectors. Thus, for every vector $\mathbf{S}' = (\mathbf{s}'_\mathbf{1}, \ldots, \mathbf{s}'_\mathbf{n})$ in $\hat{\mathbf{H}}$ we have $\mathbf{s}'_\mathbf{i} \geq \mathbf{m} - \mathbf{s}'_\mathbf{i}$ and for every vector $\mathbf{S}'' = (\mathbf{s}''_\mathbf{1}, \ldots, \mathbf{s}''_\mathbf{n})$ in $\check{\mathbf{H}}$ we have $\mathbf{s}''_\mathbf{i} \leq \mathbf{m} - \mathbf{s}''_\mathbf{i}$, for $\mathbf{i} = \mathbf{1}, \ldots, \mathbf{n}$. The cardinalities of the sets $\hat{\mathbf{H}}$ and $\check{\mathbf{H}}$ are equal to $(\frac{\mathbf{m+1}}{\mathbf{2}})^\mathbf{n}$ for odd $\mathbf{m}$ and $(\frac{\mathbf{m}}{\mathbf{2}} + \mathbf{1})^\mathbf{n}$ for even $\mathbf{m}$.
- **vertically equivalent vectors**

  Definition 2. Let $\mathbf{S}', \mathbf{S}'' \in \Xi^\mathbf{n}_\mathbf{m+1}$. $\mathbf{S}'$ and $\mathbf{S}''$ are called $\mathbf{v}$-equivalent (vertically equivalent) if $\mathbf{s}''_\mathbf{i} \in \{\mathbf{s}'_\mathbf{i}, \mathbf{m} - \mathbf{s}'_\mathbf{i}\}$ for $\mathbf{1} \leq \mathbf{i} \leq \mathbf{n}$.

  Clearly, this is an equivalence relation on $\Xi^\mathbf{n}_\mathbf{m+1}$. Let $\mathbf{V(S)}$ be the equivalence class of a vector $\mathbf{S}$.

  In $\mathbf{V(S)}$ we distinguish two vectors $\hat{\mathbf{S}} = (\hat{\mathbf{s}}_\mathbf{1}, \ldots, \hat{\mathbf{s}}_\mathbf{n})$ and $\check{\mathbf{S}} = (\check{\mathbf{s}}_\mathbf{1}, \ldots, \check{\mathbf{s}}_\mathbf{n})$—the upper and lower vectors of the class, defined as follows:

$$\hat{\mathbf{s}}_\mathbf{i} = \begin{cases} s_\mathbf{i}, & \text{if } s_\mathbf{i} \geq m - s_\mathbf{i} \\ m - s_\mathbf{i}, & \text{if } s_\mathbf{i} < m - s_\mathbf{i} \end{cases} \quad \text{and} \quad \check{\mathbf{s}}_\mathbf{i} = \begin{cases} m - s_\mathbf{i}, & \text{if } s_\mathbf{i} > m - s_\mathbf{i} \\ s_\mathbf{i}, & \text{if } s_\mathbf{i} \leq m - s_\mathbf{i}, \end{cases} \quad \mathbf{i} \in \overline{\mathbf{1}, \mathbf{n}}.$$

We call such a pair of vectors opposite, since one of them is obtained from the other by inverting all coordinates, i.e. replacing every $\mathbf{s_i}$ with $\mathbf{m} - \mathbf{s_i}$. Thus for any $\mathbf{S}$ the class $\mathbf{V(S)}$ can be constructed from its upper vector and/or lower vector by applying the coordinate inversions. $\hat{\mathbf{S}}$ and $\check{\mathbf{S}}$ are the only vectors of $\mathbf{V(S)}$ that belong to sets $\hat{\mathbf{H}}$ and $\check{\mathbf{H}}$, respectively. It is evident that the $\mathbf{v}$-equivalence classes of different vectors of $\hat{\mathbf{H}}$ (or $\check{\mathbf{H}}$) are disjoint. This provides a partition of the structure $\Xi^\mathbf{n}_\mathbf{m+1}$ into $|\hat{\mathbf{H}}|$ equivalent classes, which are uniquely defined by elements of $\hat{\mathbf{H}}$ (or $\check{\mathbf{H}}$). Thus, defining $\mathbf{v}$-equivalence class of a set as the union of $\mathbf{v}$-equivalence classes of its elements, we obtain: $\mathbf{V}(\hat{\mathbf{H}}) = \mathbf{V}(\check{\mathbf{H}}) = \Xi^\mathbf{n}_\mathbf{m+1}$.

A numerical characterization of $\mathbf{v}$-equivalence classes is obtained as follows:

for a given vector $\mathbf{S}$ let $\mathbf{k}$ denote the number of coordinates of $\mathbf{S}$ not equal to $\frac{\mathbf{m}}{\mathbf{2}}$ (for odd $\mathbf{m}$, we always have $\mathbf{k} = \mathbf{n}$). Then $|\mathbf{V}(\mathbf{S})| = \mathbf{2^k}$ and $\mathbf{V}(\mathbf{S})$ is isomorphic to the $\mathbf{k}$-dimensional unit cube; the 0-th layer of the cube contains the lower vector $\check{\mathbf{S}}$ of the class; the $\mathbf{i}$-th layer consists of all vectors of $\mathbf{V}(\mathbf{S})$ which can be obtained from the vector $\hat{\mathbf{S}}$ by applying $\mathbf{i}$ coordinate inversions.

Thus, in this section we have achieved a partition of the structure $\Xi^{\mathbf{n}}_{\mathbf{m+1}}$ into structures isomorphic to binary cubes, which are uniquely represented by elements of $\hat{\mathbf{H}}$ (or $\check{\mathbf{H}}$). It will be stated by forthcoming results, that the main problem of describing associated vectors can be reduced from $\Xi^{\mathbf{n}}_{\mathbf{m+1}}$ to $\hat{\mathbf{H}}$ (or $\check{\mathbf{H}}$) via $\mathbf{v}$-equivalence. It is important that the resulting vector set $\psi_{\mathbf{m}}$ by its structure is monotone in $\hat{\mathbf{H}}$. Unlike $\mathbf{v}$-equivalence which goes with "hops" (leaving vectors between neighboring vertices), monotonicity in $\hat{\mathbf{H}}$ means continuous inclusion of internal vertices.

It is also worth mentioning that the partition of $\Xi^{\mathbf{n}}_{\mathbf{m+1}}$ above and the $\mathbf{v}$ equivalence concept retain a monotonicity relation. A pair of vertices, which is comparable in an equivalence binary cube, is also comparable in $\Xi^{\mathbf{n}}_{\mathbf{m+1}}$ and vice versa. Therefore monotonicity in $\Xi^{\mathbf{n}}_{\mathbf{m+1}}$ implies monotonicity in all obtained binary cubes. Monotonicity related studies on $\Xi^{\mathbf{n}}_{\mathbf{m+1}}$ usually use complicated structures and algorithms, and the decomposition into the binary cubes is an efficient way of simplification.

3. Preliminary lemmas

We start with the basic properties of $\psi_{\mathbf{m}}$ in $\Xi^{\mathbf{n}}_{\mathbf{m+1}}$.

- **Property 1: $\psi_{\mathbf{m}}$ is closed with respect to the vertical equivalence.**
  This property is provided by the following lemma, which is easy to verify.

Lemma 1. 1. *If* $\mathbf{S} = (\mathbf{s_1}, \ldots, \mathbf{s_n}) \in \psi_{\mathbf{m}}$, *then* $\mathbf{V}(\mathbf{S}) \subseteq \psi_{\mathbf{m}}$.
2. *If* $\mathbf{S} \notin \psi_{\mathbf{m}}$ *then* $\mathbf{V}(\mathbf{S}) \cap \psi_{\mathbf{m}} = \emptyset$.

- **Property 2: $\psi_{\mathbf{m}}$ is monotone in $\hat{\mathbf{H}}$ and $\check{\mathbf{H}}$.**
  Property 2 is provided by the following lemma.

Lemma 2. 1. *If* $\mathbf{S}' = (\mathbf{s}'_1, \ldots, \mathbf{s}'_n), \mathbf{S}'' = (\mathbf{s}''_1, \ldots, \mathbf{s}''_n) \in \hat{\mathbf{H}}$ *and* $\mathbf{S}' > \mathbf{S}''$ *then* $\mathbf{S}' \in \psi_{\mathbf{m}}$ *implies that* $\mathbf{S}'' \in \psi_{\mathbf{m}}$.
2. *If* $\mathbf{S}' = (\mathbf{s}'_1, \ldots, \mathbf{s}'_n), \mathbf{S}'' = (\mathbf{s}''_1, \ldots, \mathbf{s}''_n) \in \check{\mathbf{H}}$ *and* $\mathbf{S}' < \mathbf{S}''$ *then* $\mathbf{S}' \in \psi_{\mathbf{m}}$ *implies that* $\mathbf{S}'' \in \psi_{\mathbf{m}}$.

Proof. 1. Let $\mathbf{S}'$ and $\mathbf{S}''$ satisfy the conditions of the lemma. Then there exists an $\mathbf{i}$, $\mathbf{1} \le \mathbf{i} \le \mathbf{n}$, such that $\mathbf{s}'_i > \mathbf{s}''_i$. Let $\mathbf{S}' \in \psi_{\mathbf{m}}$ and $\mathbf{M}'$ be an $\mathbf{m}$-subset with associated vector of partitions $\mathbf{S}'$. We may replace $\mathbf{s}'_i - \mathbf{s}''_i$ vectors in $\mathbf{M}'$ from $\mathbf{M}'_{\mathbf{x_i=1}}$ by their neighbors in $\mathbf{E}^{\mathbf{n-1}}_{\mathbf{x_i=0}} \setminus M'$, since $\mathbf{s}'_i - \mathbf{s}''_i \le \mathbf{s}'_i - (\mathbf{m} - \mathbf{s}'_i)$ and $\mathbf{S}', \mathbf{S}'' \in \hat{\mathbf{H}}$. Thus, after a number of such replacements, we obtain a subset of $\mathbf{E^n}$ with characteristic vector $\mathbf{S}''$, which implies that $\mathbf{S}'' \in \psi_{\mathbf{m}}$.

The second assertion can be proved in a similar way. □

The following is a simple corollary of Lemma 2.

Corollary 1. *Let* $\mathbf{S}' = (\mathbf{s}'_1, \ldots, \mathbf{s}'_n)$ *and* $\mathbf{S}'' = (\mathbf{s}''_1, \ldots, \mathbf{s}''_n)$ *belong to* $\hat{\mathbf{H}}(\check{\mathbf{H}})$ *and* $\mathbf{S}' > \mathbf{S}''(\mathbf{S}' < \mathbf{S}'')$. *If* $\mathbf{S}'' \notin \psi_{\mathbf{m}}$ *then* $\mathbf{S}' \notin \psi_{\mathbf{m}}$.

4. Description via the boundary elements

The lemmas given above establish monotone relations of vectors of $\psi_{\mathbf{m}}$ by coordinatewise comparisons. While Lemma 1 establishes a skeleton via the $\mathbf{v}$-equivalence, Lemma 2 expresses the overall continuity. Now it is important to consider boundary elements of these structures.

Definition 3. A vector $\mathbf{S} \in \psi_{\mathbf{m}}$ is called an upper (lower) boundary vector for $\psi_{\mathbf{m}}$ if no vector $\mathbf{R} \in \Xi^{\mathbf{n}}_{\mathbf{m+1}}$ with $\mathbf{R} > \mathbf{S}$ ($\mathbf{S} > \mathbf{R}$) belongs to $\psi_{\mathbf{m}}$.

We denote by $\hat{\psi}_{\mathbf{m}}$ and $\check{\psi}_{\mathbf{m}}$ the sets of all upper and lower boundary vectors of $\boldsymbol{\psi}_{\mathbf{m}}$, respectively. It is easy to prove that for each vector $\hat{\mathbf{S}} \in \hat{\psi}_{\mathbf{m}}$ its opposite vector $\check{\mathbf{S}}$ belongs to $\check{\psi}_{\mathbf{m}}$ and vice versa. Hence the sets $\hat{\psi}_{\mathbf{m}}$ and $\check{\psi}_{\mathbf{m}}$ contain equal numbers of elements and are symmetric (with respect to the middle layer/layers) in $\Xi^{\mathbf{n}}_{\mathbf{m+1}}$: if $\check{\mathbf{S}}$ is located on the $\mathbf{l}$-th layer, then its opposite vector $\hat{\mathbf{S}}$ is located on the $(\mathbf{m} \cdot \mathbf{n} - \mathbf{l})$-th layer. Let $\mathbf{r}$ denote the number of elements of each of these two boundary sets and let $\hat{\psi}_{\mathbf{m}} = \{\hat{\mathbf{S}}_1, \ldots, \hat{\mathbf{S}}_r\}$ and $\check{\psi}_{\mathbf{m}} = \{\check{\mathbf{S}}_1, \ldots, \check{\mathbf{S}}_r\}$. Observe that each of the boundary subsets builds an antichain in $\Xi^{\mathbf{n}}_{\mathbf{m+1}}$. It follows from Lemma 1 that $\hat{\psi}_{\mathbf{m}} \subseteq \hat{\mathbf{H}}$ and $\check{\psi}_{\mathbf{m}} \subseteq \check{\mathbf{H}}$.

For a pair of opposite vectors $\hat{\mathbf{S}} = (\hat{\mathbf{s}}_1, \ldots, \hat{\mathbf{s}}_n) \in \hat{\mathbf{H}}, \check{\mathbf{S}} = (\check{\mathbf{s}}_1, \ldots, \check{\mathbf{s}}_n) \in \check{\mathbf{H}}$ let $\mathbf{I}(\hat{\mathbf{S}})$ denote the vectors of the subcube in $\Xi^{\mathbf{n}}_{\mathbf{m+1}}$ spanned by these vectors:

$$\mathbf{I}(\hat{\mathbf{S}}) = \{\mathbf{Q}, \mathbf{Q} \in \Xi^{\mathbf{n}}_{\mathbf{m+1}} / \check{\mathbf{S}} \le \mathbf{Q} \le \hat{\mathbf{S}}\}.$$

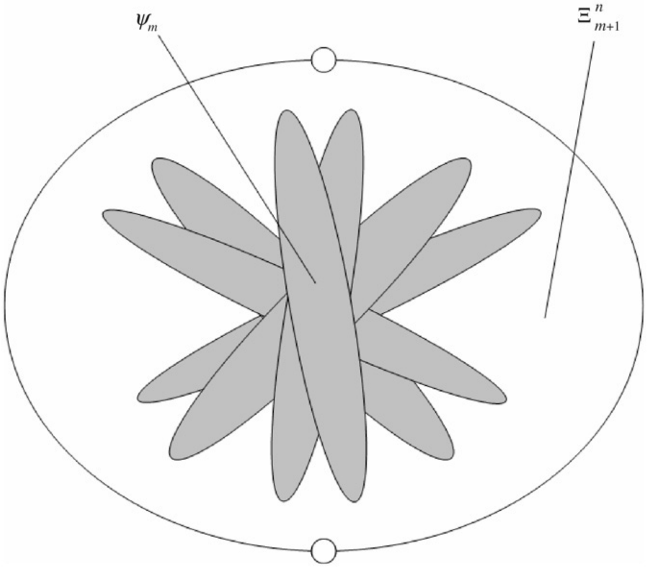

Fig. 1. Illustrates the global structure of set $\psi_\mathbf{m}$ as a union of multidimensional cubes.

The collections of subcubes $\{\mathbf{I}(\hat{\mathbf{S}}) \mid \hat{\mathbf{S}} \in \hat{\psi}_\mathbf{m}\}$ cover a certain part of $\Xi^\mathbf{n}_{\mathbf{m+1}}$. One of our main results states that this region is exactly $\psi_\mathbf{m}$ (Fig. 1).

Theorem 1. $\psi_\mathbf{m} = \bigcup_{\mathbf{j=1}}^{\mathbf{r}} \mathbf{I}(\hat{\mathbf{S}}_\mathbf{j})$.

Proof. We first prove that $\psi_\mathbf{m} \subseteq \bigcup_{\mathbf{j=1}}^{\mathbf{r}} \mathbf{I}(\hat{\mathbf{S}}_\mathbf{j})$. Suppose that there exists a vector $\mathbf{Q} \in \psi_\mathbf{m}$ such that $\mathbf{Q} \notin \bigcup_{\mathbf{j=1}}^{\mathbf{r}} \mathbf{I}(\hat{\mathbf{S}}_\mathbf{j})$. Let $\hat{\mathbf{Q}}$ and $\check{\mathbf{Q}}$ be the upper and lower vectors of the $\mathbf{v}$-equivalence class of $\mathbf{Q}$. It is easy to check that $\mathbf{Q} \notin \bigcup_{\mathbf{j=1}}^{\mathbf{r}} \mathbf{I}(\hat{\mathbf{S}}_\mathbf{j})$ implies $\hat{\mathbf{Q}}, \check{\mathbf{Q}} \notin \bigcup_{\mathbf{j=1}}^{\mathbf{r}} \mathbf{I}(\hat{\mathbf{S}}_\mathbf{j})$ and vice versa.

Therefore $\hat{\mathbf{Q}} \notin \hat{\psi}_\mathbf{m}$ and $\check{\mathbf{Q}} \notin \check{\psi}_\mathbf{m}$. Hence there exists an upper boundary vector $\hat{\mathbf{S}}_\mathbf{j} \in \hat{\psi}_\mathbf{m}$ (respectively, its opposite lower boundary vector $\check{\mathbf{S}}_\mathbf{j} \in \check{\psi}_\mathbf{m}$), such that $\hat{\mathbf{Q}} < \hat{\mathbf{S}}_\mathbf{j}$, ($\check{\mathbf{Q}} > \check{\mathbf{S}}_\mathbf{j}$). Thus we get $\check{\mathbf{S}}_\mathbf{j} < \check{\mathbf{Q}} \le \hat{\mathbf{Q}} < \hat{\mathbf{S}}_\mathbf{j}$, which implies that vectors $\hat{\mathbf{Q}}$ and $\check{\mathbf{Q}}$ belong to $\mathbf{I}(\hat{\mathbf{S}}_\mathbf{j})$ and thus $\mathbf{Q} \in \mathbf{I}(\hat{\mathbf{S}}_\mathbf{j})$, which is a contradiction.

Now we prove that $\bigcup_{\mathbf{j=1}}^{\mathbf{r}} \mathbf{I}(\hat{\mathbf{S}}_\mathbf{j}) \subseteq \psi_\mathbf{m}$. Suppose this is not the case and there exists $\mathbf{Q} \in \mathbf{I}(\hat{\mathbf{S}}_\mathbf{j})$ such that $\mathbf{Q} \notin \psi_\mathbf{m}$. By Lemma 1 we have that $\hat{\mathbf{Q}}$ and $\check{\mathbf{Q}}$ don't belong to $\psi_\mathbf{m}$. On the other hand, $\mathbf{Q} \in \mathbf{I}(\hat{\mathbf{S}}_\mathbf{j})$ implies $\hat{\mathbf{Q}}, \check{\mathbf{Q}} \in \mathbf{I}(\hat{\mathbf{S}}_\mathbf{j})$, and therefore $\hat{\mathbf{Q}} \le \hat{\mathbf{S}}_\mathbf{j}$ and $\check{\mathbf{Q}} \ge \check{\mathbf{S}}_\mathbf{j}$. Then $\hat{\mathbf{Q}} \in \psi_\mathbf{m}$, $\check{\mathbf{Q}} \in \psi_\mathbf{m}$ by Lemma 2, which leads to a contradiction. □

Theorem 1 does not provide a constructive description of $\boldsymbol{\psi}_\mathbf{m}$, since it is based on boundary elements, which are known by definition only.

Consider the set of monotone Boolean functions defined on $\mathbf{E^n}$ with exactly $\mathbf{m}$ ones. Let $\mathbf{M^1_m}$ be the set of all associated vectors of partitions of $\mathbf{m}$-subsets corresponding to the ones of monotone functions. It is easy to check that $\mathbf{M^1_m} \subseteq \hat{\mathbf{H}}$. Similarly $\mathbf{M^0_m}$ is the set of all associated vectors of partitions of $\mathbf{m}$-subsets corresponding to the zeros of monotone Boolean functions. The simple relations $\mathbf{M^1_m}, \mathbf{M^0_m} \subseteq \psi_\mathbf{m}$, $\hat{\psi}_\mathbf{m} \subseteq \mathbf{M^1_m}$ and $\check{\psi}_\mathbf{m} \subseteq \mathbf{M^0_m}$ reduce the problem of describing the set $\psi_\mathbf{m}$ to the study of the set of associated vectors corresponding to monotone Boolean functions. In fact strong inclusions $\hat{\psi}_\mathbf{m} \subset \mathbf{M^1_m}$ and $\check{\psi}_\mathbf{m} \subset \mathbf{M^0_m}$ occur, which is illustrated by the following example. Consider two monotone Boolean functions defined on $\mathbf{E^5}$, which are given by the sets of ones — $\mathbf{V}'$ and $\mathbf{V}''$ (Fig. 2):

$$V' = \{(1,1,1,1,1), (0,1,1,1,1), (1,0,1,1,1), (1,1,0,1,1), (1,1,1,0,1), (1,1,1,1,0), (0,0,1,1,1), (0,1,0,1,1), (1,0,0,1,1), (1,1,0,0,1), (1,1,0,1,0), (1,1,1,0,0), \mathbf{(0,0,0,1,1)}, \mathbf{(1,1,0,0,0)}\}$$

$$V'' = \{(1,1,1,1,1), (0,1,1,1,1), (1,0,1,1,1), (1,1,0,1,1), (1,1,1,0,1), (1,1,1,1,0), (0,0,1,1,1), (0,1,0,1,1), (1,0,0,1,1), (1,1,0,0,1), (1,1,0,1,0), (1,1,1,0,0), \mathbf{(0,1,1,1,0)}, \mathbf{(1,0,1,0,1)}\}$$

$\mathbf{S}' = \mathbf{(10, 10, 7, 10, 10)}$ and $\mathbf{S}'' = \mathbf{(10, 10, 9, 10, 10)}$ are corresponding associated vectors of partitions. Thus $\mathbf{S}', \mathbf{S}'' \in \mathbf{M^1_{14}}$. $\mathbf{S}' < \mathbf{S}''$ implies $\mathbf{S}' \notin \hat{\psi}_\mathbf{14}$.

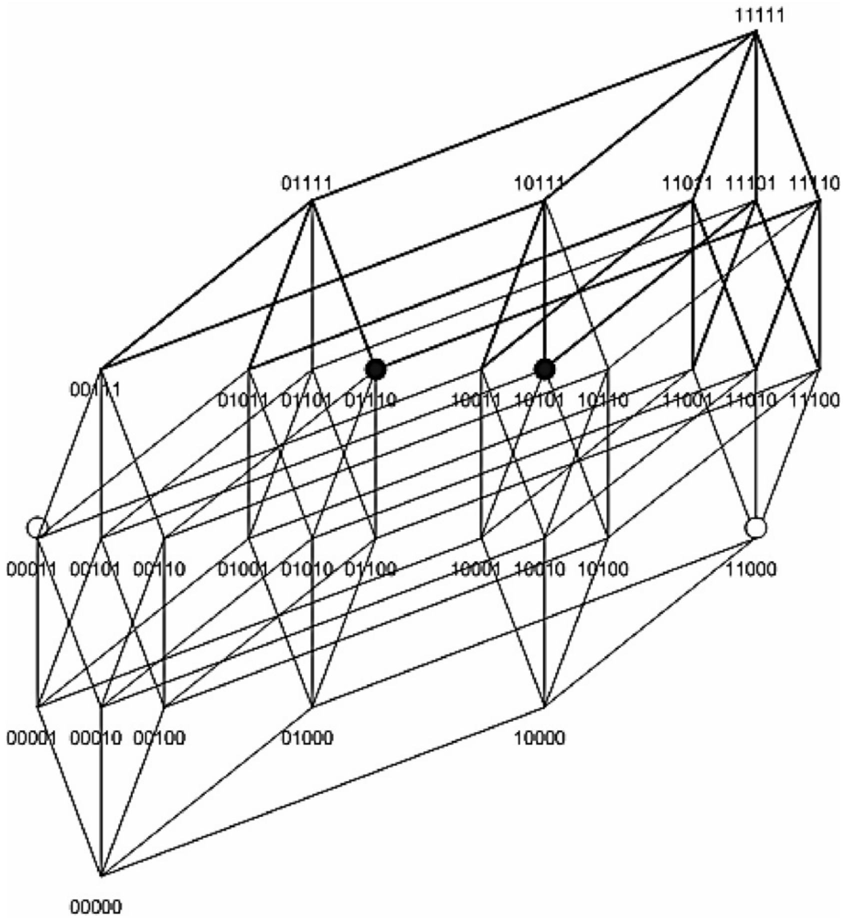

Fig. 2. Illustrates the example above graphically.

The quantities $|\mathbf{M^1_m}|$ and $|\mathbf{M^0_m}|$ are much smaller than the number of monotone Boolean functions. Unfortunately, this later number is large, known only in asymptotic estimates [15].

5. Structure and relations of associated vectors

In this section we formulate statements, which are corollaries of the results above, but also of special importance in themselves. Corollary 2 gives a constructive description for the part of $\boldsymbol{\psi}_\mathbf{m}$ belonging to $\hat{\mathbf{H}}$ and $\check{\mathbf{H}}$: $\hat{\mathbf{H}} \cap \psi_\mathbf{m}$ corresponds to the zeros of some monotone function defined on $\hat{\mathbf{H}}$, with upper zeros $\hat{\mathbf{S}}_\mathbf{1}, \ldots, \hat{\mathbf{S}}_\mathbf{r}$; and $\check{\mathbf{H}} \cap \psi_\mathbf{m}$ corresponds to the ones of some monotone function defined on $\check{\mathbf{H}}$, with lower ones $\check{\mathbf{S}}_\mathbf{1}, \ldots, \check{\mathbf{S}}_\mathbf{r}$ (Figs. 3 and 4). Corollary 3 uses Corollary 2 and $\mathbf{v}$-equivalence and restricts the problem of describing $\boldsymbol{\psi}_\mathbf{m}$ from Ξ^n_{m+1} to $\hat{\mathbf{H}}$ and/or $\check{\mathbf{H}}$.

Let $\mathbf{P}, \mathbf{Q} \in \boldsymbol{\Xi}^\mathbf{n}_{\mathbf{m}+1}$, and $\mathbf{P} \leq \mathbf{Q}$. We denote by $\mathbf{E}(\mathbf{P}, \mathbf{Q})$ the set of vertices of the subcube in $\boldsymbol{\Xi}^\mathbf{n}_{\mathbf{m}+1}$ spanned by the vectors $\mathbf{P}$ and $\mathbf{Q}$: $\mathbf{E}(\mathbf{P}, \mathbf{Q}) = \{\mathbf{S} \in \boldsymbol{\Xi}^\mathbf{n}_{\mathbf{m}+1} / \mathbf{P} \leq \mathbf{S} \leq \mathbf{Q}\}$. This way we have $\mathbf{I}(\hat{\mathbf{S}}) = \mathbf{E}(\check{\mathbf{S}}, \hat{\mathbf{S}})$.

Corollary 2. 1. $\hat{\mathbf{H}} \cap \psi_\mathbf{m} = \bigcup_{\mathbf{j=1}}^{\mathbf{r}} \mathbf{E}(\tilde{\mathbf{m}}_{\mathbf{mid}+}, \hat{\mathbf{S}}_\mathbf{j})$

2. $\check{\mathbf{H}} \cap \psi_\mathbf{m} = \bigcup_{\mathbf{j=1}}^{\mathbf{r}} \mathbf{E}(\check{\mathbf{S}}_\mathbf{j}, \tilde{\mathbf{m}}_{\mathbf{mid}-})$

Corollary 3. 1. $\psi_\mathbf{m} = \bigcup_{\mathbf{j=1}}^{\mathbf{r}} \mathbf{V}(\mathbf{E}(\tilde{\mathbf{m}}_{\mathbf{mid}+}, \hat{\mathbf{S}}_\mathbf{j}))$

2. $\psi_\mathbf{m} = \bigcup_{\mathbf{j=1}}^{\mathbf{r}} \mathbf{V}(\mathbf{E}(\check{\mathbf{S}}_\mathbf{j}, \tilde{\mathbf{m}}_{\mathbf{mid}-}))$

Similar results are true for the structure of the complement $\boldsymbol{\Xi}^\mathbf{n}_{\mathbf{m}+1} \setminus \psi_\mathbf{m}$ of $\boldsymbol{\psi}_\mathbf{m}$.

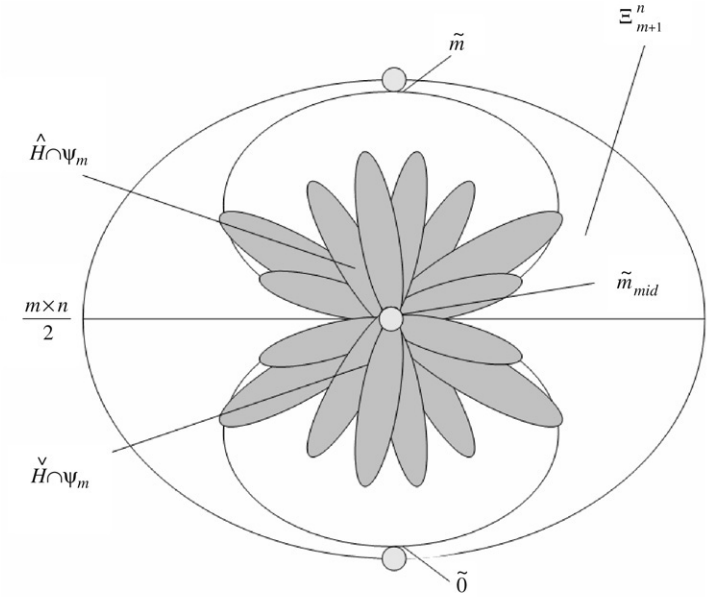

Fig. 3. Illustrates the sets $\hat{\mathbf{H}} \cap \psi_{\mathbf{m}}$ and $\check{\mathbf{H}} \cap \psi_{\mathbf{m}}$ for even $\mathbf{m}$.

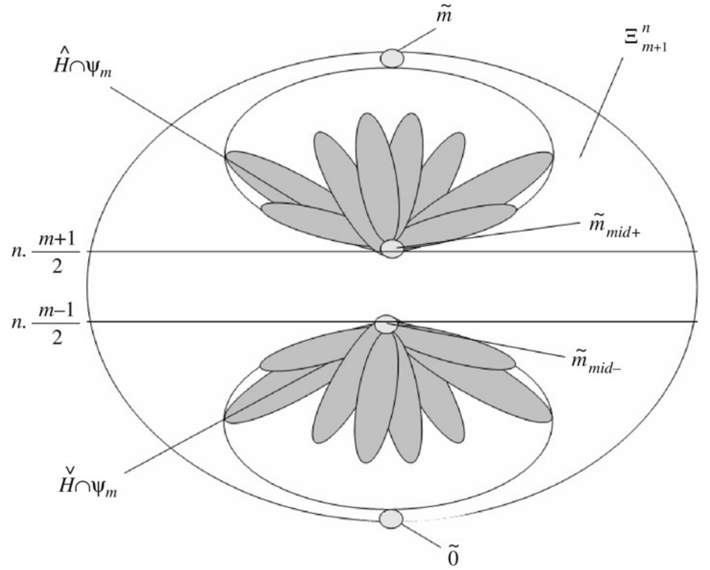

Fig. 4. Illustrates the sets $\hat{\mathbf{H}} \cap \psi_{\mathbf{m}}$ and $\check{\mathbf{H}} \cap \psi_{\mathbf{m}}$ for odd $\mathbf{m}$.

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