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Operations on Hypermaps, and Outer Automorphisms

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We present a group $\mathfrak{H}$, isomorphic to $PGL(2, \mathbb{Z})$, of operations on hypermaps, and a group $\tilde{\mathfrak{H}}$, isomorphic to $GL(2, \mathbb{Z})$, of operations on oriented hypermaps without boundary. These are induced by the group of automorphisms of a certain group G whose transitive permutation representations correspond to hypermaps.

1. INTRODUCTION

Wilson [18] and Lins [12] have described six operations on maps on surfaces (including duality and the identity operation) which preserve certain important features such as the number of flags and the automorphism group of the map. These form a group isomorphic to S_3. Jones and Thornton [11] showed that these operations arise naturally in algebraic map theory: maps may be regarded as transitive permutation representations of a certain group Γ, and the outer automorphism group

$$\mathrm{Out}(\Gamma) = \mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma) \simeq S_3$$

permutes these representations, inducing six operations on maps. Maps on n-manifolds may be regarded as transitive permutation representations of a certain group Γ_n. In [6] it was shown that, for $n > 2$,

$$\mathrm{Out}(\Gamma_n) = \mathrm{Aut}(\Gamma_n)/\mathrm{Inn}(\Gamma_n) \simeq D_4$$

where D_4 is the dihedral group of order eight. Vince [16] has shown that hypermaps may be regarded as transitive permutation representations of a certain group G. Machi [14] has described a group of six operations on oriented hypermaps without boundary; these extend to *all* hypermaps, and they correspond to permuting hypervertices, hyperedges and hyperfaces. These operations are induced by six outer automorphisms of G. The object of this note is to show that

$$\mathfrak{H} \simeq \mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G) \simeq PGL(2, \mathbb{Z})$$

and

$$\tilde{\mathfrak{H}} \simeq \mathrm{Out}\ (G^+) = \mathrm{Aut}(G^+)/\mathrm{Inn}(G^+) \simeq GL(2, \mathbb{Z})$$

where $\mathfrak{H}$ is the group of operations on hypermaps induced by the action of $\mathrm{Out}(G)$ on transitive permutation representations of the group G, and $\tilde{\mathfrak{H}}$ is the group of operations on oriented hypermaps without boundary induced by the action of $\mathrm{Out}(G^+)$ on transitive permutation representations of a certain subgroup $G^+ \lhd G$. For a survey of all of these results, see [9].

2. PRELIMINARIES

Throughout this paper we regard functions as acting on the right, except matrix multiplication which we regard as acting on the left.

For the proof of Theorem 5.2 we mention here the group $AGL(1, p)$, p prime. This is the group of all affine transformations

$$f \to af + b \qquad (a, b \in GF(p), \quad a \neq 0)$$

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of the field of order p, $GF(p)$. It has a regular normal subgroup $T \simeq (GF(p), +)$ consisting of the translations

$$t_b: f \to f + b \qquad (b \in GF(p))$$

complemented by S, the stabiliser of 0, consisting of the scalar transformations

$$s_a: f \to af \qquad (a \in GF(p)\backslash\{0\}),$$

and isomorphic to the multiplicative group $GF(p)\backslash\{0\}$.

Each element of $AGL(1, p)$ can be written uniquely in the form $g = s_a t_b$; we define μ: $AGL(1, p) \to GF(p)\backslash\{0\}$ to be the epimorphism $g \to a$. It can be shown that every automorphism of $AGL(1, p)$ is inner [8].

3. Hypermaps

In this section we outline an algebraic theory of hypermaps drawn from the theory of maps presented in [1] and [10] and the theory of hypermaps presented in [2], [3], [5] and [16].

By a *hypermap* $\mathscr{H}$ on a surface without boundary we mean a connected regular graph $\mathscr{G}$ of degree 3 (possibly with multiple edges) imbedded (without crossings) in a connected surface $\mathscr{S}$ (possibly non-orientable, non-compact) without boundary such that each of the faces (the connected components of $\mathscr{S}/\mathscr{G}$) is homeomorphic to an open disc, together with a colouring of the faces by $\{0, 1, 2\}$ such that every edge borders two different coloured faces.

Thus, for example, we might take the coloured cube shown in Fig. 1.

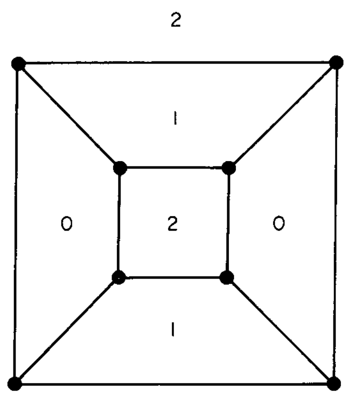

Figure 1.

Or we could form a hypermap from a map on a surface, each of whose faces are coloured 2, by replacing its edges by digons coloured 1 and by then placing a small disc coloured 0 around each vertex. For example, take the regular imbedding of K_4 in a sphere, shown in Fig. 2. This gives rise to the hypermap shown in Fig. 3. In general, the faces coloured 0, 1, 2 are called the *hypervertices*, *hyperedges* and *hyperfaces* respectively. Thus each hypermap has an underlying imbedded hypergraph.

Our third example is obtained from the unique orientable regular imbedding of K_7 shown in Fig. 4. If we colour its faces by $\{0, 1\}$ and place a small disc coloured 2 around each vertex then we obtain the hypermap on a torus shown in Fig. 5. The underlying hypergraph can be seen in the Fano plane, shown in Fig. 6, by regarding the points as hypervertices and the lines as hyperedges. This imbedding was described by Walsh in [17]. Singerman [15] described a second imbedding of the Fano plane; we show this in Fig. 7. This is on an

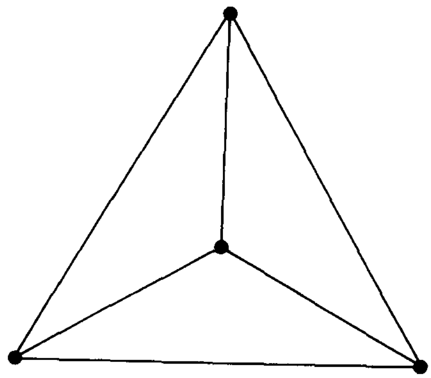

FIGURE 2.

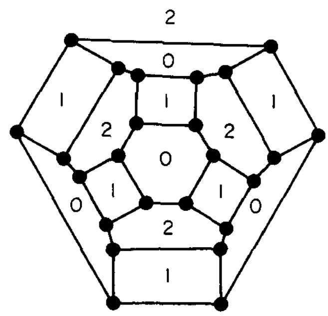

FIGURE 3.

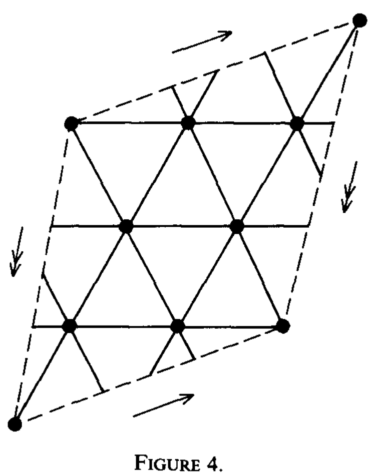

FIGURE 4.

orientable surface of genus 3. If we contract each hypervertex to a point then we obtain the unique orientable non-regular edge-symmetric imbedding of K_7 [7].

For the most part we are content to define hypermaps as being on surfaces without boundary. However, we may more generally allow $\mathscr{S}$ a boundary. In this case we allow the faces to be homeomorphic to a half-disc and allow $\mathscr{G}$ to have free edges, that is, edges homeomorphic to [0, 1] with only one of the ends belonging to the vertex set. (For a more precise definition of a map on a surface with boundary see [1].) We make the further conditions that no vertex of $\mathscr{G}$ is to lie on the boundary of $\mathscr{S}$, and that every free edge of $\mathscr{G}$ is to meet the boundary of $\mathscr{S}$. Figure 8 gives an example of a hypermap on a disc. We

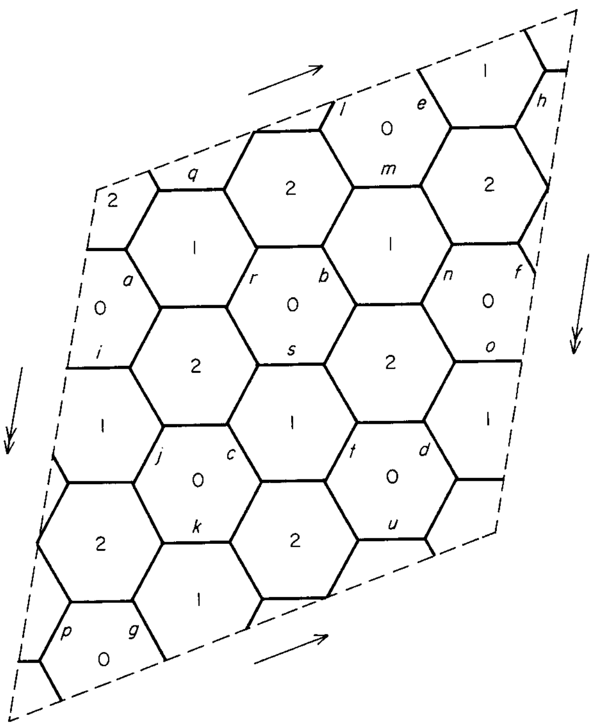

FIGURE 5.

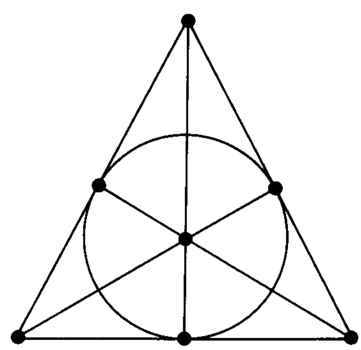

FIGURE 6.

refer to the vertices as *hyperblades* and to the edges that border a hypervertex and a hyperedge as *hyperdarts*.

Our definition of a topological hypermap differs slightly from those of [2], [16] and [17]. Cori [2] takes the surface to be oriented, compact and without boundary, and contracts each hyperdart to a point called a 'brin'. Walsh [17] also takes an oriented, compact, boundaryless surface and considers the dual of the map coloured $\{0, 1\}$ formed by contracting each hyperface to a point. Vince [16] takes the map coloured $\{1, 2\}$ formed by contracting each hypervertex to a point.

We define three permutations of the set of hyperblades, Ω, as follows: We colour each edge of $\mathscr{G}$ by the complement of the colours of the faces which it borders and, for $i = 0, 1, 2$, ϱ_i transposes each pair of hyperblades that form the ends of an edge coloured i. Clearly these permutations satisfy the relations $\varrho_i^2 = 1$ and by the connectedness of $\mathscr{G}$ they generate a transitive group of permutations of Ω, so we have a transitive permutation

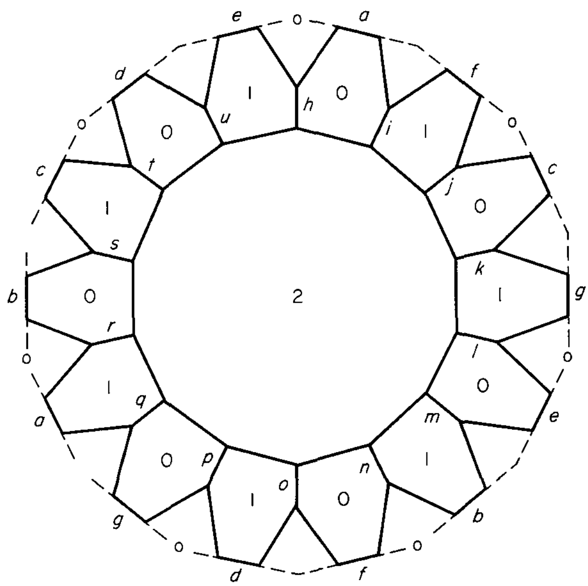

FIGURE 7.

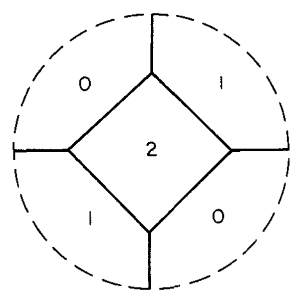

FIGURE 8.

representation $\pi\colon G \to S^{\Omega}$ of the group

$$G = \langle r_0, r_1, r_2 \mid r_0^2 = r_1^2 = r_2^2 = 1 \rangle.$$

This permutation representation is isomorphic to the action of G (by right multiplication) on the cosets Hg of a subgroup $H \leqslant G$; this subgroup H, the *hypermap subgroup* associated with $\mathscr{H}$, is the stabiliser in G of an element of Ω, and is uniquely determined up to conjugacy. In fact, the edge-coloured graph $\mathscr{G}$ is just the Schreier coset graph of H in G (with loops replaced by free edges) and so we have a bijection between hypermaps and transitive permutation representation of G (or more strictly between isomorphism classes in each category).

Hypermap coverings $\mathscr{H}_1 \to \mathscr{H}_2$ correspond to inclusions $H_1 \leqslant H_2$, and the automorphism group $\mathrm{Aut}(\mathscr{H})$ can be realised as the action of $N_G(H)/H$ on the right cosets of H, where $N_G(H)$ is the normaliser of H in G, acting by left multiplication.

It can also be shown that: $\mathscr{S}$ is without boundary if and only if H is torsion free; $\mathscr{S}$ is orientable and without boundary if and only if H lies in G^+, the subgroup of index 2 in G that consists of words of even length, with free basis $x_1 = r_0 r_2$ and $x_2 = r_1 r_2$; and $\mathscr{S}$ is compact if and only if H has finite index in G.

In fact, if $\mathscr{S}$ is oriented and without boundary then the hyperblades of $\mathscr{H}$ are two-colourable. The orientation of each hypervertex defined by starting at a particular hyperblade and proceeding along an edge of $\mathcal{G}$ coloured 1 will be positive for the hyperblades of one colour and negative for those of the other. (Thus, choosing an orientation for $\mathcal{S}$ amounts to choosing a colouring of the vertices of $\mathcal{G}$.) If we represent each hyperdart by the incident hyperblade of positive colour then we can define an action of x_1^π and x_2^π on the hyperdarts of the oriented hypermap. Under this action the cycles of x_1^π are the hyperdarts bounding the hyperedges taken in a negative direction, and the cycles of x_2^π are the hyperdarts bounding the hypervertices taken in a positive direction. This action of x_1^π and x_2^π on the hyperdarts corresponds to the permutations α and σ of brins as defined in [2]. The brins taken in a positive direction around the hyperfaces correspond to the cycles of $\sigma^{-1}\alpha$. Thus an oriented hypermap without boundary corresponds to a transitive permutation representation of the group

$$G^+ = \langle x_1, x_2 | - \rangle$$

If H is the stabiliser in G of a positive hyperblade then this transitive permutation representation is isomorphic to the action of G^+ (by right multiplication) on the right cosets of H in G^+. The group of (orientation-preserving) automorphisms of the oriented hypermap $\mathcal{H}$ can be realised as the action of $N_{G^+}(H)/H$ (by left multiplication) on the right cosets of H in G^+.

A hypermap $\mathcal{H}$ is *reflexible* if $\mathrm{Aut}(\mathcal{H})$ acts transitively on the set of hyperblades or, equivalently, $H \lhd G$. An orientable hypermap $\mathcal{H}$ without boundary is *regular* if its group of orientation-preserving automorphisms acts transitively on the set of hyperdarts or, equivalently, $H \lhd G^+$. Both imbeddings of the Fano plane shown in Figures 5 and 7 are regular [15]. For a detailed account of regular hypermaps, see [3].

For example, let $\mathcal{H}_1$ be the hypermap shown in Fig. 5 and let $\mathcal{H}_2$ be the hypermap shown in Fig. 7. For $i = 1, 2$ let α_i, σ_i be the corresponding permutations of brins (as defined in [2]) according to a clockwise orientation. Thus

$$\alpha_1 = (\mathrm{hue})(jif)(lkg)(bnm)(dpo)(arq)(sct);$$

$$\sigma_1 = (aih)(ckj)(eml)(fon)(gqp)(rbs)(tdu);$$

$$\alpha_2 = (euh)(fij)(gkl)(mnb)(opd)(qra)(tcs);$$

$$\sigma_2 = (aih)(ckj)(eml)(fon)(gqp)(rbs)(tdu).$$

Note that $\alpha_2 = \alpha_1^{-1}$ and $\sigma_2 = \sigma_1$.

4. Operations

Any permutation, ψ, of $\{0, 1, 2\}$ clearly induces an operation, $\bar{\psi}$, on hypermaps that permutes hypervertices, hyperedges and hyperfaces. These operations were first described by Machi [14] for oriented hypermaps. The corresponding group automorphism $r_i \to r_{i\psi}$ of G, also denoted by ψ, induces the operation $\bar{\psi}$ by its left action on the associated transitive permutation representations of G. We define an *operation* on hypermaps to be any transformation that is induced by an automorphism of G in this way. This definition was first made by Jones and Thornton [11] for maps on surfaces (see also [6], in which higher dimensions are considered).

Many topological and combinatorial properties are equivalent to algebraic properties that are preserved by $\mathrm{Aut}(G)$. In particular, the induced operations preserve coverings, automorphism groups, reflexibility, boundaries, and compactness of hypermaps. We shall see later that $\mathrm{Aut}(G)$ restricts to G^+ and so the induced operations restrict to orientable hypermaps without boundary. In particular, they restrict to regular hypermaps.

If a hypermap $\mathscr{H}$ has associated transitive permutation representation π and if $\phi \in \mathrm{Aut}(G)$ then we denote the hypermap with associated transitive permutation representation $\phi^{-1}\pi$ by $\mathscr{H}^{\phi}$. If $\mathscr{H}$ has hypermap subgroup H then $\mathscr{H}^{\phi}$ has hypermap subgroup H^{ϕ}. We make analogous definitions for operations on oriented hypermaps without boundary. We let $\mathfrak{H}$ denote the group of operations on hypermaps induced by the left action of $\mathrm{Aut}(G)$ on the transitive permutation representations of G and let $\tilde{\mathfrak{H}}$ denote the group of operations on oriented hypermaps without boundary induced by the left action of $\mathrm{Aut}(G^+)$ on transitive permutation representations of G^+.

Each inner automorphism of G acts trivially on hypermaps, and so we have an induced action of the outer automorphisms group $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$. Similarly, we have an induced action of $\mathrm{Out}(G^+) = \mathrm{Aut}(G^+)/\mathrm{Inn}(G^+)$ on oriented hypermaps without boundary, and the following diagrams commute:

$$\begin{array}{ccc} \mathrm{Aut}(G) & & \\ \downarrow & \searrow & \\ \mathrm{Out}(G) & \to & \mathfrak{H} \end{array} \qquad\qquad \begin{array}{ccc} \mathrm{Aut}(G^+) & & \\ \downarrow & \searrow & \\ \mathrm{Out}(G^+) & \to & \tilde{\mathfrak{H}} \end{array}$$

We shall see later that

$$\mathfrak{H} \simeq \mathrm{Out}(G) \simeq PGL(2, \mathbb{Z})$$

and that

$$\tilde{\mathfrak{H}} \simeq \mathrm{Out}(G^+) \simeq GL(2, \mathbb{Z})$$

but first we introduce a new operation.

Let τ be the automorphism that conjugates r_0 by r_2. Then $\bar{\tau} \in \mathfrak{H}$ may be described as follows:

(1) shrink each hyperface of $\mathscr{H}$ to a point;
(2) make a directed cut along each hyperdart;
(3) rejoin corresponding sides in opposing directions;
(4) place a small disc coloured 2 around each resultant vertex to obtain the hypermap $\mathscr{H}^{\tau}$.

This operation is illustrated in Fig. 9.

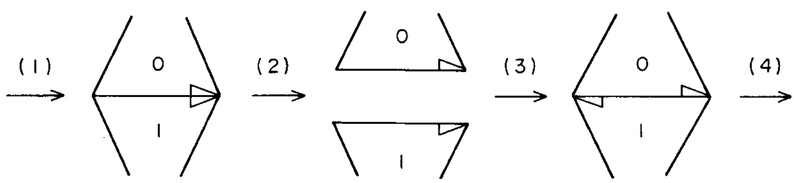

FIGURE 9.

A corresponding operation was first described by Wilson [18] and Lins [12] in the context of maps on surfaces. Notice that $\bar{\tau}$ preserves the hypervertices and hyperedges of $\mathscr{H}$, together with their incidence, so that $\mathscr{H}$ and $\mathscr{H}^{\tau}$ have the same underlying hypergraph.

For example, let $\mathscr{H}$ be the hypermap shown in Fig. 1. If we shrink each hyperface of $\mathscr{H}$ then we obtain the map coloured $\{0, 1\}$ shown in Fig. 10. This is easily seen to be invariant under the cutting operation that follows and so $\mathscr{H}^{\tau} = \mathscr{H}$.

The automorphism τ restricts to an automorphism of G^+ and so induces an operation $\tilde{\tau} \in \tilde{\mathfrak{H}}$. This swaps oriented hypermaps $\mathscr{H}_1$ and $\mathscr{H}_2$ where

$$\alpha_2 = x_1^{\pi_2} = x_1^{\tau^{-1}\pi_1} = (r_0 r_2)^{\tau^{-1}\pi_1} = (r_2 r_0)^{\pi_1} = (x_1^{\pi_1})^{-1} = \alpha_1^{-1}$$

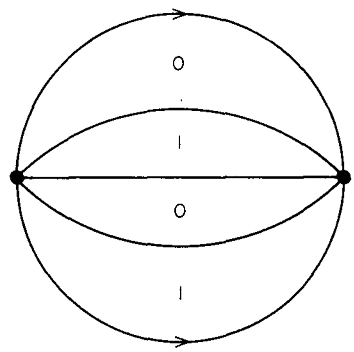

FIGURE 10.

and

$$\sigma_2 = x_2^{\pi_2} = x_2^{\tau^{-1}\pi_1} = (r_1 r_2)^{\tau^{-1}\pi_1} = (r_1 r_2)^{\pi_1} = x_2^{\pi_1} = \sigma_1;$$

whence $\tilde{\tau}$ swaps the hypermaps shown in Figs 5 and 7.

5. Group Automorphisms

In this section we construct an epimorphism from $\mathrm{Aut}(G)$ onto $PGL(2, \mathbb{Z})$ with kernel $\mathrm{Inn}(G)$. We then show that this is also the kernel of the action of $\mathrm{Aut}(G)$ on hypermaps.

If $\phi \in \mathrm{Aut}(G)$ then each r_i^ϕ is conjugate to some r_j by the torsion theorem for free products [13, IV.1.6] whence ϕ restricts to an automorphism of the subgroup G^+. We shall exploit the proof [13, I.4.5] that $\mathrm{Out}(G^+) \simeq GL(2, \mathbb{Z})$.

For $i, j = 1, 2$ let ε_i: $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be the epimorphism which takes the ith co-ordinate and let ϱ: $G^+ \to \mathbb{Z} \times \mathbb{Z}$ be the abelianising epimorphism defined by $x_j^{\varrho\varepsilon_i} = \delta_{ij}$. We have an induced homomorphism $\tilde{\varrho}$: $\mathrm{Aut}(G^+) \to GL(2, \mathbb{Z})$ defined by $\phi \to (\phi_{ij})^{-1}$, where $\phi_{ij} = x_j^{\phi\varrho\varepsilon_i}$ (inversion is necessary since we regard automorphisms as acting on the right but matrices as acting on the left). From $\tilde{\varrho}$ we obtain the obvious homomorphism $\bar{\varrho}$: $\mathrm{Aut}(G) \to PGL(2, \mathbb{Z})$.

Let A_1, A_2 and A_3 be the matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

respectively. Then $\bar{A}_1$, $\bar{A}_2$ and $\bar{A}_3$ generate $PGL(2, \mathbb{Z})$ with defining relations

$$\bar{A}_1^2 = \bar{A}_2^2 = \bar{A}_3^2 = (\bar{A}_1\bar{A}_2)^3 = (\bar{A}_1\bar{A}_3)^2 = 1$$

where $\bar{A}_i$ denotes the image of A_i in $PGL(2, \mathbb{Z})$ [4, 7.2]. For $\{i, j, k\} = \{0, 1, 2\}$ let ψ_i be the automorphism of G that transposes r_j and r_k, let τ_{ij} be that which conjugates r_i by r_j and let $a_1 = \psi_2$, $a_2 = \tau_{12}\psi_1\tau_{12}$ and $a_3 = \tau_{02}$. It is easily verified that: $a_i^{\bar{\varrho}} = \bar{A}_i$ for $i = 1, 2, 3$, whence $\bar{\varrho}$ is an epimorphism; that a_1^2, a_2^2, a_3^2, $(a_1 a_2)^3$, $(a_1 a_3)^2 \in \mathrm{Inn}(G)$, whence $\mathrm{Ker}(\bar{\varrho}) \subseteq \mathrm{Inn}(G)$; and that $r_0, r_1, r_2 \in \mathrm{Ker}(\bar{\varrho})$, whence $\mathrm{Inn}(G) \subseteq \mathrm{Ker}(\bar{\varrho})$. Thus we have:

THEOREM 5.1. $\mathrm{Out}(G) \simeq PGL(2, \mathbb{Z})$ *and* $\mathrm{Aut}(G)$ *is generated by the automorphisms that permute* r_0, r_1 *and* r_2 *and the automorphism that conjugates* r_0 *by* r_2.

Suppose that $\phi \in \mathrm{Aut}(G)$ acts trivially on compact, regular hypermaps. We shall show that $\phi \in \mathrm{Inn}(G)$. It will follow that $\mathfrak{H} \simeq \mathrm{Out}(G)$. Let p be any odd prime and let r by any primitive root of p. For $i, j = 1, 2$ let H_i be the kernel of the epimorphism θ_i: $G^+ \to AGL(1, p)$ defined by $x_i^{\theta_i} = s_r$ and $x_j^{\theta_i} = t_1$, $i \neq j$. Then for $g \in G$ we have $g^{\theta_i\mu} = r^{g^{\varrho\varepsilon_i}}$. Whence $\mathrm{Ker}(\theta_i\mu)$ consists of those $g \in G$ for which $g^{\varrho\varepsilon_i} \equiv 0 \bmod (p - 1)$.

Since H_i is a normal subgroup of G^+ of finite index, it is associated with a compact, regular hypermap which, by hypothesis, must be invariant under $\bar{\phi}$, whence $H_i^{\phi} = H_i^{r_2^{\delta_i}}$ for some $\delta_i \in \{0, 1\}$. Let $\hat{\phi}_i$: $AGL(1, p) \to AGL(1, p)$ be defined by $\theta_i \hat{\phi}_i = r_2^{\delta_i}\phi^{-1}\theta_i$. Clearly, $\hat{\phi}_i \in \mathrm{Aut}(AGL(1, p))$. We recall that every automorphism of $AGL(1, p)$ is inner and that μ only depends upon the conjugancy classes of $AGL(1, p)$. Thus $\hat{\phi}_i\mu = \mu$.

Consider the case $i = 1$. If $\delta_1 = 0$ then $x_1^{\phi^{-1}\theta_1\mu} = x_1^{\theta_1\hat{\phi}_1\mu} = x_1^{\theta_1\mu}$ and $x_2^{\phi^{-1}\theta_1\mu} = x_2^{\theta_1\hat{\phi}_1\mu} = x_2^{\theta_1\mu}$; whence $x_1^{\phi^{-1}}x_1^{-1}$, $x_2^{\phi^{-1}}x_2^{-1} \in \mathrm{Ker}(\theta_1\mu)$, and so

$$x_1^{\phi^{-1}\varrho\varepsilon_1} - 1 = (x_1^{\phi^{-1}}x_1^{-1})^{\varrho\varepsilon_1} \equiv 0, \qquad \mathrm{mod}(p - 1)$$

and

$$x_2^{\phi^{-1}\varrho\varepsilon_1} = (x_2^{\phi^{-1}}x_2^{-1})^{\varrho\varepsilon_1} \equiv 0, \qquad \mathrm{mod}(p - 1).$$

Similarly, if $\delta_1 = 1$ then

$$x_1^{\phi^{-1}\varrho\varepsilon_1} + 1 = (x_1^{\phi^{-1}}x_1)^{\varrho\varepsilon_1} \equiv 0, \qquad \mathrm{mod}(p - 1)$$

and

$$x_2^{\phi^{-1}\varrho\varepsilon_1} = (x_2^{\phi^{-1}}x_2)^{\varrho\varepsilon_1} \equiv 0, \qquad \mathrm{mod}(p - 1).$$

Taking p sufficiently large determines a value δ_1 for which $x_1^{\phi^{-1}\varrho\varepsilon_1} = (-1)^{\delta_1}$ and $x_2^{\phi^{-1}\varrho\varepsilon_1} = 0$. Similar arguments in the case $i = 2$ yield a value δ_2 for which $x_1^{\phi^{-1}\varrho\varepsilon_2} = 0$ and $x_2^{\phi^{-1}\varrho\varepsilon_2} = (-1)^{\delta_2}$. Thus we have

$$\phi^{\bar{\varrho}} = \pm\begin{pmatrix} (-1)^{\delta_1} & 0 \\ 0 & (-1)^{\delta_2} \end{pmatrix} = \bar{A}_3^{(\delta_1+\delta_2)} = \tau_{02}^{(\delta_1+\delta_2)\bar{\varrho}}$$

Whence $\phi \in \mathrm{Inn}(G)\langle\tau_{02}\rangle$. But then, by symmetry, $\phi \in \bigcap_{i\neq j}\mathrm{Inn}(G)\langle\tau_{ij}\rangle = \mathrm{Inn}(G)$, as required. Thus:

THEOREM 5.2. $\mathfrak{H} \simeq \mathrm{Out}(G) \simeq PGL(2, \mathbb{Z})$.

REMARK. The above argument is easily adapted to show that

$$\tilde{\mathfrak{H}} \simeq \mathrm{Out}(G^+) \simeq GL(2, \mathbb{Z})$$

Now the restriction $\mathrm{Aut}(G) \to \mathrm{Aut}(G^+)$ is onto, whence Theorem 5.1 provides generators for $\tilde{\mathfrak{H}}$. For example, it is easily verified that $\mathscr{H}_1$ (Fig. 5) lies in an orbit of size 8 consisting of itself and its mirror image together with the six colourings of $\mathscr{H}_2$ (Fig. 7) by 0, 1 and 2.

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