ELSEVIER

Discrete Mathematics 261 (2003) 399–411

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Strict colorings of Steiner triple and quadruple systems: a survey

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Received 18 June 2001; accepted 30 September 2001

Dedicated to Alexander Rosa on the occasion of his 65th birthday

Abstract

The paper surveys problems, results and methods concerning the coloring of Steiner triple and quadruple systems viewed as mixed hypergraphs. In this setting, two types of conditions are considered: each block of the Steiner system in question has to contain (i) a monochromatic pair of vertices, or, more, restrictively, (ii) a triple of vertices that meets precisely two color classes.

1. Introduction

1.1. Mixed hypergraphs

The concept of *mixed hypergraph* was introduced in 1993 [30,31]. A mixed hypergraph is a triple $\mathscr{H}=(X,\mathscr{C},\mathscr{D})$, where X is a finite set of vertices, $|X|=n\geqslant 1$, and $\mathscr{C}$ and $\mathscr{D}$ are two arbitrary families of subsets of X. The elements of $\mathscr{D}$ are called

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[1] Research supported by INDAM.

[2] Research supported in part by Hungarian Scientific Research Fund under grants OTKA T-026575 and T-023969.

[3] Research supported by INDAM and CRDF BGP Award MM2-3018.

0012-365X/03/$ - see front matter
PII: S0012-365X(02)00485-5

$\mathscr{D}$*-edges*, while those of $\mathscr{C}$ are called $\mathscr{C}$*-edges*. The size of every $\mathscr{C}$-edge and every $\mathscr{D}$-edge is at least 2.

A *proper* k*-coloring* of a mixed hypergraph is a mapping from X into a set of k colors, $\{1,2,\ldots,k\}$, so that each $\mathscr{C}$-edge has at least two vertices with a $\mathscr{C}$ommon color and each $\mathscr{D}$-edge has at least two vertices with $\mathscr{D}$istinct colors. A mixed hypergraph is *colorable* if it has at least one proper coloring, and is k*-colorable* if it has a proper coloring with at most k colors. A mixed hypergraph is called *uncolorable* if it admits no proper coloring. A *strict* k*-coloring* is a proper k-coloring using all the k colors. In a colorable mixed hypergraph $\mathscr{H}$, the minimum number of colors in a proper coloring is its *lower chromatic number* $\chi(\mathscr{H})$; and the maximum number of colors in a strict coloring is its *upper chromatic number* $\bar{\chi}(\mathscr{H})$.

The "classical" weak coloring of the vertices of a hypergraph introduced by Erdös and Hajnal in 1966 [1,7] can be seen as a proper coloring of a mixed hypergraph in which there are only $\mathscr{D}$-edges. In the language of mixed hypergraphs, classical hypergraphs are called $\mathscr{D}$*-hypergraphs*, while mixed hypergraphs with only $\mathscr{C}$-edges are called $\mathscr{C}$*-hypergraphs*.

In a $\mathscr{D}$-hypergraph, the lower chromatic number coincides with the (weak) chromatic number [1,7] and the upper chromatic number trivially equals n. In a $\mathscr{C}$-hypergraph, the lower chromatic number trivially equals 1 but the upper chromatic number represents a value that is hard to determine. As one can see, in mixed hypergraphs, the upper chromatic number (in contrast to the lower chromatic number) becomes well-defined only if strict colorings are considered.

Mixed hypergraphs with $\mathscr{D}=\mathscr{C}$ are called *bi-hypergraphs*, and the subsets of X in consideration are called *bi-edges*. In any proper coloring of bi-hypergraphs, each bi-edge is neither *monochromatic* (because it is a $\mathscr{D}$-edge) nor *polychromatic* (because it is a $\mathscr{C}$-edge).

Another important concept referring to the colorings of mixed hypergraphs is that of uncolorability. Uncolorable mixed hypergraphs may easily be constructed. For example, a complete $\mathscr{D}$-graph $K_n, n\geqslant 3$ of $\mathscr{D}$-edges contained in a $\mathscr{C}$-edge of size n, or any mixed hypergraph having a bi-edge of cardinality 2 are uncolorable. The structure of uncolorable mixed hypergraphs is very general. The first paper on this topic is [29]. The problem of uncolorability did not originally exist in the theory of hypergraph coloring [1], as it arises only because of the interaction between $\mathscr{D}$-edges and $\mathscr{C}$-edges when a proper coloring is sought.

For each k, let r_k be the number of partitions of the vertex set into k nonempty parts (color classes) such that the coloring constraint is satisfied on each $\mathscr{C}$- and each $\mathscr{D}$-edge. Such partitions are called *feasible*. This means that in any feasible partition each $\mathscr{C}$-edge has at least two vertices in some $\mathscr{C}$ommon class of partition, and each $\mathscr{D}$-edge has at least two vertices which fall into two $\mathscr{D}$istinct classes of a partition. In fact r_k coincides with the number of strict k-colorings if we do not count permutations of the colors.

The integer vector

$$R(\mathscr{H})=(r_1,\ldots,r_n)=(0,\ldots,0,r_{\chi},\ldots,r_{\bar{\chi}},0,\ldots,0)$$

is the *chromatic spectrum* of $\mathscr{H}$.

The set of values k for which $\mathscr{H}$ has a strict k-coloring is the *feasible set* of $\mathscr{H}$, written as $S(\mathscr{H})$; this is the set of indices i such that $r_i > 0$.

The chromatic spectrum was introduced in [30]. In [11,12] it was shown that it may be *broken* (may have gaps), i.e. it may happen that $r_i = 0$ even for some $\chi < i < \bar{\chi}$. The existence of gaps in a chromatic spectrum is a peculiarity of strict colorings of mixed hypergraphs that was never encountered before. Note that gaps are not possible if we consider proper colorings. In [11,12] it was shown that the minimum number of the vertices of a mixed hypergraph with broken chromatic spectrum is 6. Current research is trying to define classes of mixed hypergraphs with the presence (or absence) of gaps in their chromatic spectrum.

"Classical" coloring theory also deals with subsets of X called *stable* (or, *independent*) *sets* which, by definition, contain no edges. The parameter α, called the *stability* (*independence*) *number*, is the maximum cardinality of a stable set.

In mixed-hypergraph coloring theory, it is possible to define three different types of subsets of X, called $\mathscr{D}$*-stable sets*, $\mathscr{C}$*-stable sets* and *bi-stable sets*, which characterize subsets of vertices of $\mathscr{H}$ that contain no $\mathscr{D}$-edges (corresponding to stable sets), subsets containing no $\mathscr{C}$-edges, and subsets containing neither $\mathscr{D}$-edges nor $\mathscr{C}$-edges. The parameters $\alpha_{\mathscr{D}}$, $\alpha_{\mathscr{C}}$ and α_{bi} refer to these subsets and indicate those with maximum cardinality, called $\mathscr{D}$-stability, $\mathscr{C}$-stability and bi-stability numbers, respectively. Although, when considered separately, these subsets all are stable, in colorings of mixed hypergraphs they play different roles.

In [30] the following relation was presented between the upper chromatic number $\bar{\chi}$ and $\alpha_{\mathscr{C}}$ for any mixed hypergraph $\mathscr{H}$:

$$\bar{\chi}(\mathscr{H}) \leqslant \alpha_{\mathscr{C}}(\mathscr{H}).$$

It is clear that in bi-hypergraphs the three different families of stable sets coincide, as do the three parameters $\alpha_{\mathscr{C}}$, $\alpha_{\mathscr{D}}$ and α_{bi}.

1.2. Steiner triple and quadruple systems

In this paper, we present the most significant results obtained in the study of the strict colorings of particular *Steiner systems*: Steiner triple systems, also written as STSs for short, and Steiner quadruple systems, or SQSs.

A Steiner system $S(t,k,v)$ is a pair $(X,\mathscr{B})$, where X is a finite set of vertices and $\mathscr{B}$ is a family of subsets of X whose elements are called *blocks* and which has the following two properties: each block in $\mathscr{B}$ has a cardinality k and each subset of t vertices of X is contained in precisely one block of $\mathscr{B}$. Systems of the type $S(2,3,v)$ are *Steiner triple systems*, often denoted STS(v), where v is the number of vertices in X, and it is known that $v \equiv 1$ or $3 \pmod 6$ is necessary and sufficient for an STS(v) to exist. Systems of the type $S(3,4,v)$ represent *Steiner quadruple systems*, or SQS(v), where the condition for existence is $v \equiv 2$ or $4 \pmod 6$ [10]. An $S(t,k,v)$ is therefore a particular hypergraph and it can also be viewed as a mixed hypergraph.

When coloring an $S(t,k,v)$ in which each block is considered just as a $\mathscr{C}$-edge, we have a $\mathscr{C}$-hypergraph; in the case of STS we denote it by CSTS(v). In the case of SQS we denote it by CSQS(v). If, on the other hand, each block is assumed to be both a

$\mathscr{C}$-edge and a $\mathscr{D}$-edge, we have a bi-hypergraph, or a BSTS(v) for triple systems and a BSQS(v) for quadruple systems [19,16,20,17,5].

The paper is divided into four sections. In the second section we survey the most significant results obtained for BSTSs and CSTSs, and in the third those obtained for CSQSs and BSQSs. In the fourth section we present the parameter mb that characterizes particular strict colorings of CSTSs and CSQSs [18].

2. BSTSs and CSTSs

Let us consider a triple system STS(v). If it is viewed as a CSTS(v), then evidently, it is colorable, and each one of its blocks can be colored either with one color, or with two colors. If it is viewed as a BSTS(v), then the problem of colorability arises. In a colorable BSTS(v), in every proper coloring each of the blocks will be colored with precisely two colors.

It is clear that if each block of an STS(v) is a $\mathscr{D}$-edge, the system will be colored according to "classical" (or weak) coloring; the most significant results obtained on such colorings can be found in [4,23,24,25,27].

BSTSs and CSTSs were recently studied in [5,3,14,15,16,17,18,19,20]. Suppose that a BSTS(v) or a CSTS(v) is colorable with a strict coloring $\mathscr{P}$ using h colors, and let the X_i, with $1 \leqslant i \leqslant h$, be the color classes of $\mathscr{P}$, i.e. the set of vertices colored with the color i. In addition, let $n_i = |X_i|$. For the sake of convenience we can re-order the labels i of X_i in such a way that the cardinalities of the color classes form a non-decreasing sequence, i.e. that $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_h$. Henceforward, given a triple or quadruple system $\mathscr{H}$, we will use the integer vector $c = (n_1, n_2, \ldots, n_h)$ to indicate the set of strict colorings of $\mathscr{H}$ which use h colors and whose color classes have the same cardinality as the components of the vector c.

In [5,14,18,19] the following necessary conditions for the existence of a strict coloring of a BSTS or a CSTS were proven, for any subset I of the set $\{1, 2, 3, \ldots, h\}$.

Theorem 1 (Milazzo [18], Milazzo and Tuza [19]). *Let $\mathscr{P}$ be a strict coloring of a* BSTS *or* CSTS *and $S_k = \bigcup_{j \in I} X_j$ be the union of $k = |I|$ color classes of $\mathscr{P}$ with respective cardinalities $|X_j| = n_j$ for all $j \in I$. Denoting $s_k = |S_k|$, the inequalities*

$$s_k(s_k - 1) \leqslant 3 \sum_{j \in I} n_j(n_j - 1) \qquad (k)$$

hold whenever $2 \leqslant |I| \leqslant h$.

Proof. For the number b_k of blocks containing more than one element of S_k we have $b_k \geqslant s_k(s_k - 1)/6$. Since each of these blocks must contain a monochromatic pair *inside* S_k (for otherwise the block would be 3-colored), and each monochromatic pair is contained in just one block, the upper bound $b_k \leqslant \sum_{j \in I} n_j(n_j - 1)/2$ also holds, yielding (k). □

Other necessary conditions for the existence of a strict coloring of a BSTS or CSTS can be found in [5,14,16,19,17,20]. The theory of strict coloring in Steiner systems is mainly based on the cardinalities of the classes of colors, which will be dealt with in greater detail in Section 3. The following theorem gives an indication of the color class parity in a strict coloring of a BSTS or a CSTS.

Theorem 2 (Colbourn et al. [5], Lo Faro et al. [14]). *If $\mathscr{P}$ is a strict coloring of a* BSTS *or* CSTS, *then there is only one color class with an odd cardinality.*

In [19] Zs. Tuza achieved an important technical result related to number theory, which made it possible to determine the best upper bound of $\bar{\chi}$ for any CSTS or BSTS that is "strictly" colorable. Using Lemma 2 of [19], Theorem 1 yields the following result.

Theorem 3. *If $\mathscr{P}$ is a strict coloring of a* BSTS *or* CSTS *using h colors, then $n_i \geqslant 2^{i-1}$ for all $1 \leqslant i \leqslant h$.*

From historical point of view it can be mentioned that the idea to look at Steiner systems as mixed hypergraphs was first formulated by the third author in 1993 in Catania at the research seminar of Mario Gionfriddo. The next theorem is the first result obtained in the study of strict colorings in Steiner systems. It was obtained by Zs. Tuza in 1993, following a series of discussions between M. Gionfriddo, Zs. Tuza and V. Voloshin concerning the possible existence of strict colorings in triple and quadruple systems. The result initially referred to CSTSs and was later applied to BSTSs, after Milazzo and Voloshin defined them in Chişinău in 1994. The theorem gives the best upper bound for the upper chromatic number of any colorable BSTS or CSTS.

Theorem 4 (Milazzo and Tuza [19]). *For any* CSTS(v), *and any colorable* BSTS(v), *if $v \leqslant 2^h - 1$, then $\bar{\chi} \leqslant h$.*

Proof. Let $\mathscr{P}$ be a strict coloring of a BSTS(v) or CSTS(v), which uses the maximum number of colors. Both Theorems 1 and 3 are valid, and a comparison between the first and last term in the following sequence proves the theorem:

$$2^h - 1 \geqslant v = \sum_{i=1}^{\bar{\chi}} n_i \geqslant \sum_{i=1}^{\bar{\chi}} 2^{i-1} = 2^h - 1 = 2^{\bar{\chi}} - 1. \qquad \square$$

Corollary 1 (Milazzo and Tuza [19]). *If $\mathscr{H}$ is a* BSTS(v) *or* CSTS(v) with $v \leqslant 2^h - 1$ *and $\bar{\chi}(\mathscr{H}) = h$, then*

1. $v = 2^h - 1$;
2. *the h color classes of $\mathscr{H}$ have the following cardinalities:*

   $2^0,\ 2^1,\ 2^2, \ldots, 2^{h-1}$

   and they are all bi-stable sets if $\mathscr{H} =$ BSTS *or $\mathscr{C}$-stable sets if $\mathscr{H} =$* CSTS;

Table 1

BSTS(7)	$\bar{\chi}=3$
BSTS(9)	$\bar{\chi}=3$
BSTS(13)	$\bar{\chi}=3$
BSTS(15)*	$\bar{\chi}=3$
BSTS(19)*	$\bar{\chi}=3,4$
BSTS(21)*	$\bar{\chi}=3$

3. *$\mathscr{H}$ is obtained by means of a sequence of "double-plus-one" constructions, starting from the* STS(3). □

Corollary 1 gives the possibility to determine an infinite class of colorable BSTSs and CSTSs, both having order 2^h-1 and upper chromatic number $\bar{\chi}(\mathscr{H})=h$. In addition, this result allows us to state that the upper bound for $\bar{\chi}$ determined in Theorem 4 is the best possible.

In general, it is possible to obtain infinite classes of BSTSs or CSTSs by exploiting the following theorem, presented in [20].

Theorem 5 (Milazzo and Tuza [20]). *If $\mathscr{P}$ is a strict coloring of a* BSTS(v) *or a* CSTS(v) *of the type $(n_1,n_2,\dots,n_h,n_{h+1},\dots,n_k)$, and if the coloring $\mathscr{P}'$ of type $(n_1,n_2,\dots,n_h)$ is a strict coloring of a* BSTS(v'), *then $n_i \geqslant 2^{i-h-1}\cdot(v'+1)$ for all $i>h$.*

In [17,20], by exploiting Theorem 5, four infinite classes of BSTS(v) and CSTS(v) of orders $v=10\times 2^h-1$ and $v=14\times 2^h-1$ were determined, obtained from sequences of double-plus-one constructions starting from the system BSTS(9) and the two non-isomorphic systems BSTS(13), all with $\bar{\chi}(\mathscr{H})=h+3$.

In [14,17,20] and [8,26] (for BSTS(15)) all the upper chromatic numbers for the following "small" colorable BSTSs were determined:

In Table 1 the asterisks indicate that there exist some uncolorable BSTS(v). Similar results were obtained for "small" CSTSs [14,17,20]; it is recalled that for these systems $\chi=1$.

Since mixed hypergraphs may have gaps in their chromatic spectra, the problem of finding a BSTS or a BSQS with broken chromatic spectrum naturally arose. Obviously, the problem of identifying triple or quadruple systems of this type is highly significant and up to now no Steiner system with these features has been determined. (The only known designs with gaps in the chromatic spectrum are the so-called BP_3-designs, found recently by L. Gionfriddo [9].) In [3,14] the authors studied the value of the lower chromatic number and the spectra of "small" BSTSs.

The following theorem determines the first of the BSTSs belonging to the class identified by Corollary 1 whose upper and lower chromatic numbers coincide.

Theorem 6 (Buratti et al. [3]). *For* BSTS(v), *with $v=2^h-1$, obtained from a sequence of double-plus-one constructions starting from* STS(3), *if $h<10$, then $\bar{\chi}=\chi=h$.*

M. Buratti proved in [3] that for any h, a BSTS(2^h-1) with $\bar{\chi}=\chi=h$ does exist.

The problem of whether Theorem 6 holds for any value of h or whether there exists a particular $\bar{h}$ for which a BSTS$(2^{\bar{h}}-1)$ with $\chi\neq\bar{\chi}=\bar{h}$ exists, remains open.

In [14] other "small" BSTSs in which the upper chromatic number is equal to the lower chromatic number were studied, and the lowest order with which $\chi\neq\bar{\chi}$ was determined.

Theorem 7 (Lo Faro et al. [14]). *For all* BSTS(v) *of order* $v<19$, $\chi=\bar{\chi}$ *holds, while with* $v=19$ *there exist some* BSTS(19) *with* $\chi=3$ *and* $\bar{\chi}=4$.

A deep investigation of coloring properties of BSTS (termed as bicoloring of STS) was undertaken in [5]. Colbourn et al. presented particular constructions determining BSTSs that have a strict coloring with three colors.

Theorem 8 (Colbourn et al. [5]). *If there exists a* BSTS(u) *that can be colored with a strict coloring* (a,b,c), *where* $c=\max\{a,b,c\}$ *and* $c\leqslant a+b$, *and there also exists a* BSTS(v) *that can be colored with a strict coloring* (x,y,z), *then there exists a* BSTS$(u\cdot v)$ *that can be colored with a strict coloring using three colors*,

$$(ax+by+cz, ay+bz+cx, az+bx+cy).$$

In [5] other similar constructions are presented, determining BSTSs that can be colored with strict colorings using four and five colors.

Modifying Stinson's [28] "hill-climbing" algorithm with $v\leqslant 1000$, $v\leqslant 157$ and $v\leqslant 105$, Colbourn et al. [5] determined BSTS(v) that can be colored with three, four and five colors, respectively, and also presented three conjectures on the existence of BSTSs that are three-, four- and five-colorable in the strict sense [5].

Obviously, all CSTSs are colorable, but there are uncolorable BSTSs. Historically, the first result in this sense can be found in [19], where an infinite family of uncolorable BSTS(v) is proved to exist.

Theorem 9 (Milazzo and Tuza [19]). *All* BSTS(v) *that do not contain bi-stable sets of cardinality at least* $v/\lceil\log_2(v+1)\rceil$ *are uncolorable*.

De Brandes and Rödl [2] proved that there exist infinitely many values of v admitting STS(v) of order v that do not possess stable sets with a cardinality of $c\sqrt{v}\ln v$. (Here any $c>4$ is a suitable choice.) For these systems, $\chi\geqslant\sqrt{v}/(c\ln v)$, but according to Theorem 4, $\bar{\chi}$ is upper-bounded by $\lceil\log_2(v+1)\rceil$, hence for large enough v we obtain the contradiction $\bar{\chi}<\chi$. This implies the existence of an infinite uncolorable family of BSTSs.

The first explicitly described uncolorable BSTS(15) was found by Ganter in 1997 [8].

Since the inequality $\bar{\chi}(\mathscr{H})\leqslant\alpha_{\mathscr{C}}(\mathscr{H})$ is valid for any mixed hypergraph $\mathscr{H}$, and in bi-hypergraphs the $\mathscr{C}$-stable sets are the same as the stable sets in the corresponding ($\mathscr{D}$-) hypergraphs, the STSs with stable sets of bounded size deserve a special attention with respect to the uncolorability of BSTS. More detailed analysis showed the following:

Theorem 10 (Colbourn and Rosa [6]). *If* $\alpha_{\mathscr{C}}(\mathrm{BSTS}(v)) \leqslant v/3$, *then* BSTS($v$) *is uncolorable*.

Proof. Let $\alpha_{\mathscr{C}}(\mathrm{BSTS}(v)) \leqslant v/3$ and suppose there exists a strict k-coloring of BSTS(v) in which the color classes have the sizes $n_1, n_2, \ldots, n_k$ where $k \geqslant 3$, $\Sigma_{i=1}^{k} n_i = v$, and $n_i \leqslant v/3$ for all i. Since each block contains precisely two vertices with the same color, the number of blocks equals $\Sigma_{i=1}^{k} \binom{n_i}{2}$. Let $v = 6t+3$. The maximum value of $\Sigma_{i=1}^{k} \binom{n_i}{2}$ is attained when $k = 3$ and $n_1 = n_2 = n_3 = 2t+1$. But then

$$\Sigma_{i=1}^{k} \binom{n_i}{2} = 3 \binom{2t+1}{2} < (3t+1)(2t+1),$$

which is the number of blocks in BSTS(6t+3), a contradiction. The case when $v = 6t+1$ is handled similarly. □

There are uncolorable BSTSs of all orders $\geqslant 15$. Namely, further simple calculations based on [2] (Rosa, private communication, 1998) give that any BSTS(15) with $\alpha_{\mathscr{C}} \leqslant 7$ is uncolorable. This implies that a BSTS(15) is colorable if and only if it contains BSTS(7) as a subsystem, and hence is obtained by the double-plus-one construction. Furthermore, any BSTS(19) with $\alpha_{\mathscr{C}} = 7$, any BSTS(21) with $\alpha_{\mathscr{C}} \leqslant 8$, any BSTS(25) with $\alpha_{\mathscr{C}} \leqslant 9$, any BSTS(27) with $\alpha_{\mathscr{C}} \leqslant 11$, any BSTS(31) with $\alpha_{\mathscr{C}} \leqslant 13$ and any BSTS(33) with $\alpha_{\mathscr{C}} \leqslant 13$ are uncolorable. As one can see, Theorem 10 provides a necessary but not sufficient condition of colorability, so the criteria of (un)colorability in the general case are yet to be found.

3. BSQS and CSQS

3.1. BSQS

In this section we will use the same notation as in the previous one.

A BSQS(v) is a quadruple system in which each block is a $\mathscr{C}$-edge and a $\mathscr{D}$-edge at the same time. The following result exhibits a necessary condition for the existence of strict colorings.

Theorem 11 (Milazzo [16]). *If* $\mathscr{P}$ *is a strict coloring for* BSQS(v) *with h colors*, *then*

$$\sum_{i=1}^{h} \binom{n_i}{2} (v - n_i) \geqslant \frac{1}{2} \binom{v}{3} + \sum_{i=1}^{h} \binom{n_i}{3}.$$

In a strict coloring of a BSQS each block can only be colored with two or three colors, and in three different ways. The numbers of the blocks colored in these three different ways have been determined and only depend on the cardinality of the color classes.

Table 2

BSQS(8)	$\bar{\chi}=3$
BSQS(10)	$\bar{\chi}=4$
BSQS(16)	$\bar{\chi}=3$

Theorem 12 (Milazzo [16]). *If $\mathscr{P}$ is a strict coloring for a* BSQS(v), *then*:

1. $\sum_{i=1}^{h}\binom{n_i}{3}$ *blocks are colored with two colors unequally; more specifically, three vertices in each of these blocks are colored with the same color;*
2. $(c'_h/2)$ *blocks are colored with three colors, where c'_h is the number of all the different trichromatic triples in X colored with the colors of $\mathscr{P}$;*
3. $|\mathscr{B}|-c'_h/2-\sum_{i=1}^{h}\binom{n_i}{3}$ *blocks are colored with two colors equally; more specifically, there are two monochromatic pairs of vertices in each of these blocks.*

In [14,16,19,17] the following results were obtained for "small" BSQSs:

The systems in Table 2 are all colorable, and it seems to us that the case of $v=14$ has not yet been investigated, despite it would be very natural, and it does not look like a very hard problem. Proof of the colorability of all the BSQS(16) is of a greater significance as it proves the colorability of a very large number of systems (in the range of thousands). It is also interesting that the intermediate value of $v=10$ yields a larger upper chromatic number.

Theorem 13 (Lo Faro et al. [15]). *Each* BSQS(16) *is colorable and* $\bar{\chi}=3$.

This result also gives us information about the chromatic spectrum of the quadruple systems BSQS(16). For systems containing blocking sets, in fact, $\chi=2$ holds, while for the others we have $\bar{\chi}=\chi=3$ and $|S(\mathscr{H})|=1$.

No uncolorable BSQS has been found yet, and this problem seems to be linked to the determination of SQS(v) with particular independent sets.

3.2. CSQS

A CSQS is a quadruple system in which each block is a C-edge. If $\mathscr{P}$ is a strict coloring of such a system, then each block can be colored with one, two or three colors. In [16] the following necessary conditions for the existence of a strict coloring of any CSQS were determined.

Theorem 14. *If $\mathscr{P}$ is a strict coloring of a* CSQS, *then*

$$\sum_{j=1}^{i}\binom{n_j}{3}+2\sum_{j=1}^{i}\binom{n_j}{2}(s_i-n_j)\geqslant\binom{s_i}{3} \qquad (i)$$

for all $1\leqslant i\leqslant h$.

Theorem 15. *If $\mathscr{P}$ is a strict coloring of a* CSQS$(v)=(X,\mathscr{B})$, *then*

$$|\mathscr{B}| \leqslant \sum_{i=1}^{h} \binom{n_i}{2} \frac{v-2}{2} - \sum_{i=1}^{h} \frac{5}{4} \binom{n_i}{3}.$$

For an infinite class of CSQS(2^k), obtained by means of a sequence of "*double-construction*" [13] starting from the trivial system SQS(4), it can easily be proved that there always exists a strict coloring using $k+1$ colors, so we have that $\bar{\chi} \geqslant k+1$ for this class of CSQSs.

The following two theorems summarize results obtained in [17,18].

Theorem 16. *For any* CSQS(2^k) *with* $k=2,3,4$ *or* 5, $\bar{\chi} \leqslant k+1$.

In the systems considered in Theorem 16, it is possible to determine a characterization of the strict colorings, that use $k+1$ colors, by means of the cardinalities of the color classes. This result is similar to the one in part *2* of Corollary 1, but is only obtained for a finite number of values k.

Theorem 17. *A coloring of a* CSQS(2^k), *where* $k=2,3,4$, *using* $k+1$ *colors is of the type* $(2^0,2^0,2^1,2^2,\ldots,2^{k-1})$.

The problem of determining an upper bound for the upper chromatic number when $k>5$ remains an open issue. More specifically, if there exists some colorable CSQS(2^k) with more than $k+1$ colors, then the following result holds.

Theorem 18 (Milazzo and Tuza [19]). *If a* CSQS(2^k) *is colored by a strict coloring using more than* $k+1$ *colors, then each color class has a cardinality greater than one*.

Nevertheless, it may happen to be the case that no such colorings exist, and the preceding two theorems are valid for all k.

In general, for Steiner systems of the type S$(t,t+1,v)$ in which all the blocks are $\mathscr{C}$-edges, it has been proved that the asymptotic behavior of the upper chromatic number of S$(t,t+1,v)$ is at most logarithmic with respect to the order of the system [21]. We do not know whether this upper bound is tight—apart from a multiplicative constant—for each (or, any) $t \geqslant 4$. This problem seems to be hopelessly hard at present, because so far only finitely many S$(t,t+1,v)$ are known with block size $t+1=5$ and 6, and none with larger than 6.

4. The "monochromatic block number"

A frequent problem in strict colorings has been that of determining the number of monochromatic blocks ($\mathscr{C}$-edges) in the strict colorings of $\mathscr{H}$ that use the maximum number of colors, or in general in strict colorings that use h colors. For Steiner systems it has been proved, for example, that if CSQS(8) is colored with four colors, then there exists one and only one monochromatic block [19].

In [18], Steiner systems whose blocks are always $\mathscr{C}$-edges, were studied in general for a CS(t,k,v). The parameter mb, called "*monochromatic block number*" was defined, representing the number of monochromatic blocks in a strict coloring $\mathscr{P}$ of a CS(t,k,v).

In a CSTS(v) the parameter mb has been determined for any coloring $\mathscr{P}$.

Theorem 19 (Milazzo [18]). *If $\mathscr{P}$ is a strict coloring of a* CSTS(v), *then*

$$\mathrm{mb}(\mathscr{P}) = \frac{1}{2}\sum_i \binom{n_i}{2} - \frac{v(v-1)}{12}.$$

As an important consequence, Theorem 19 yields that the parameter mb depends only on the cardinalities of the color classes—i.e., on the type of the coloring $\mathscr{P}$—and not on the actual positions of the classes with respect to the blocks.

A more complex problem is that of determining mb for CSQS(v) systems. In these systems, in fact, only the following result has been obtained for the CSQS(2^k) colored with a particular strict coloring $\bar{\mathscr{P}}$.

Theorem 20 (Milazzo [18]). *If a* CSQS(2^k), *where* $k \geqslant 3$, *is colored with a strict coloring $\bar{\mathscr{P}}$ using $k+1$ colors and in $\bar{\mathscr{P}}$ there is a color class with cardinality one, then*

$$\frac{c_k}{2} \leqslant \mathrm{mb}(\bar{\mathscr{P}}) \leqslant \sum_{i=2}^{k-1} |B(2^i)|, \tag{1}$$

where the quantity $|B(2^i)| = \frac{1}{4}\binom{2^i}{3}$ *is the number of blocks in an* SQS(2^i) *and*

$$c_k = \binom{2^{k-1}}{3} - \left[\sum_{i=2}^{k-2}\binom{2^i}{3} + \sum_{i=1}^{k-2}\binom{2^i}{2}(2^{k-1}-2^i)\right].$$

Moreover, the inequality on the right-hand side of (1) *is the best possible.*

In [18] it was proved that the strict coloring $\bar{\mathscr{P}}$ is a very particular coloring because it is of the type $(2^0, 2^0, 2^1, 2^2, \ldots, 2^{k-1})$, already encountered in Theorem 17. In [19] an infinite class of the CSQS(2^k), all colorable with $\bar{\mathscr{P}}$, was determined.

The left-hand side inequality in (1) is best possible for CSQS(8) and CSQS(16), as shown in [18]. The problem of determining whether this inequality is the best possible for all or only some values of k remains an open problem.

5. Concluding remarks

In this paper, we have presented the main results obtained in the field of strict colorings for BSTSs, CSTSs, BSQSs and CSQSs.

Early research was oriented towards studying the upper chromatic number and then problems of colorability.

Following the work of Jiang et al. [11,12], research began to focus on the lower chromatic number and, above all, the characterization of the chromatic spectrum of these systems.

Strict colorings of Steiner systems of the STS and SQS types have given rise to the study of different types of colorings for these systems, especially those of the type S$(2,4,v)$ and SQS [22].

It is worthwhile pointing out in this section the most significant problems that still remain to be investigated.

1. Finding a criterion of colorability of BSTS.
2. Determination of uncolorable BSQSs.
3. Determination of an upper bound for the upper chromatic number of a BSQS or a CSQS.
4. The study of BSTSs and BSQSs where the upper and lower chromatic numbers coincide.
5. Identification of the chromatic spectrum $(r_1, r_2, \ldots, r_v)$ of BSTSs and BSQSs, determining the values of r_i.
6. Determination of BSTSs and BSQSs with a broken spectrum (or, to prove that all BSTS (BSQS) have a continuous spectrum).
7. The monochromatic block number for CSQSs.
8. The study of constructions of STSs and SQSs that are colorable or uncolorable.
9. Determination of a colorable or uncolorable BSTS(v) or BSQS(v), for a certain possible v.

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