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On the Wadge reducibility of k-partitions[☆]

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Dedicated to Dieter Spreen on the occasion of his 60th birthday

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ABSTRACT

We establish some results on the Wadge degrees and on the Boolean hierarchy of k-partitions of some spaces, where k is a natural number. The main attention is paid to the Baire space, Baire domain and their close relatives. For the case of $\mathbf{\Delta}^0_2$-measurable k-partitions the structures of Wadge degrees are characterized completely. For many degree structures, undecidability of the first-order theories is shown, for any $k \geq 3$.

1. Introduction

The Wadge reducibility of subsets of the Baire space [26,27] is a classical object of descriptive set theory. In this paper, which is a slight extension of the conference paper [23], we consider a generalization of the Wadge reducibility from the case of subsets A of a topological space X (identified with the characteristic functions $c_A : X \to \{0, 1\}$) to the case of k-partitions $\nu : X \to k$ of X (such functions ν are in a natural bijective correspondence with tuples $(A_0, \ldots, A_{k-1})$ of pairwise disjoint sets with $A_0 \cup \cdots \cup A_{k-1} = X$) for an integer $k \geq 2$ which is identified with the set $\{0, \ldots, k-1\}$. Study of the Wadge reducibility of k-partitions was initiated in [3,25,13,5,6,16,17,19,22].[1]

We establish some results on the Wadge degrees and on the Boolean hierarchy of k-partitions of some spaces, where k is a natural number. The main attention is paid to the Baire space, Baire domain and their close relatives. For the case of $\mathbf{\Delta}^0_2$-measurable k-partitions the structures of Wadge degrees are characterized completely. For many degree structures, undecidability of the first-order theories is shown, for any $k \geq 3$.

We start in Section 2 with introducing some notation. In Sections 3 and 4 we describe some relevant classes of posets, remind some known and establish some new observations about them. Section 5 recalls definition and properties of the Boolean hierarchy of k-partitions. In Section 6 we discuss some substructures of the structure of Wadge degrees in the Baire and Cantor spaces. In Section 7 we establish some facts on the Wadge reducibility in two natural classes of ω-algebraic domains while in Section 8 we provide additional information about this structure in the Baire and Cantor domains. We conclude in Section 9 with a short discussion and open questions.

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2. Notation

Levels of the Borel hierarchy in a space X are denoted $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Delta}^0_\alpha$, for $\alpha < \omega_1$, so, in particular, $\mathbf{\Pi}^0_\alpha = co\text{-}\mathbf{\Sigma}^0_\alpha$ is the set of complements of $\mathbf{\Sigma}^0_\alpha$-sets and $\mathbf{\Delta}^0_\alpha = \mathbf{\Sigma}^0_\alpha \cap \mathbf{\Pi}^0_\alpha$. By $\mathbf{B}$ we denote the class of Borel sets. If the space is not clear from the context, we may use more exact notation like $\mathbf{\Sigma}^0_n(X)$ or $\mathbf{B}(X)$.

Let X be a space, $\mu, \nu : X \to k$ be k-partitions of X and $\mathcal{C}$ a class of k-partitions of X. We say that μ is *Wadge reducible* to ν (in symbols, $\mu \leq_W \nu$) if $\mu = \nu \circ f$ for some continuous function f on X. For $k = 2$ this definition coincides with the Wadge reducibility of subsets of X. Let $\mathcal{C} \leq_W \nu$ denote that any element of $\mathcal{C}$ is Wadge reducible to ν, and $\nu \equiv_W \mathcal{C}$ denote that ν is Wadge complete in $\mathcal{C}$, i.e. $\nu \in \mathcal{C}$ and $\mathcal{C} \leq_W \nu$.

Since for many natural spaces (e.g., for the space of reals) the structure of Wadge degrees of $\mathbf{\Delta}^0_2$ is complicated [6] we restrict our attention mainly to the Baire space, Baire domain and some of their close relatives. For such spaces we give a complete characterization of the structure of Wadge degrees of $\mathbf{\Delta}^0_2$-measurable k-partitions. We extend the main facts about the Hausdorff difference hierarchy of sets in the Baire space [7] and in the ω-algebraic domains [18] to the case of k-partitions. We also show that many substructures of the Wadge degrees of k-partitions have undecidable first-order theories for $k \geq 3$. Recall that *first-order theory FO*(A) of a structure A of signature σ is the set of first-order sentences of signature σ which are true in A. A theory of signature σ is *hereditary undecidable* if any of its subtheories of the same signature σ is undecidable. Of course, any hereditary undecidable theory is undecidable.

We use standard set-theoretic notation. The class of subsets of X is denoted $P(X)$. For any class $\mathcal{C} \subseteq P(X)$, let $co\text{-}\mathcal{C}$ be the class of all complements of sets in $\mathcal{C}$, $BC(\mathcal{C})$ the Boolean closure of $\mathcal{C}$ and $\mathcal{C}_k$ the set of $\mathcal{C}$-partitions (or, more exactly, $\mathcal{C}$-measurable k-partitions), i.e. partitions $\nu \in k^X$ such that $\nu^{-1}(i) \in \mathcal{C}$ for each $i < k$. We assume the reader to be familiar with the notion of ordinal, in particular with the first non-countable ordinal ω_1.

Let us recall definition of the Baire and Cantor spaces and domains that are of primary importance for mathematics and computer science. Let ω^* be the set of finite sequences (strings) of natural numbers. Let ω^+ be the set of finite non-empty strings of natural numbers. The empty string is denoted by $\emptyset$, the concatenation of strings σ, τ by $\sigma^\frown\tau$ or just by $\sigma\tau$. By $\sigma \sqsubseteq \tau$ we denote that the string σ is an initial segment of the string τ (please be careful in distinguishing $\sqsubseteq$ and $\subseteq$). Let ω^ω be the set of all infinite sequences of natural numbers (i.e., of all functions $\xi : \omega \to \omega$). For $\sigma \in \omega^*$ and $\xi \in \omega^\omega$, we write $\sigma \sqsubseteq \xi$ to denote that σ is an initial segment of the sequence ξ. Define a topology on ω^ω by taking arbitrary unions of sets of the form $\{\xi \in \omega^\omega \mid \sigma \sqsubseteq \xi\}, \sigma \in \omega^*$, as open sets. The space ω^ω with this topology is known as the *Baire space*, and the subspace 2^ω of ω^ω is known as the *Cantor space*. It is well known that 2^ω is homeomorphic to the space n^ω for each n, $2 \leq n < \omega$.

The *Baire domain* is the set $\omega^{\leq\omega} = \omega^* \cup \omega^\omega$ of finite and infinite strings of natural numbers, with the unions of sets of the form $\{\xi \in \omega^{\leq\omega} \mid \sigma \sqsubseteq \xi\}, \sigma \in \omega^*$, as open sets. For any $2 \leq n < \omega$, the *Cantor domain* is the set $n^{\leq\omega} = n^* \cup n^\omega$ of finite and infinite words over the alphabet n considered as the subspace of the Baire domain. Note that the Cantor domains $n^{\leq\omega}$ and $m^{\leq\omega}$ are not homeomorphic for distinct n and m. A computability theory for the Baire domain was developed in [24].

We use some standard notation and terminology on partially ordered sets (posets) which may be found e.g. in [2]. We will not be very cautious when applying notions about posets also to preorders; in such cases we mean the corresponding quotient-poset of the preorder. A poset $(P; \leq)$ will be often shorter denoted just by P. Any subset of a poset P may be considered as a poset with the induced partial order. In particular, this applies to the "upper cones" $\check{x} = \{y \in P \mid x \leq y\}$ defined by any $x \in P$. A *well preorder* is a preorder P that has neither infinite descending chains nor infinite antichains. For such preorders (as well as for the well-founded preorders) there is a canonical rank function rk_P assigning ordinals to the elements of P; rank of P is by definition the supremum of ranks of its elements. With any well preorder P we associate also its *width* $w(P)$ defined as follows: if P has antichains with any finite number of elements, then $w(P) = \omega$, otherwise $w(P)$ is the greatest natural number n for which P has an antichain with n elements.

We conclude this section with introducing some more special terminology. By *a base in* X we mean a class $\mathcal{L} \subseteq P(X)$ closed under finite unions and intersections. A base $\mathcal{L}$ is a σ-base if it is closed also under countable unions. As is well-known, any level $\mathbf{\Sigma}^0_\alpha$, $\alpha > 0$, of the Borel hierarchy in X is a σ-base. A base $\mathcal{L}$ is *reducible* if it has the reduction property [7], i.e. for all $C_0, C_1 \in \mathcal{L}$ there are disjoint $C'_0, C'_1 \in \mathcal{L}$ such that $C'_i \subseteq C_i$ for both $i < 2$ and $C_0 \cup C_1 = C'_0 \cup C'_1$. A base $\mathcal{L}$ is σ*-reducible*, if for each countable sequence $C_0, C_1, \ldots$ in $\mathcal{L}$ there is a countable sequence $C'_0, C'_1, \ldots$ in $\mathcal{L}$ (called a reduct of $C_0, C_1, \ldots$) such that $C'_i \cap C'_j = \emptyset$ for all $i \neq j$ and $\bigcup_{i<\omega} C'_i = \bigcup_{i<\omega} C_i$. It is well-known [7] that any class $\mathbf{\Sigma}^0_\alpha$, $\alpha > 1$, is σ-reducible. For the class $\mathbf{\Sigma}^0_1$ of open sets the situation is more subtle: it is σ-reducible for some natural spaces (e.g., for the Baire and Cantor spaces and domains) while it is not reducible for some other natural spaces (e.g., for the space of reals).

3. Discrete weak semilattices

In this section we summerize some auxiliary algebraic notions and facts. Recall that *semilattice* is a structure $(P; \leq, \cup)$ consisting of a preorder $(P; \leq)$ and a binary operation $\cup$ of supremum in $(P; \leq)$ (thus, we consider only upper semilattices). By σ*-semilattice* we mean a semilattice in which every countable set of elements has a supremum. With a slight abuse of notation, we apply the operation $\cup$ also to finite non-empty subsets of P. This causes no problem because the supremum of any non-empty finite set is unique up to the equivalence relation $\equiv$ induced by $\leq$.

We start with a definition which is a slight modification of the corresponding notions introduced in [14,15].

Definition 3.1. Let I be a non-empty set. By I-discrete weak semilattice (dws, for short) we mean a structure $(P; \leq, \{P_i\}_{i \in I})$ with $P_i \subseteq P$ such that:

(i) $(P; \leq)$ is a preorder;

(ii) for all $n < \omega$, $x_0, \ldots, x_n \in P$ and $i \in I$ there exists $u_i = u_i(x_0, \ldots, x_n) \in P_i$ which is a supremum for $x_0, \ldots, x_n$ in P_i, i.e. $\forall j \leq n(x_j \leq u_i)$ and for any $y \in P_i$ with $\forall j \leq n(x_j \leq y)$ it holds $u_i \leq y$;

(iii) for all $n < \omega$, $x_0, \ldots, x_n \in P$, $i \neq i' \in I$ and $y \in P_{i'}$, if $y \leq u_i(x_0, \ldots, x_n)$ then $y \leq x_j$ for some $j \leq n$.

By σ-dws we mean a dws $(P; \leq, P_0, \ldots, P_{k-1})$ that has the same properties also for all ω-sequences $x_0, x_1, \ldots$ in P.

Throughout the paper, we are interested in the case when $I = k$ for some integer $k \geq 2$; in this case we write the dws also in the form $(P; \leq, P_0, \ldots, P_{k-1})$. Note that the operations u_i above may be considered as n-ary operations on P for each $n > 0$ (in σ-dws's even as ω-ary operations). These operations are associative and commutative. The following properties of dws's are immediate (see [14,15]).

Proposition 3.2. *Let $(P; \leq, P_0, \ldots, P_{k-1})$ be a dws and $y, x_0, \ldots, x_n \in P_0 \cup \cdots \cup P_{k-1}$.*

(i) *If $x_j \leq y$ for all $j \leq n$ then $u_i(x_0, \ldots, x_n) \leq y$ for some $i < k$.*

(ii) *If $y \leq u_i(x_0, x \ldots, x_n)$ for all $i < k$ then $y \leq x_j$ for some $j \leq n$.*

(iii) *If $\{x_0, \ldots, x_n\}$ has no greatest element then it has no supremum in $P_0 \cup \cdots \cup P_{k-1}$.*

Note that if $(P; \leq, P_0, \ldots, P_{k-1})$ is a σ-dws then the last proposition holds true also for ω-sequences $x_0, x_1, \ldots \in P_0 \cup \cdots \cup P_{k-1}$.

The next easy assertion shows that considering of only binary operations u_i is sufficient to recover the structure of a dws.

Proposition 3.3

(i) *Let $(P; \leq, P_0, \ldots, P_{k-1})$ be a dws. Then the binary operations $u_0, \ldots, u_{k-1}$ on P have for all $x, y, z, t \in P$ and distinct $i, j < k$ the following properties: $x, y \leq u_i(x, y)$; $x, y \leq u_i(z, t) \to u_i(x, y) \leq u_i(z, t)$; $u_j(z, t) \leq u_i(x, y) \to (u_j(z, t) \leq y \vee u_j(z, t) \leq z)$.*

(ii) *Let $(P; \leq)$ be a preorder and $u_0, \ldots, u_{k-1}$ binary operations on P satisfying the properties in (i). Then $(P; \leq, P_0, \ldots, P_{k-1})$, where $P_i = \{u_i(x, y) \mid x, y \in P\}$, is a dws.*

(iii) *The maps $(P; \leq, P_0, \ldots, P_{k-1}) \mapsto (P; \leq, u_0, \ldots, u_{k-1})$ and back are inverses of each other, up to isomorphism of the quotient-structures.*

By the last proposition, we may apply the term "dws" also to the structures $(P; \leq, u_0, \ldots, u_{k-1})$ satisfying the three properties in (i). Note that the class of dws's written in this form is universally axiomatizable, so any substructure of a dws $(P; \leq, u_0, \ldots, u_{k-1})$ is also a dws.

Note that for any dws the unary operations u_i are closure operators on $(P; \leq)$, i.e. they satisfy $\forall x(x \leq u_i(x))$, $\forall x \forall y(x \leq y \to u_i(x) \leq u_i(y))$ and $\forall x(u_i(u_i(x)) \leq u_i(x))$. They have also the discreteness property: $\forall x \forall y(u_i(x) \leq u_j(y) \to u_i(x) \leq y)$, for all $i \neq j$. This shows a close relation of dws's to the semilattices with discrete closures (dc-semilattices, for short) introduced in [15].

Definition 3.4. By semilattice with discrete closures (dc-semilattice for short) we mean a structure $(S; \leq, \cup, p_0, \ldots, p_{k-1})$ satisfying the following axioms:

(1) $(S; \cup)$ is an upper semilattice, i.e. it satisfies $(x \cup y) \cup z = x \cup (y \cup z)$, $x \cup y = y \cup x$ and $x \cup x = x$, for all $x, y, z \in S$.

(2) $\leq$ is the partial order on S induced by $\cup$, i.e. $x \leq y$ iff $x \cup y = y$, for all $x, y \in S$.

(3) Every p_i, $i < k$, is a closure operation on $(S; \leq)$, i.e. it satisfies $x \leq p_i(x)$, $x \leq y \to p_i(x) \leq p_i(y)$ and $p_i(p_i(x)) \leq p_i(x)$, for all $x, y \in S$.

(4) The operations p_i have the following discreteness property: for all distinct $i, j < k$, $p_i(x) \leq p_j(y) \to p_i(x) \leq y$, for all $x, y \in S$.

(5) Every $p_i(x)$ is join-irreducible, i.e. $p_i(x) \leq y \cup z \to (p_i(x) \leq y \vee p_i(x) \leq z)$, for all $x, y, z \in S$.

By $dc\sigma$-semilattice we mean a dc-semilattice $(S; \leq, \cup, p_0, \ldots, p_{k-1})$ such that $(S; \cup)$ is a σ-semilattice and the axiom 5) of dc-semilattices holds also for supremums of countable subsets of S, i.e. $p_i(x) \leq \bigcup_{j<\omega} y_j$ implies that $p_i(x) \leq y_j$ for some $j < \omega$; we express this by saying that $p_i(x)$ is σ-join-irreducible.

The next easy assertion shows that dws's that are semilattices essentially coincide with the dc-semilattices.

Proposition 3.5

(i) *Let $(P; \leq, P_0, \ldots, P_{k-1})$ be a dws and $(P; \leq, \cup)$ is a semilattice. Then the structure $(P; \leq, \cup, u_0, \ldots, u_{k-1})$ with the unary operations u_i on P is a dc-semilattice.*

(ii) *If $(P; \leq, \cup, p_0, \ldots, p_{k-1})$ is a dc-semilattice then $(P; \leq, P_0, \ldots, P_{k-1})$, where $P_i = \{p_i(x) \mid x \in P\}$ is a dws.*

(iii) *The maps $(P; \leq, \cup, P_0, \ldots, P_{k-1}) \mapsto (P; \leq, \cup, u_0, \ldots, u_{k-1}))$ and back are inverses of each other, up to isomorphism of the quotient-structures.*

(iv) *Similar relationship exists between $dc\sigma$-semilattices and σ-dws's.*

In [15] we considered also some variations of dws's and dc-semilattices. By *2-dws* we mean a structure $(P; \leq, \{P_i^j\}_{i,j \in I})$ with the properties similar to those of dws's with the only exception: this time the property (iii) states that for all $n < \omega$, $x_0, \ldots, x_n \in P$, $i \neq i', j \neq j'$ and $y \in P_{i'}^{j'}$, if $y \leq u_i^j(x_0, \ldots, x_n)$ then $y \leq x_l$ for some $l \leq n$. By a *2-dc-semilattice* we mean a structure $(P; \leq, \cup, \{r_i^j\}_{i,j \in I})$ satisfying the same properties as *dc*-semilattices with a similar modification of the discreteness property: for all $x, y \in P, i \neq i', j \neq j', r_i^j(x) \leq r_{i'}^{j'}(y) \to r_i^j(x) \leq y$. Analogs of Propositions 3.2–3.5 are easily seen to be true also for 2-dws's and 2-*dc*-semilattices. We state also the following evident relationship between the introduced notions.

Proposition 3.6. (i) *If* $(P; \leq, \{P_i^j\}_{i,j \in I})$ *is a 2-dws then* $(P; \leq, \{P_i^i\}_{i \in I})$ *is a dws.*

(ii) *If* $(P; \leq, \cup, \{r_i^j\}_{i,j \in I})$ *is a 2-dc-semilattice then* $(P; \leq, \cup, \{r_i^i\}_{i \in I})$ *is a dc-semilattice.*

(iii) *Similar relationship exists between 2-σdc-semilattices and 2-σ-dws's.*

In [21] it was shown that most non-trivial dws's and 2-dws's have undecidable first-order theories. In particular, the following fact holds true.

Proposition 3.7. *Let $k \geq 3$ and P be a dws or a 2-dws that is not linearly ordered. Then FO(P) is hereditary undecidable.*

4. The homomorphic preorder

In this section we recall some definitions and facts about the so called homomorphic preorders studied in [5,6,8,9,16, 19,10,11], and make some additional remarks. The homomorphic preorders provide minimal models for some theories discussed in the previous section. Most posets considered here are assumed to be (at most) countable and without infinite chains. The absence of infinite chains in a poset $(P; \leq)$ is of course equivalent to well-foundednes of both $(P; \leq)$ and $(P; \geq)$.

By *a forest* we mean a poset without infinite chains in which every upper cone $\check{x}$ is a chain. *A tree* is a forest having the biggest element (called *the root* of the tree). Let $(T; \leq)$ be a tree without infinite chains; in particular, it is well-founded. As for each well-founded partial order, there is a canonical rank function rk_T from T to ordinals defined by $rk_T(x) = sup\{rk_T(y) + 1 \mid y < x\}$. The rank $rk(T)$ of $(T; \leq)$ is by definition the ordinal $rk_T(r)$, where r is the root of $(T; \leq)$. It is well-known that rank of any countable tree without infinite chains is a countable ordinal, and any countable ordinal is the rank of such a tree. Since $(\omega^*; \sqsubseteq)$ is the infinite branching tree, any tree (resp., forest) $(P; \leq)$ without infinite chains is isomorphic to a tree (resp., forest) $(P'; \sqsupseteq)$ where P' is an initial segment of $(\omega^*; \sqsubseteq)$ (resp., of $(\omega^+; \sqsubseteq)$).

A *k-poset* is a triple $(P; \leq, c)$ consisting of a poset $(P; \leq)$ and a labeling $c : P \to k$. Rank of a k-tree (or a k-poset) $(T; \leq, c)$ is by definition the rank of $(T; \leq)$. A *morphism* $f : (P; \leq, c) \to (P'; \leq', c')$ between k-posets is a monotone function $f : (P; \leq) \to (P'; \leq')$ respecting the labelings, i.e. satisfying $c = c' \circ f$. Let $\widetilde{\mathcal{P}}_k$, $\widetilde{\mathcal{F}}_k$, $\widetilde{\mathcal{T}}_k$ and $\widetilde{\mathcal{T}}_k^i$ denote the classes of all countable k-posets, countable k-forests, countable k-trees and countable i-rooted k-trees without infinite chains, respectively. The homomorphic preorder $\leq$ on $\widetilde{\mathcal{P}}_k$ is defined as follows: $(P, \leq, c) \leq (P', \leq', c')$, if there is a morphism from $(P, \leq, c)$ to $(P', \leq', c')$. Let $\mathcal{P}_k$, $\mathcal{F}_k$, $\mathcal{T}_k$ and $\mathcal{T}_k^i$ be the subsets of the corresponding tilde-sets formed by finite posets only. Note that the empty poset $\emptyset$ is assumed to be in $\mathcal{F}_k$ but not in $\widetilde{\mathcal{T}}_k$; it is the smallest element of $(\widetilde{\mathcal{P}}_k; \leq)$.

As observed in [8,16], the structure $(\widetilde{\mathcal{P}}_k; \leq)$ has for $k > 2$ infinite antichains and infinite descending chains. In contrast, the following result from [19] shows that the structure of k-forests has much better properties. This is of interest for our topic because the structure is closely related to the Wadge reducibility of k-partitions. We call a k-tree $(T; \leq, c) \in \widetilde{\mathcal{T}}_k$ *repetition free* if $c(x) \neq c(y)$ whenever y is an immediate successor of x in $(T; \leq)$.

Proposition 4.1. (i) *For any $k \geq 2$, $(\widetilde{\mathcal{F}}_k; \leq)$ is a well preorder of rank ω_1.*

(ii) *For any $k \geq 2$, $(\mathcal{F}_k; \leq)$ is an initial segment of $(\widetilde{\mathcal{F}}_k; \leq)$ that consists exactly of the elements of finite rank.*

(iii) *$w(\widetilde{\mathcal{F}}_2) = 2$ and $w(\widetilde{\mathcal{F}}_k) = \omega$ for $k > 2$.*

(iv) *For any $k \geq 2$, the quotient structure of $(\widetilde{\mathcal{F}}_k; \leq)$ is a distributive lattice and a σ-semilattice.*

(v) *Every $T \in \widetilde{\mathcal{T}}_k$ is equivalent to some repetition free $S \in \widetilde{\mathcal{T}}_k$.*

Let $P \sqcup Q$ be the join of k-posets P, Q and $\bigsqcup_i P_i = P_0 \sqcup P_1 \sqcup \cdots$ the join of an infinite sequence $P_0, P_1, \ldots$ of k-posets. For a k-forest F and $i < k$, let $p_i(F)$ be the k-tree obtained from F by adjoining a new biggest element and assigning the label i to this element. It is clear that the introduced operations respect the homomorphic equivalence and that any finite (countable) k-forest is equivalent to a finite (respectively, countable) term of signature $\{\sqcup, p_0, \ldots, p_{k-1}, 0, \ldots, k-1\}$ without free variables (the constant symbol i in the signature is interpreted as the singleton tree carrying the label i). For k-trees $T_0, T_1, \ldots$ and $i < k$, define the k-tree $U_i(T_0, T_1, \ldots) = p_i(T_0 \sqcup T_1 \sqcup \cdots)$. The following facts were observed in [16,19].

Proposition 4.2. (i) *The quotient structure of* $(\widetilde{\mathcal{F}}_k; \sqcup, p_0, \ldots, p_{k-1})$ (*resp., of* $(\mathcal{F}_k; \sqcup, p_0, \ldots, p_{k-1})$) *is a dc*$\sigma$*-semilattice* (*resp., a dc-semilattice*).

(ii) *The quotient structure of* $(\widetilde{\mathcal{T}}_k; \leq, \widetilde{\mathcal{T}}_k^0, \ldots, \widetilde{\mathcal{T}}_k^{k-1})$ (*of* $(\mathcal{T}_k; \leq, \mathcal{T}_k^0, \ldots, \mathcal{T}_k^{k-1})$) *is a* σ*-dws* (*resp., a dws*).

For a *dc*σ-semilattice S, let $\sigma ji(S)$ ($ji(S)$) denote the set of σ-join-irreducible (resp., join-irreducible) elements of S, then of course $\sigma ji(S) \subseteq ji(S)$. The next assertion characterizes irreducible elements in $\widetilde{\mathcal{F}}_k$ and gives a canonical representation of elements in this lattice. By *canonical representation* of $x \in \widetilde{\mathcal{F}}_k$ we mean a representation $x = \bigsqcup \mathcal{Y}$ for some finite antichain $\mathcal{Y} \subseteq ji(\widetilde{\mathcal{F}}_k)$. The following fact was established in [22], Proposition 6.3.

Proposition 4.3. (i) $\sigma ji(\widetilde{\mathcal{F}}_k) = \widetilde{\mathcal{T}}_k$.

(ii) $ji(\widetilde{\mathcal{F}}_k)$ *coincides with the set of elements of the form* $T_0 \sqcup T_1 \sqcup \cdots$, *for some ascending chain* $T_0 \leq T_1 \leq \cdots$ *in* $\widetilde{\mathcal{T}}_k$.

(iii) *Any element of* $\widetilde{\mathcal{F}}_k$ *has a unique canonical representation*.

The following result shows that the structures from Proposition 4.2 have natural minimality properties.

Proposition 4.4. (i) *Let* $(S; \leq, \cup, p_0, \ldots, p_{k-1})$ *be a dc*σ*-semilattice and let a be an element of S such that* $a < p_i(a)$ *for all* $i < k$. *Then the sub-dc*σ*-semilattice* (a) *of S generated by a is isomorphic to the quotient structure of* $(\widetilde{\mathcal{F}}_k; \leq, \sqcup, p_0, \ldots, p_{k-1})$. *A similar assertion holds true for the dc-semilattices and* $\mathcal{F}_k$.

(ii) *Let* $(S; \leq, u_0, \ldots, u_{k-1})$ *be a* σ*-dws and* $\{a_0, \ldots, a_{k-1}\}$ *an antichain in* $(S; \leq)$. *Then the sub-*σ*-dws* $(a_0, \ldots, a_{k-1})$ *of S generated by* $\{a_0, \ldots, a_{k-1}\}$ *is isomorphic to the quotient structure of* $(\widetilde{\mathcal{T}}_k; \leq, U_0, \ldots, U_{k-1})$. *A similar assertion holds true for the dws's and* $\mathcal{T}_k$.

Proof (sketch). The assertion (i) was proved in [19]. First define a function $f : \widetilde{\mathcal{T}}_k \to (a)$ by induction on the rank of trees as follows: if T is a singleton tree carrying the label i then $f(T) = p_i(a)$; if $T = p_i(T_0 \sqcup T_1 \sqcup \cdots)$ is not singleton and $T_j \in \widetilde{\mathcal{T}}_k \setminus \widetilde{\mathcal{T}}_k^i$ (such a representation exists by Proposition 4.1(v)) then $f(T) = p_i(f(T_0) \sqcup f(T_1) \sqcup \cdots)$. Now extend f to the set $\widetilde{\mathcal{F}}_k$ in the natural way: for every countable set $\mathcal{U} \subseteq \widetilde{\mathcal{T}}_k$, set $f(\sqcup \mathcal{U}) = \sqcup \{f(T) \mid T \in \mathcal{U}\}$. It is not hard to see that in this way we obtain a correctly defined function f from the quotient structure of $\widetilde{\mathcal{F}}_k$ onto (a). An induction shows that this function is indeed a desired isomorphism. Note that the function f actually preserves countable unions of arbitrary elements.

(ii) is checked in a similar way, so we define only the function $f : \widetilde{\mathcal{T}}_k \to (a_0, \ldots, a_{k-1})$ by induction on the rank of trees as follows: if T is a singleton tree carrying the label i then $f(T) = a_i$; if $T = p_i(T_0 \sqcup T_1 \sqcup \cdots)$ is not singleton and $T_j \in \widetilde{\mathcal{T}}_k \setminus \widetilde{\mathcal{T}}_k^i$ then $f(T) = U_i(f(T_0), f(T_1), \ldots)$. $\square$

In [10,11] some facts about definability, automorphisms and undecidability in the introduced structures were established, e.g.:

Proposition 4.5. *For any* $k \geq 3$, *each element of the quotient structure of* $(\mathcal{F}_k; \leq, 0, \ldots, k-1)$ *is first-order definable. The same is true for the quotient structure of* $(\mathcal{T}_k; \leq, 0, \ldots, k-1)$.

In [12] we show that similar definability result holds true also for $(\widetilde{\mathcal{F}}_k; \leq, 0, \ldots, k-1)$ and $(\widetilde{\mathcal{T}}_k; \leq, 0, \ldots, k-1)$, only in this case we have to replace first-order definability by $L_{\omega_1,\omega}$-definability.

Let $\mathbf{S}_k$ be the symmetric group on k elements, i.e. the group of permutations of the elements $0, \ldots, k-1$. Let $Aut(A)$ denote the automorphism group of a structure A. By $\simeq$ we denote the isomorphism relation. The next result is a straightforward generalization of the corresponding fact in [11].

Proposition 4.6. (i) *For any* $k \geq 2$ *we have* $Aut(\mathcal{F}_k; \leq) \simeq Aut(\mathcal{T}_k; \leq)$ *and* $Aut(\widetilde{\mathcal{F}}_k; \leq) \simeq Aut(\widetilde{\mathcal{T}}_k; \leq)$.

(ii) $Aut(\mathcal{T}_2; \leq) \simeq \mathbf{S}_2^{\omega}$ *and* $Aut(\widetilde{\mathcal{T}}_2; \leq) \simeq \mathbf{S}_2^{\omega_1}$.

(iii) *For any* $k \geq 3$, $Aut(\mathcal{F}_k; \leq) \simeq \mathbf{S}_k \simeq Aut(\widetilde{\mathcal{F}}_k; \leq)$.

(iv) *For all* $k \geq 2$ *and* $i < k$, $Aut(\mathcal{T}_k^i; \leq) \simeq \mathbf{S}_{k-1} \simeq Aut(\widetilde{\mathcal{T}}_k^i; \leq)$.

The next fact strengthens Proposition 3.7 for the structures considered here.

Proposition 4.7. (i) *For all* $k > 2$ *and* $i < k$, *the first-order theories of the quotient structures of* $(\mathcal{F}_k; \leq), (\mathcal{T}_k^i; \leq)$ *and* $(\mathcal{T}_k; \leq)$ *are computably isomorphic to the first-order arithmetic* $FO(\omega, +, \cdot)$.

(ii) *For all* $k > 2$ *and* $i < k$, *the theory* $FO(\omega; +, \cdot)$ *is m-reducible to any of the theories* $FO(\widetilde{\mathcal{F}}_k; \leq), FO(\widetilde{\mathcal{T}}_k; \leq), FO(\widetilde{\mathcal{T}}_k^i; \leq)$.

Proof. (i) is established in [10].

(ii) We consider only $FO(\widetilde{\mathcal{F}}_k; \leq)$, the other cases are similar. By (i), it suffices to m-reduce $FO(\mathcal{F}_k; \leq)$ to $FO(\widetilde{\mathcal{F}}_k; \leq)$. For this it suffices to show that the set $\mathcal{F}_k$ is first-order definable in $(\widetilde{\mathcal{F}}_k; \leq)$. By Proposition 4.1, $\mathcal{F}_k$ coincides with the set of elements $x \in \widetilde{\mathcal{F}}_k$ such that any $y \leq x$ is either minimal or has an immediate predecessor; this informal definition of $\mathcal{F}_k$ may be written as a first-order formula of signature $\{\leq\}$. $\square$

Note that, in contrast with the last proposition, the first-order theories of all structures above are decidable for $k = 2$.

5. The Boolean hierarchy

In this section we recall definitions and some facts related to the Boolean hierarchy of k-partitions studied for the case of finite k-posets in [8,9,16] and for the countable case in [19,22].

Let $P = (P; \leq)$ be a countable poset without infinite chains, X a space and $\mathcal{L}$ a σ-base in X (see Section 2). Functions of the form $S : P \to \mathcal{L}$ are called P-families and are denoted also by $\{S_p\}_{p \in P}$. A P-family is *monotone* if it is a monotone function from $(P; \leq)$ into $(\mathcal{L}; \subseteq)$. A P-family S is *admissible* if $\bigcup_p S_p = X$ and $S_p \cap S_q = \bigcup\{S_r \mid r \leq p, q\}$ for all $p, q \in P$. Note that any admissible P-family is monotone. Note also that if P is a forest then a P-family S is admissible iff it is monotone, $\bigcup_p S_p = X$ and $S_p \cap S_q = \emptyset$ for all p, q incomparable in P. For any P-family S, define a map $\tilde{S} : P \to P(X)$ by $\tilde{S}_p = S_p \setminus \bigcup_{q<p} S_q$. It is easy to see that if S is admissible then $\{\tilde{S}_p\}_{p \in P}$ is a partition of X.

For a countable k-poset (P, c) without infinite chains, let $\mathcal{L}[P, c] = \{c \circ \tilde{S} \mid S \in H(P, \mathcal{L})\}$ where $H(P, \mathcal{L})$ is the set of admissible P-families and $\tilde{S}$ is identified with the function from X to P sending $x \in X$ to the unique $p \in P$ with $x \in \tilde{S}_p$. Note that $\mathcal{L}[P, c] \subseteq k^X$, i.e. $\mathcal{L}(P, c)$ is a class of k-partitions of X. *The Boolean hierarchy of k-partitions over $\mathcal{L}$* is by definition the family $\{\mathcal{L}[P]\}_{P \in \widetilde{\mathcal{P}}_k}$; by $BH_k(\mathcal{L})$ we denote the collection $\{\mathcal{L}[P] \mid P \in \widetilde{\mathcal{P}}_k\}$ of levels of this hierarchy. We consider also a smaller collection of classes of k-partitions $FBH_k(\mathcal{L}) = \{\mathcal{L}[P] \mid P \in \widetilde{\mathcal{F}}_k\}$ defined by the k-forests.

In [8,19] it was observed that levels of the Boolean hierarchy are closely related to the homomorphic preorder, namely for all countable k-posets P and Q without infinite chains $P \leq Q$ implies $\mathcal{L}(P) \subseteq \mathcal{L}(Q)$. Since, by the preceding section, the homomorphic preorder of k-posets is far from being a well preorder, the Boolean hierarchy of k-partitions defined above does not in general have properties one expects from a hierarchy. In [16,19] it was shown that the situation is better for the Boolean hierarchies over σ-reducible bases.

Proposition 5.1. *Over any σ-reducible σ-base $\mathcal{L}, BH_k(\mathcal{L}) = FBH_k(\mathcal{L})$, and hence the poset $(BH_k(\mathcal{L}); \subseteq)$ is a well preorder of rank $\leq \omega_1$.*

The last result applies to the base $\mathcal{L} = \mathbf{\Sigma}^0_1$ of open sets in the Baire and Cantor spaces because this base is well-known to be σ-reducible (see Theorem 22.16 in [7]). For $k = 2$, the Boolean hierarchy over this base coincides with the Hausdorff difference hierarchy. For $k \geq 3$ we obtain an extension of the difference hierarchy of sets to the case of k-partitions considered in [22]).

The next easy fact establishes the reduction property for the Baire and Cantor domains.

Proposition 5.2. *The base $\mathbf{\Sigma}^0_1$ of open sets in the Baire and Cantor domains is σ-reducible.*

Proof. We consider the Baire domain but the argument works for the Cantor domains as well. Let $A_n \in \mathbf{\Sigma}^0_1$ for all $n < \omega$. Let B be the set of minimal elements in $(\omega^* \cap (\bigcup_{n<\omega} A_n); \sqsubseteq)$. For any $n < \omega$, set

$$B_n = \{u \in B \cap A_n \mid \forall m < n(u \notin A_m)\}, \ A'_n = \{\xi \in \omega^{\leq \omega} \mid \exists u \in B_n(u \sqsubseteq \xi)\}.$$

Then $\{B_n\}_{n<\omega}$ is a partition of B and $\{A'_n\}_{n<\omega}$ is a reduct of $\{A_n\}_{n<\omega}$. $\square$

In Sections 7 and 8 we develop a complete theory of the Boolean hierarchy and Wadge reducibility for some classes of domains.

6. Wadge reducibility in Baire and Cantor spaces

Here we consider the Wadge reducibility of k-partitions for the Baire and Cantor spaces. To our knowledge, the first result about the Wadge reducibility of k-partitions of the Baire and Cantor spaces is Theorem 3.2 in [3]. The following assertion is a particular case of that theorem.

Proposition 6.1. *Let $X \in \{\omega^\omega, 2^\omega\}$. Then the structure $(\mathbf{B}(X); \leq_W)$ of Borel-measurable k-partitions is a well preorder.*

This assertion gives important information about the structure $(\mathbf{B}(X); \leq_W)$ but it leaves open many questions. Let us introduce some algebraic structure on this preorder. Define an operation $\mu \oplus \nu$ on k-partitions of ω^ω by $(\mu \oplus \nu)(0 \cdot \xi) = \mu(\xi)$ and $(\mu \oplus \nu)(i \cdot \xi) = \nu(\xi)$ for all $0 < i < \omega$ and $\xi \in \omega^\omega$. For a sequence $\nu_0, \nu_1, \ldots$ of k-partitions of ω^ω, define the k-partition $\nu = \bigoplus_{i<\omega} \nu_i$ by $\nu(i \cdot \xi) = \nu_i(\xi)$, for all $i < \omega$ and $\xi \in \omega^\omega$. Note that the definition of the binary join operation $\mu \oplus \nu$ applies also to the Cantor space but the ω-ary one does not. This leads to some minor distinctions in the structures of Wadge degrees.

For a k-partition ν of ω^ω and $i < k$, define a k-partition $p_i(\nu)$ of ω^ω as follows: $[p_i(\nu)](\xi) = i$, if $\xi \notin 0^*1\omega^\omega$, and $[p_i(\nu)](\xi) = \nu(\eta)$, if $\xi = 0^n1\eta$ (here we use the self-evident notation in the style of regular expressions in automata theory). Note that the definition of p_i applies also to the Cantor space.

The next fact was established in [16,19].

Proposition 6.2. (i) *The quotient structures of* $(k^{\omega^\omega}; \leq_W, \oplus, p_0, \ldots, p_{k-1})$ *and of* $((\mathbf{\Delta}^0_2)_k; \leq_W, \oplus, p_0, \ldots, p_{k-1})$ *in the Baire space are* $dc\sigma$*-semilattices, and in the Cantor space they are dc-semilattices.*

(ii) *The quotient structure of* $((BC(\mathbf{\Sigma}^0_1))_k; \leq_W, \oplus, p_0, \ldots, p_{k-1})$ *in the Baire and Cantor spaces is a dc-semilattice.*

Now we turn to characterizing of some ideals of the Wadge preorder in the Baire space. Adjoining a new smallest element $\perp$ to $((\mathbf{\Delta}^0_2)_k$ with $p_i(\perp) = i$ we obtain (by Proposition 6.2) a $dc\sigma$-semilattice $((\mathbf{\Delta}^0_2)_k \cup \{\perp\}; \leq_W, \oplus, p_0, \ldots, p_{k-1})$. By Proposition 6.2 and the proofsketch of Proposition 4.4, there is a natural embedding $\mu : \widetilde{\mathcal{F}}_k \to (\mathbf{\Delta}^0_2)_k \cup \{\perp\}$ of $dc\sigma$-semilattices with $\mu(\emptyset) = \perp$. For a proof of the next result see [22].

Theorem 6.3. *The map* μ *induces an isomorphism of the quotient structure of* $(\widetilde{\mathcal{F}}_k \setminus \{\emptyset\}; \leq, \sqcup, p_0, \ldots, p_{k-1})$ *onto that of* $((\mathbf{\Delta}^0_2(\omega^\omega))_k; \leq_W, \oplus, p_0, \ldots, p_{k-1})$*. Moreover,* $\mu(F) \equiv_W \mathbf{\Sigma}^0_1[F]$ *for each* $F \in \widetilde{\mathcal{F}}_k$*.*

Note that from earlier unpublished work of P. Hertling (see Satz 6.2 b) in [5] and Theorem 2.2.4 in [6]) it follows that the quotient structures of $((BC(\mathbf{\Sigma}^0_1))_k; \leq_W)$ and of $(\mathcal{F}_k; \leq)$ are isomorphic. From the results above it follows that μ induces such an isomorphism and preserves the operations $p_0, \ldots, p_{k-1}$. Thus, we have

Theorem 6.4. *Let* $X \in \{\omega^\omega, 2^\omega\}$*. Then the quotient structure of structures* $((BC(\mathbf{\Sigma}^0_1))_k; \leq_W, \oplus, p_0, \ldots, p_{k-1})$ *and* $(\mathcal{F}_k; \leq, \sqcup, p_0, \ldots, p_{k-1})$ *are isomorphic.*

Taking into account that the Cantor space is compact it is not hard to see that the proof in [22] implies the following fact in which $(\widetilde{\mathcal{T}}_k)$ denotes the dc-semilattice generated by $\widetilde{\mathcal{T}}_k$ in the dc-semilattice $(\widetilde{\mathcal{F}}_k)$.

Corollary 6.5. *The restriction of* μ *to* $(\widetilde{\mathcal{T}}_k)$ *induces an isomorphism of the quotient structure of* $((\widetilde{\mathcal{T}}_k); \leq, \sqcup, p_0, \ldots, p_{k-1})$ *onto the quotient structure of* $S = ((\mathbf{\Delta}^0_2(2^\omega))_k; \leq_W, \oplus, p_0, \ldots, p_{k-1})$*. Moreover,* $\mu(T) \equiv_W \mathbf{\Sigma}^0_1[F]$ *for each* $F \in \widetilde{\mathcal{T}}_k$*. Thus,* $ji(S) = \sigma ji(S)$*.*

The next result follows from Theorem 6.3, Corollary 6.4 and Proposition 4.6.

Corollary 6.6. *For the Baire and Cantor spaces we have*:

(i) $Aut(BC(\mathbf{\Sigma}^0_1); \leq_W) \simeq \mathbf{S}^\omega_2$ *and* $Aut(\mathbf{\Delta}^0_2; \leq_W) \simeq \mathbf{S}^{\omega_1}_2$.

(ii) *For any* $k \geq 3, Aut((BC(\mathbf{\Sigma}^0_1))_k; \leq) \simeq \mathbf{S}_k \simeq Aut((\mathbf{\Delta}^0_2)_k; \leq)$.

The next result follows from Theorem 6.3, Corollary 6.4 and proposition 4.7.

Corollary 6.7. *For the Baire and Cantor spaces we have*:

(i) *For any* $k \geq 3$*, the theory* $FO((BC(\mathbf{\Sigma}^0_1))_k; \leq_W)$ *is undecidable and, moreover, it is computably isomorphic to the first-order arithmetic* $FO(\omega; +, \cdot)$.

(ii) *For any* $k \geq 3, FO((\mathbf{\Delta}^0_2)_k; \leq_W)$ *is undecidable and, moreover, the theory* $FO(\omega; +, \cdot)$ *is m-reducible to* $FO((\mathbf{\Delta}^0_2)_k; \leq_W)$.

We conclude this section with a characterization of $\mathbf{\Delta}^0_2$-measurable k-partitions ν in terms of their ranks $rk(\nu)$ in the well preorder $((\mathbf{B}(\omega^\omega))_k; \leq_W)$. We consider only the Baire space; a slightly different proof establishes the similar fact for the Cantor space.

Theorem 6.8. *Let* ν *be a Borel measurable k-partition of the Baire space.*

(i) ν *is* $\mathbf{\Delta}^0_2$*-measurable iff* $rk(\nu) < \omega_1$.

(ii) ν *is* $BC(\mathbf{\Sigma}^0_1)$*-measurable iff* $rk(\nu) < \omega$.

Proof. (i) Let ν be $\mathbf{\Delta}^0_2$-measurable. Obviously, $(\mathbf{\Delta}^0_2)_k$ is an initial segment of $(\mathbf{B}(\omega^\omega); \leq_W)$, hence $rk(\nu)$ coincides with the rank of ν in $((\mathbf{\Delta}^0_2)_k; \leq_W)$. By Theorem 6.3 and Proposition 4.1(i), $rk(\nu) < \omega_1$.

It remains to show that if ν is not $\mathbf{\Delta}^0_2$-measurable then $rk(\nu) \geq \omega_1$. Let $i < k$ satisfy $\nu^{-1}(i) \notin \mathbf{\Delta}^0_2$. By a well known property of the Wadge reducibility of sets, $\mathbf{\Sigma}^0_2 \leq_W \nu^{-1}(i)$ or $\mathbf{\Pi}^0_2 \leq_W \nu^{-1}(i)$, hence there is a sequence $\{A_\alpha\}_{\alpha<\omega_1}$ of σ-join-irreducible subsets of the Baire space such that $A_\alpha \leq_W \nu^{-1}(i)$ and $(\bigoplus_{\gamma\leq\alpha} A_\gamma) <_W A_\beta$ for all $\alpha < \beta < \omega_1$ (e.g., take A_α as a Wadge complete set in the α-th non-self-dual level of the difference hierarchy). For any $\alpha < \omega_1$, let f_α be a continuous function on the Baire space that Wadge reduces A_α to $\nu^{-1}(i)$, and let $\mu_\alpha = \nu \circ f_\alpha$. Then $\mu_\alpha \leq_W \nu$ and $\mu_\beta \not\leq_W (\bigoplus_{\gamma\leq\alpha} \mu_\gamma)$ for all $\alpha < \beta < \omega_1$

(because $A_\alpha = \mu_\beta^{-1}(i)$). For the sequence $\rho_\alpha = \bigoplus_{\gamma\leq\alpha}\mu_\gamma$ we then have $\rho_\alpha \leq_W \nu$ and $\rho_\alpha <_W \rho_\beta$ for all $\alpha < \beta < \omega_1$. Therefore, $\omega_1 \leq rk(\nu)$.

(ii) Follows from (i), Theorem 6.4 and Proposition 4.1 (ii). $\square$

7. Wadge reducibility in domains

Here we discuss the Wadge reducibility in the ω-algebraic domains that are central objects of the domain theory (for definitions and general properties of these objects see e.g. [1]). For an ω-algebraic domain X, let $F(X)$ denote the countable set of finitary (or compact) elements. The specialization order is denoted by $\leq$, and the bottom element of $(X;\leq)$ is denoted by $\perp$.

Let X be an ω-algebraic domain. A set $A \subseteq X$ is called *approximable* [17] if for any $x \in A$ there is a finitary element $p \leq x$ with $[p,x] \subseteq A$, where $[p,x] = \{y \in X | p \leq y \leq x\}$. Call a k-partition ν of X *approximable* if $\nu^{-1}(i)$ is approximable for each $i < k$. By a *repetition-free ω-chain* for ν we mean a sequence $\{p_n\}_{n<\omega}$ of finitary elements such that $p_0 \leq p_1 \leq \ldots$ and $\nu(p_n) \neq \nu(p_{n+1})$ for each $n < \omega$.

Proposition 7.1. *Let X be an ω-algebraic domain and ν a k-partition of X.*

(i) *ν is $\mathbf{\Delta}^0_2$-measurable iff ν is approximable.*

(ii) *If ν is $\mathbf{\Delta}^0_2$-measurable then it has no repetition-free ω-chain.*

Proof. (i) Follows from the result in [18] that a set $A \subseteq X$ is in $\mathbf{\Delta}^0_2$ iff both A and its complement are approximable.

(ii) Suppose the contrary: ν has a repetition-free ω-chain. Then for some $i < n$ the characteristic function of $\nu^{-1}(i)$ has a repetition-free ω-chain. By a result in [18], $\nu^{-1}(i) \notin \mathbf{\Delta}^0_2$, a contradiction. $\square$

Let X be an ω-algebraic domain, ν a k-partition of X and $T = (T,\leq,t) \in \widetilde{\mathcal{T}}_k$. By a *$\nu$-representation* of T we mean a monotone function $g : (T;\leq) \to (F(X);\leq)$ such that $t = \nu \circ g$; T is *ν-representable* if there exists a ν-representation of T.

Proposition 7.2. *Let X be an ω-algebraic domain $T = (T,\leq,t) \in \widetilde{\mathcal{T}}_k$, and ν a k-partition of X. If T is ν-representable then $\mathbf{\Sigma}^0_1[T] \leq_W \nu$.*

Proof. Let $g : T \to F(X)$ be a ν-representation of T. Let $\mu \in \mathbf{\Sigma}^0_1[T]$, then $\mu = t \circ \tilde{S}$ for some admissible $S : T \to \mathbf{\Sigma}^0_1$. Define a function f on X by $f(x) = g(a)$ where a is the unique element of T with $x \in \tilde{S}_a$. Then f is continuous and $\mu(x) = t(a) = \nu g(a) = \nu f(x)$, hence $\mu \leq_W \nu$. $\square$

Proposition 7.3. *Let X be an ω-algebraic domain and ν a $\mathbf{\Delta}^0_2$-measurable k-partition of X. Then there exists a ν-representable $T \in \widetilde{\mathcal{T}}_k$ such that $S \leq T$ for each ν-representable $S \in \widetilde{\mathcal{T}}_k$.*

Proof. Let us construct initial segments $T_0 \subseteq T_1 \subseteq \cdots$ of $(\omega^*;\sqsubseteq)$ and functions $f_n : T_n \to F(X)$ by induction on n as follows. Set $T_0 = \{\emptyset\}, f_0 = \{(\emptyset,\perp)\}$ and assume that T_n, f_n are already defined. For any leaf τ of T_n, let $\{p^\tau_i\}_{i<k(\tau)},\ k(\tau) \leq \omega$, be an enumeration without repetitions of the set

$$\{q \in F(X) \mid f_n(\tau) \leq q \wedge \nu(f_n(\tau)) \neq \nu(q)\}$$

(recall that $F(X)$ is always assumed countable). Set

$$T_{n+1} = T_n \cup \{\sigma^\frown i | \sigma \in leaf(T_n), i < k(\tau)\}$$

and

$$f_{n+1} = f_n \cup (\cup\{(\sigma^\frown i, p^\tau_i) | \sigma \in leaf(T_n), i < k(\tau)\}).$$

Finally, set $T = \bigcup_n T_n$ and $f = \bigcup_n f_n$. The tree $(T;\sqsubseteq)$ has no infinite chains because if $\emptyset \sqsubseteq i_0 \sqsubseteq i_0 i_1 \sqsubseteq \cdots$ were such a chain then $f(\emptyset) \leq f(i_0) \leq f(i_0 i_1) \leq \cdots$ would be a repetition-free ω-chain for ν, contradicting Proposition 7.1. Therefore, the k-tree $T = (T,\sqsubseteq,\nu\circ f) \in \widetilde{\mathcal{T}}_k$ is ν-representable. Now let $S \in \widetilde{\mathcal{T}}_k$ be arbitrary ν-representable tree. By Proposition 4.1(v). w.l.o.g. we may assume S to be repetition-free. One easily constructs a homomorphism $h : S \to T$ (by a similar induction) that witnesses $S \leq T$. $\square$

Next we discuss two classes of ω-algebraic domains introduced and studied in [17]. By a *reflective domain* we mean an ω-algebraic domain X such that for some continuous functions $q_0, e_0, q_1, e_1 : X \to X$ there hold $q_0 e_0 = q_1 e_1 = id_X$, and $e_0(X), e_1(X)$ are disjoint open sets. Examples of reflective domains are the Baire and Cantor domains, the domain $\omega^\omega_\perp$ of partial functions $g : \omega \rightharpoonup \omega$, and many other natural (in particular, functional) domains, see [17].

Define continuous functions $s_k, r_k (k < \omega)$ on X by $s_0 = e_0,\ s_{k+1} = e_1 s_k$ and $r_0 = q_0,\ r_{k+1} = r_k q_1$. Let also $D_k = s_k(X)$. In [17] we observed that in any reflective domain X the following properties of the introduced objects hold true: for any $k < \omega$, $r_k s_k = id_X$; the sets D_k are open, pairwise disjoint and satisfy $D_k = \{x | s_k(\perp) \leq x\}$; $\{\overline{\cup_k D_k}, D_0, D_1 \ldots\}$ is a partition of X.

For any $i < k$, define an operation u_i sending sequences of k-partitions $\nu_0, \nu_1, \ldots$ of X to k-partitions of X by letting $u_i = u_i(\nu_0, \nu_1, \ldots)$ to be the map

$$u_i(x) = \begin{cases} i & \text{if} \quad x \notin \bigcup_k D_k \\ \nu_k r_k(x) & \text{if} \quad x \in D_k. \end{cases}$$

In [17] we observed that the operations u_i witness the following

Proposition 7.4. *Let X be a reflective domain and $\mathcal{P}_i = \{\nu \in k^X \mid \nu(\bot) = i\}$ for any $i < k$. Then $(k^X; \leq_W, \{\mathcal{P}_s\}_{s \in S})$ is a σ-dws.*

Note that, by Section 3, any subset of k^X closed under $u_0, \ldots, u_{k-1}$ is a σ-dws as well. In particular, this applies to $\mathbf{\Delta}^0_\alpha$-measurable k-partitions for $\alpha \geq 2$ and to $BC(\mathbf{\Sigma}^0_\alpha)$-measurable k-partitions for $\alpha \geq 1$. The next result generalizes Theorem 5.8 in [17] from the case of sets to the case of k-partitions.

Theorem 7.5. *In any reflective domain X, the Boolean hierarchy of k-partitions does not collapse, i.e., for all $S, T \in \widetilde{\mathcal{T}}_k$, $\mathbf{\Sigma}^0_1[S] \subseteq \mathbf{\Sigma}^0_1[T]$ iff $S \leq T$. Moreover, any level of the Boolean hierarchy has a Wadge complete k-partition.*

Proof. Assume w.l.o.g. that S, T are repetition-free and are initial segments of $(\omega^*; \sqsubseteq)$. We need to check only the implication from left to right. Suppose that $S \not\leq T$, then, by Propositions 7.4 and by the proof sketch of Proposition 4.4(ii), $\mu_S \not\leq \mu_T$ where $\mu_S = f(S)$ for the unique isomorphic embedding $f : (\widetilde{\mathcal{T}}_k; \leq) \to (k^X; \leq_W)$ sending the one-element tree i to the constant partition $\lambda x.i$. So it suffices to show that μ_S is Wadge complete in $\mathbf{\Sigma}^0_1[S]$ (and μ_T is Wadge complete in $\mathbf{\Sigma}^0_1[T]$). The relation $\mu_S \in \mathbf{\Sigma}^0_1[S]$ is checked by a straightforward induction on the rank of S. It remains to show that $\mathbf{\Sigma}^0_1[S] \leq_W \mu_S$. By Proposition 7.2, it suffices to show that S is μ_S-representable.

For any $\sigma \in \omega^*$, define continuous functions s_σ and r_σ on X by $s_\emptyset = id_X$, $s_{\sigma k} = s_\sigma s_k$ and $r_\emptyset = id_X$, $r_{\sigma k} = r_k r_\sigma$. It is easy to see that $r_\sigma s_\sigma = id_X$, the sets $D_\sigma = s_\sigma(X) = \{x | s(\bot) \leq x\}$ are open, $s_\sigma(\bot) \in F(X)$, $D_\sigma \supset D_{\sigma n}$ and $D_{\sigma m} \cap D_{\sigma n} = \emptyset$ for all $m \neq n$. Therefore, $\sigma \mapsto s_\sigma(\bot)$ is an embedding of $(\omega^*; \sqsubseteq)$ into $(F(X); \leq)$. Moreover, the restriction of this embedding to S is a μ_S-representation of S. $\square$

In [17] also another class of domains was considered. By a *2-reflective domain* we mean an ω-algebraic domain X with a top element $\top$ such that there exist continuous functions $q_0, e_0, q_1, e_1 : X \to X$ and open sets B_0, C_0, B_1, C_1 with the following properties: $q_0 e_0 = q_1 e_1 = id_X$; $B_0 \supseteq C_0$ and $B_1 \supseteq C_1$; $e_0(X) = B_0 \setminus C_0$ and $e_1(X) = B_1 \setminus C_1$; $B_0 \cap B_1 = C_0 \cap C_1$. Examples of 2-reflective domains are the domain $P\omega$, and many other natural (in particular, functional) domains, see [17]. The following fact is Theorem 7.5 in [17].

Proposition 7.6. *Let X be a 2-reflective domain. Then $(k^X; \leq_W, \{\mathcal{P}^j_i\}_{i,j<k})$, where $\mathcal{P}^j_i = \{\nu \in k^X | \nu(\bot) = i \wedge f(\top) = j\}$ for all $i, j < k$, is a 2-dws.*

As an immediate corollary of Propositions 7.6, 3.6 and 3.7 we obtain

Theorem 7.7. *Let X be a reflective or a 2-reflective domain, $k \geq 3$ and $\mathcal{C}$ is one of the classes $P(X)$, $\mathbf{B}$, $BC(\mathbf{\Sigma}^0_1)$, $\mathbf{\Sigma}^0_n$, $\mathbf{\Pi}^0_n$, $BC(\mathbf{\Sigma}^0_n)$, $\mathbf{\Delta}^0_{n+1}$ in X, where $n > 1$. Then $FO(\mathcal{C}_k; \leq_W)$ is hereditary undecidable.*

8. Wadge Reducibility in Baire and Cantor Domains

In this section we consider in more detail the Wadge reducibility of k-partitions of the Baire and Cantor domains. Since these domains are reflective, the results of the previous section apply to them. The main result of this section is the following

Theorem 8.1. *Let X be the Baire or a Cantor domain. Then the quotient structure of $((\mathbf{\Delta}^0_2(X))_k; \leq_W)$ is isomorphic to the quotient structure of $(\widetilde{\mathcal{T}}_k; \leq)$.*

Proof. We give the proof only for the Baire domain but the argument applies to the other case as well. By Propositions 7.4 and 4.4 (ii), the map $T \mapsto \mu_T$ from the proof of Theorem 7.5 induces an isomorphic embedding of the quotient-structure of $(\widetilde{\mathcal{T}}_k; \leq)$ into the quotient structure of $((\mathbf{\Delta}^0_2(\omega^\omega))_k; \leq_W)$. It remains to show that any $\mathbf{\Delta}^0_2$-measurable k-partition ν of $\omega^{\leq\omega}$ is Wadge equivalent to μ_T for some $T \in \widetilde{\mathcal{T}}_k$. Let T be the biggest ν-representable k-tree that exists by Proposition 7.3. Let $f : T \to \omega^*$ be the ν-representation of T constructed in the same way as in the proof of Proposition 7.3, only this time we require in the induction step that $\{p^\tau_i\}_{i<k(\tau)}$, $k(\tau) \leq \omega$, is an enumeration without repetitions of the set of minimal elements in $(\{q \in \omega^* \mid f_n(\tau) \leq q \wedge \nu(f_n(\tau)) \neq \nu(q))\}; \sqsubseteq)$. Since T is ν-representable, $\mathbf{\Sigma}^0_1[T] \leq_W \nu$ by Proposition 7.2. Since $\mu_T \in \mathbf{\Sigma}^0_1[T]$ by Theorem 7.5, $\mu_T \leq_W \nu$.

It remains to show that $\nu \leq_W \mu_T$. Since μ_T is Wadge complete in $\mathbf{\Sigma}^0_1[T]$ by Theorem 7.5, it suffices to show that $\nu \in \mathbf{\Sigma}^0_1[T]$. Define $S : T \to \mathbf{\Sigma}^0_1$ by $S(\sigma) = \{\xi \in \omega^{\leq\omega} \mid f(\sigma) \sqsubseteq \xi\}$. Since f is an embedding of T into $(\omega^*; \sqsubseteq)$, S is admissible. So it remains to show that $\xi \in \tilde{S}(\sigma)$ implies $\nu(\xi) = \nu(f(\sigma))$. Suppose the contrary, so $f(\sigma) \sqsubseteq \xi, f(\sigma i) \not\sqsubseteq \xi$ for all $\sigma i \in T$, and $\nu(\xi) \neq \nu(f(\sigma))$. Since ν is approximable by Proposition 7.1(i), $\nu([\tau, \xi]) = \nu(\xi)$ for some $\tau \sqsubseteq \xi$, $\tau \in \omega^*$. Take the minimal such τ. We have $f(\sigma) \sqsubseteq \tau$ or $\tau \sqsubseteq f(\sigma)$. The second case is impossible because $\nu(\xi) \neq \nu(f(\sigma))$, hence $f(\sigma) \sqsubseteq \tau$. By construction of f, $\tau = f(\sigma i)$ for some $i < m$. A contradiction. $\square$

Restricting the argument above to finite k-trees we obtain

Corollary 8.2. *Let X be the Baire or a Cantor domain. Then the quotient structure of $(\mathcal{T}_k; \leq)$ is isomorphic to that of $((BC(\mathbf{\Sigma}^0_1(X)))_k; \leq_W)$.*

The next result follows from Theorem 8.1, Corollary 8.2 and Proposition 4.6.

Corollary 8.3. *For the Baire and Cantor domains we have*:

(i) $Aut(BC(\mathbf{\Sigma}^0_1); \leq_W) \simeq \mathbf{S}^{\omega}_2$ *and* $Aut(\mathbf{\Delta}^0_2; \leq_W) \simeq \mathbf{S}^{\omega_1}_2$.

(ii) *For any* $k \geq 3$, $Aut((BC(\mathbf{\Sigma}^0_1)_k; \leq) \simeq \mathbf{S}_k \simeq Aut((\mathbf{\Delta}^0_2)_k; \leq)$.

The next result follows from Theorem 8.1, Corollary 8.2 and Proposition 4.7.

Corollary 8.4. *Let X be the Baire or a Cantor domain.*

(i) *For any* $k \geq 3$, *the theory* $FO((BC(\mathbf{\Sigma}^0_1(X)))_k; \leq_W)$ *is computably isomorphic to the first-order arithmetic* $FO(\omega; +, \cdot)$.

(ii) *For any* $k \geq 3$, $FO(\omega; +, \cdot)$ *is m-reducible to* $FO((\mathbf{\Delta}^0_2)(X)))_k; \leq_W)$.

9. Conclusion

This paper extends essentially all previously known results on the difference hierarchy of sets in the Baire and Cantor spaces to the case of k-partitions. Several facts on the Wadge hierarchy of sets are also extended to the case of k-partitions. Interestingly, many natural substructures of the structure of Wadge degrees become undecidable for $k \geq 3$.

At the same time, many natural open question about the Wadge reducibility of k-partitions remain open. Though the results of Section 6 provide a complete extension of the theory of Wadge degrees of $\mathbf{\Delta}^0_2$-sets (see Section C of Chapter I of [27]) to the $\mathbf{\Delta}^0_2$-measurable k-partitions, very little is known outside this class. We believe that actually all the main facts about the Wadge reducibility of Borel sets in [27] may be lifted to the case of k-partitions but this of course requires a lot of additional work.

Section 8 develops a complete theory of the Boolean hierarchy and of the Wadge reducibility of $\mathbf{\Delta}^0_2$-measurable k-partitions in the Baire and Cantor domains. Beyond this class, almost nothing is known. In particular, we do not currently know whether the structure of Borel (or even $\mathbf{\Delta}^0_3$) sets in the Baire domain is a well preorder.

In [24], a computability theory on the Baire domain was developed. This theory suggests a natural effective version of the Wadge reducibility on the Baire domain (namely, the many-one reducibility by computable functions). Most probably, as is usual in computability theory, the corresponding degree structures are complicated even for topologically simple sets and partitions. But for subclasses with sufficient computability constraints the degree structures may turn out manageable and useful. It seems natural to make a systematic search for structures of this kind.

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