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Modal Predicates and Coequations

Alexander Kurz [1,2]

CWI
P.O. Box 94079, 1090 GB Amsterdam
The Netherlands

Jiří Rosický [3,4]

Masaryk University
Janáčkovo nám. 2a, 662 95 Brno
Czech Republic

Abstract

We show how coalgebras can be presented by operations and equations. This is a special case of Linton's approach to algebras over a general base category $\mathcal{X}$, namely where $\mathcal{X}$ is taken as the dual of sets. Since the resulting equations generalise coalgebraic coequations to situations without cofree coalgebras, we call them coequations. We prove a general co-Birkhoff theorem describing covarieties of coalgebras by means of coequations. We argue that the resulting coequational logic generalises modal logic.

Introduction

Let us start with recalling that universal algebras are defined as sets equipped with operations subjected to equations. Operations can be infinitary. Given a set X, a mapping $f : A^X \to A$ is an X-ary operation on a set A. One is often working with Y-tuples $f_y : A^X \to A$, $y \in Y$, of X-ary operations on a set A. These Y-tuples uniquely correspond to mappings $f : A^X \to A^Y$. Starting with a set of operations one always has free algebras. But there are important examples of universal algebras given by a class of operations which still have free algebras (complete semilattices, compact Hausdorff spaces). Linton

[1] Partially supported by the Ministry of Education of the Czech Republic under the project MSM 143100009.
[2] Email: kurz@cwi.nl
[3] Supported by the Grant Agency of the Czech Republic under the grant 201/01/0148.
[4] Email: rosicky@math.muni.cz

showed in [14] that equationally defined universal algebras are, under the existence of free algebras, precisely the monadic categories over Set. Moreover, in [15], he generalised the result from sets to any base category $\mathcal{X}$. In that work, operations are still mappings

$$f : A^X \to A^Y$$

where A, X, and Y are objects in $\mathcal{X}$ and A^X is the set of all morphisms $X \to A$. Note that, in Set, A^X coincides with the X-fold product of A. In general, however, it is important to consider A^X as a set of morphisms because the other approach would be too special for a general base category $\mathcal{X}$. In particular, it would be too special for $\mathcal{X} = \mathsf{Set}^{\mathrm{op}}$. Davis [7] used Linton's approach for introducing universal coalgebras over Set even without assuming the existence of cofree coalgebras (i.e., free algebras over $\mathsf{Set}^{\mathrm{op}}$). The second author then considered Linton's algebras over a general base category, without the existence of free algebras, in [23].

There is another way of defining universal algebras over a general base category. One starts with an endofunctor $F : \mathcal{X} \to \mathcal{X}$ and defines F-algebras as objects A equipped with a morphism $\alpha : FA \to A$. These algebras are called F-dynamics in Manes [17] and were extensively studied by Trnková and her students in Prague (cf. [4]). Notably, Reiterman [21] compared F-algebras with algebras given by operations and equations.

There is a revived interest in universal coalgebra motivated by its connections with the theory of systems (see [24]). Coalgebras are here understood as F-algebras over $\mathcal{X} = \mathsf{Set}^{\mathrm{op}}$, i.e., as sets A equipped with a mapping $\alpha : A \to FA$ where $F : \mathsf{Set} \to \mathsf{Set}$. Our aim is to show the potential of defining coalgebras by means of operations and equations (Section 2). Our equations for coalgebras dualise equations for algebras and generalise previous concepts of coalgebraic coequations (cf. [24,8,12,10]) to situations without cofree coalgebras. We prove a general co-Birkhoff theorem showing that covarieties of coalgebras are always definable by coequations (Section 3) and we present a full explanation of Davis's characterisation of coequational categories (Section 4). Finally, we show that the dual of operations $f : A^X \to A^Y$ are modal predicates, i.e., predicates that are invariant under bisimulation. This gives rise to general notions of modal predicate and modal operator and shows that our coequational logic is a generalised modal logic (Section 5).

We work in Gödel-Bernays set theory with the axiom of choice for classes. It means that all proper classes are isomorphic, in particular to the class *Ord* of all ordinals. Categories are assumed to be locally small, which means that they have a class of objects and sets of morphisms between any two given objects. Occasionally we encounter categories which do not satisfy this requirement and we call them *illegitimate*.

1 Preliminaries

Given an endofunctor $F : \mathcal{X} \to \mathcal{X}$, a *$F$-coalgebra* $\mathsf{A} = (A, \alpha)$ consists of an object $A \in \mathcal{X}$ and an arrow $\alpha : A \to FA$. F-coalgebras form a category $\mathsf{Coalg}(F)$ where a *F-morphism* $f : (A, \alpha) \to (B, \beta)$ is an arrow $f : A \to B \in \mathcal{X}$ such that $Ff \circ \alpha = \beta \circ f$:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] & FA \arrow[d, "Ff"] \\ B \arrow[r, "\beta"'] & FB \end{tikzcd}$$

The *forgetful functor* $U : \mathsf{Coalg}(F) \to \mathcal{X}$ maps a coalgebra (A, α) to A and a morphism $f : (A, \alpha) \to (B, \beta)$ to the arrow $f : A \to B$ in $\mathcal{X}$. U creates and hence preserves colimits.

Set denotes the category of sets and functions and $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$ the covariant powerset functor. [5] We also use the convention $2 = \{0, 1\}$ and call the elements of 2 truth values.

Given $A, X \in \mathcal{X}$, the set of arrows $A \to X$ is denoted by X^A. For $f : X \to Y$, the function $f^A : X^A \to Y^A$ is defined as $f^A(g : A \to X) = f \circ g$. For $f : A \to B$, the function $X^f : X^B \to X^A$ is defined as $X^f(g : B \to X) = g \circ f$. [6] The assignment $- \mapsto X^-$ gives rise to a functor $X^- : \mathcal{X}^{\mathrm{op}} \to \mathsf{Set}$. We write X^U for the functor $\mathcal{A}^{\mathrm{op}} \to \mathsf{Set}$ obtained from composing $U : \mathcal{A} \to \mathcal{X}$ and $X^- : \mathcal{X}^{\mathrm{op}} \to \mathsf{Set}$.

A *concrete category* is a faithful functor $U : \mathcal{K} \to \mathsf{Set}$. A functor $F : \mathcal{K} \to \mathcal{K}'$ is a concrete functor between the concrete categories $U : \mathcal{K} \to \mathsf{Set}$ and $U : \mathcal{K}' \to \mathsf{Set}$ iff $U'F = U$. Concrete categories are isomorphic if the isomorphisms are concrete functors. In case that $\mathcal{K}$ has coproducts, a *covariety* in $\mathcal{K}$ is a full subcategory which is closed under coproducts, subobjects and quotients. [7]

A *measurable cardinal* κ is a cardinal on which a non-principal κ-complete ultrafilter exists. An ultrafilter is κ-complete if it is closed under intersections of cardinality $< \kappa$ and it is non-principal if it does not contain a singleton-set.

Each category of coalgebras comes equipped with a notion of bisimulation or, as we prefer to call it, behavioural equivalence. There are different but equivalent ways to define this notion, the following one seems to be ap-

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[5] $\mathcal{P}X = \{Y \mid Y \subseteq X\}$ and $\mathcal{P}(f : X \to X') : \mathcal{P}X \to \mathcal{P}X'$, $X \supseteq Y \mapsto \{f(y) \mid y \in Y\}$.

[6] For instance, with $\mathcal{X} = \mathsf{Set}$, 2^A is the set of subsets of A and $2^f : 2^B \to 2^A$ is the inverse-image-map of $f : A \to B$.

[7] We say that B is a *subobject* of A if there is $f : \mathsf{B} \to \mathsf{A}$ such that Uf is injective; B is a *quotient* of A if there is $f : \mathsf{A} \to \mathsf{B}$ such that Uf is surjective.

propriate in our setting (in the case $\mathcal{X} = \mathsf{Set}$). It formalises the idea that *behavioural equivalence is the smallest equivalence relation that is invariant under coalgebra morphisms*, see [13]. We define two notions, the second one taking 'colourings' into account: Given C, a *colouring* v for a coalgebra A is an arrow $U\mathsf{A} \to C$. Instead of $U : \mathsf{Coalg}(F) \to \mathsf{Set}$, we only require a functor $U : \mathcal{A} \to \mathsf{Set}$ in our definition of behavioural equivalence.

Definition 1.1 (Behavioural Equivalence) *Consider a functor* $U : \mathcal{A} \to \mathsf{Set}$ *and* $C \in \mathsf{Set}$. *In the following* A, B *range over* $\mathcal{A}$, v, w *over valuations* $U\mathsf{A} \to C$, $U\mathsf{B} \to C$, *respectively, and* a, b *over elements of* $U\mathsf{A}, U\mathsf{B}$, *respectively, and* f *over morphisms* $\mathsf{A} \to \mathsf{B}$.

(i) $\sim$ *is the equivalence relation generated by all*

$$(\mathsf{A}, a) \sim (\mathsf{B}, Uf(a)).$$

(ii) $\sim_C$ *is the equivalence relation generated by all*

$$(\mathsf{A}, w \circ Uf, a) \sim_C (\mathsf{B}, w, Uf(a)).$$

If $(\mathsf{A}, a) \sim (\mathsf{B}, b)$ *we say that* a *and* b *are behaviourally equivalent. If* $(\mathsf{A}, v, a) \sim_C (\mathsf{B}, w, b)$ *we say that* a *and* b *are* C*-behaviourally equivalent.*

Remark 1.2

(i) In case that $\mathcal{A} = \mathsf{Coalg}(F)$ and F preserves weak pullbacks, $(\mathsf{A}, a) \sim (\mathsf{B}, b)$ iff a and b are related by a bisimulation in the sense of Aczel and Mendler [1].

(ii) In particular, in case of $\mathsf{Coalg}(\mathcal{P})$, behavioural equivalence is the familiar bisimulation between (unlabelled) transition systems.

The following gives an alternative characterisation of behavioural equivalence.

Proposition 1.3 *Suppose* $U : \mathcal{A} \to \mathsf{Set}$ *creates colimits.* $(\mathsf{A}_1, v_1, a_1) \sim_C (\mathsf{A}_2, v_2, a_2)$ *iff there are* B, w *and morphisms* $f_i : \mathsf{A}_i \to \mathsf{B}$ *such that*

$$\begin{tikzcd}
 & C & \\
U\mathsf{A}_1 \arrow[ur, "v_1"] \arrow[r, "Uf_1"'] & U\mathsf{B} \arrow[u, "w"] & U\mathsf{A}_2 \arrow[ul, "v_2"'] \arrow[l, "Uf_2"]
\end{tikzcd} \tag{1}$$

commutes and $(Uf_1)(a_1) = (Uf_2)(a_2)$.

Proof Let $\approx$ denote the relation defined by condition (1). $\approx \subseteq \sim$ is immediate. For the converse, note that $\approx$ contains the generating pairs of $\sim$ and is reflexive and symmetric. $\approx$ is transitive, since $\mathcal{A}$ has pushouts and U preserves these. $\square$

2 Coequational Categories

As we have explained in the introduction, coalgebras can be introduced as sets A equipped with operations

$$f : X^A \to Y^A$$

where X, Y are sets and X^A denotes the set of all mappings $A \to X$ (because X^A in Set is A^X in $\mathsf{Set}^{\mathrm{op}}$).

Definition 2.1 (Coalgebras for a signature) *A signature Σ is a class of operation symbols σ each equipped with a pair (X, Y) of sets. We call σ a (X, Y)-ary operation symbol. A Σ-coalgebra A is a set A together with mappings*

$$\sigma_{\mathsf{A}} : X^A \to Y^A$$

for each (X, Y)-ary operation symbol $\sigma \in \Sigma$. A homomorphism of Σ-coalgebras is defined as a mapping $h : A \to B$ such that the following square commutes

$$\begin{tikzcd} X^A \arrow[r, "\sigma_{\mathsf{A}}"] & Y^A \\ X^B \arrow[u, "X^h"] \arrow[r, "\sigma_{\mathsf{B}}"] & Y^B \arrow[u, "Y^h"'] \end{tikzcd}$$

for all σ in Σ. The resulting (illegitimate) category of coalgebras is denoted by $\mathsf{Coalg}(\Sigma)$.

Remark 2.2 If Σ consists of a single (X,Y)-ary operation symbol σ then $\mathsf{Coalg}(\Sigma) \cong \mathsf{Coalg}(Y^{(X^-)})$. (It generalises the presentation of topological spaces together with open continuous maps as coalgebras, see [9].) Hence, for a set Σ of operation symbols, $\mathsf{Coalg}(\Sigma) \cong \mathsf{Coalg}(F)$ for some functor $F : \mathsf{Set} \to \mathsf{Set}$ (by taking a coproduct of (Y^{X^-})'s).

As usual, a signature Σ gives rise to **terms**. Terms are also equipped with arities and defined as follows:

(i) every (X, Y)-ary operation symbol is an (X, Y)-ary term,

(ii) every mapping $f : X \to Y$ determines an (X, Y)-ary term x_f,

(iii) having an (X, Y)-ary term t_1 and an (Y, Z)-ary term t_2, we get an (X, Z)-ary term $t_2 \cdot t_1$.

For each (X, Y)-ary term t and a Σ-coalgebra A we get the mapping

$$t_{\mathsf{A}} : X^A \to Y^A$$

as follows:

(ii) $(x_f)_{\mathsf{A}}(v) = f \circ v$ for $v : A \to X$,

(iii) $(t_2 \cdot t_1)_{\mathsf{A}} = (t_2)_{\mathsf{A}} \circ (t_1)_{\mathsf{A}}$.

We can now define coequations:

Definition 2.3 (Coequations) *A coequation is a pair (t_1, t_2) of (X, Y)-ary terms. We write $t_1 = t_2$. A Σ-coalgebra A satisfies this coequation iff $(t_1)_{\mathsf{A}} = (t_2)_{\mathsf{A}}$.*

A **coequational theory** E is a class of coequations. The category of all Σ-coalgebras satisfying all coequations from E is denoted by $\mathsf{Coalg}(E)$. It might be an illegitimate category. We are interested in legitimate categories $\mathsf{Coalg}(E)$. Each such category is equipped with a forgetful functor $U : \mathsf{Coalg}(\Sigma) \to \mathsf{Set}$ and thus it is a concrete category.

Definition 2.4 (Coequational category) *A concrete category will be called coequational if it is isomorphic to $\mathsf{Coalg}(E)$ for some coequational theory E.*

We show that $\mathsf{Coalg}(F)$ is always coequational. This result is due to Reiterman. His paper [21], p. 62, formulates it, without proof, over $\mathsf{Set}^{\mathrm{op}}$ only (i.e. for algebras over Set) and thus we present Reiterman's proof sent to the second author in the late 70s.

Proposition 2.5 (Reiterman) *Let $F : \mathsf{Set} \to \mathsf{Set}$ be a functor. Then $\mathsf{Coalg}(F)$ is coequational.*

Proof Let Σ be a signature consisting of (X, FX)-ary operation symbols σ^X for every set X and let E consist of coequations

$$\sigma^Y \cdot x_f = x_{Ff} \cdot \sigma^X \tag{2}$$

for every mapping $f : X \to Y$. There is a functor $G : \mathsf{Coalg}(F) \to \mathsf{Coalg}(E)$ given as follows: For $\mathsf{A} = (A, \alpha)$, $G(\mathsf{A})$ is the E-coalgebra $(A, \sigma^X_{G\mathsf{A}})$ where

$$\sigma^X_{G\mathsf{A}}(v) = Fv \circ \alpha$$

for $v : A \to X$. On the other hand there is a functor $H : \mathsf{Coalg}(E) \to \mathsf{Coalg}(F)$ sending an E-coalgebra $\mathsf{A} = (A, \sigma^X_{\mathsf{A}})$ to the F-coalgebra $(A, \sigma^A_{\mathsf{A}}(\mathrm{id}_A))$.

We have $HG = \mathrm{Id}$ because, for each F-coalgebra $\mathsf{A} = (A, \alpha)$ we have $\sigma^A_{G\mathsf{A}}(\mathrm{id}_A) = F\mathrm{id}_A \circ \alpha = \alpha$. Conversely, $GH = \mathrm{Id}$ because, for each E-coalgebra $\mathsf{A} = (A, \sigma^X_{\mathsf{A}})$, we have $GH\mathsf{A} = G(A, \sigma^A_{\mathsf{A}}(\mathrm{id}_A)) = (A, F(-) \circ \sigma^A_{\mathsf{A}}(\mathrm{id}_A))$ which equals $(A, \sigma^X_{\mathsf{A}})$ because for all $v : A \to X$

$$\begin{tikzcd}
A^A \arrow[r, "\sigma^A_{\mathsf{A}}"] \arrow[d, "(x_v)_{\mathsf{A}}"'] & (FA)^A \arrow[d, "(x_{Fv})_{\mathsf{A}}"] \\
X^A \arrow[r, "\sigma^X_{\mathsf{A}}"] & (FA)^A
\end{tikzcd}$$

commutes (due to the coequation $\sigma^X \cdot x_v = x_{Fv} \cdot \sigma^A$) and thus $Fv \circ \sigma^A_{\mathsf{A}}(\mathrm{id}_A) = \sigma^X_{\mathsf{A}}(v \circ \mathrm{id}_A) = \sigma^X_{\mathsf{A}}(v)$. $\square$

A disadvantage of the just described procedure is that one needs a proper class of operation symbols. Let (M) denote the following set-theoretic statement (see [3]).

(M) There do not exist arbitrarily large measurable cardinals.

Proposition 2.6 *Assume (M) and let $F : \mathsf{Set} \to \mathsf{Set}$ be a functor preserving cofiltered limits. Then $\mathsf{Coalg}(F)$ is coequational in a signature consisting of a single operation symbol.*

Proof Let P be an infinite set whose cardinality is greater than any measurable cardinal. Then the full subcategory P of Set having a single object P is codense in Set (see [3] A.5). Let Σ consist of a single (P, FP)-ary operation symbol σ and E consist of coequations (2) for $f : P \to P$ (where $\sigma = \sigma^P$). Analogously to Proposition 2.5 , we get a functor $G : \mathsf{Coalg}(F) \to \mathsf{Coalg}(E)$ and our task is to define $H : \mathsf{Coalg}(E) \to \mathsf{Coalg}(F)$. Let $(A, \sigma_{\mathsf{A}}) \in \mathsf{Coalg}(E)$. Since P is infinite, the comma-category $(A{\downarrow}\mathsf{P})$ is cofiltered ($\langle u, v\rangle$ serves as a lower bound for $u, v : A \to P$). The codensity of P means that $f : A \to P$ forms a limit cone to the projection $Q : (A{\downarrow}\mathsf{P}) \to \mathsf{Set}$. Since F preserves cofiltered limits, $Ff : FA \to FP$ forms a limit cone to FQ. Coequations in E say that $\sigma_{\mathsf{A}}(f) : A \to FP$ is a cone to FQ. Thus there is a unique mapping $\alpha : A \to FA$ such that $Ff \circ \alpha = \sigma_{\mathsf{A}}(f)$ for each $f : A \to P$. We put $H\mathsf{A} = (A, \alpha)$. The rest is analogous to Proposition 2.5. $\square$

Remark 2.7

(i) We can assume that there are no measurable cardinals. If V is a model of ZFC in which measurable cardinals exist, let κ be the smallest such. Then V_κ, the restriction of V to sets of rank $< \kappa$, is a model of ZFC that contains no measurable cardinals.

(ii) If there are no measurable cardinals, we can take a countable set for P.

(iii) Since cofiltered limits are connected, Proposition 2.6 applies to polynomial functors $F : \mathsf{Set} \to \mathsf{Set}$.

We have seen that categories of coalgebras for a functor are coequational. But often, one is more interested in covarieties of these categories.

Problem 2.8 *Is every covariety in $\mathsf{Coalg}(F)$, where $F : \mathsf{Set} \to \mathsf{Set}$, coequational? More generally, is every covariety of a coequational category coequational?*

Let $U : \mathcal{K} \to \mathsf{Set}$ be a functor. A *U-split equaliser* is an equaliser in $\mathcal{K}$

$$\mathsf{A} \xrightarrow{e} \mathsf{B} \underset{g}{\overset{f}{\rightrightarrows}} \mathsf{C}$$

such that its U-image splits, i.e., it is equipped with $t : U\mathsf{C} \to U\mathsf{B}$ and $s : U\mathsf{B} \to U\mathsf{A}$ such that $s \circ Ue = \mathrm{id}_{U\mathsf{A}}$, $t \circ Uf = \mathrm{id}_{U\mathsf{B}}$, and $t \circ Ug = Ue \circ s$. U *creates U-split equalisers* if it creates equalisers of pairs f, g for which Uf, Ug has a split equaliser in Set. Beck's theorem [16] says that U is comonadic iff it has a right-adjoint and creates U-split equalisers. We will say that a concrete category $\mathcal{K}$ is **co-Beck** if U creates colimits and U-split equalisers.

Proposition 2.9 *Each covariety of a coequational category is co-Beck.*

Proof Straightforward, cf. [23]. □

Remark 2.10

(i) Propositions 2.5 and 2.9 also hold if we replace Set by an arbitrary category.

(ii) We can now give an easy proof of Linton's theorem [15], § 9, which says that coequational categories with cofree coalgebras (i.e., U has a right adjoint) coincide with comonadic categories. A proof that comonadic categories are coequational follows from Proposition 2.5. If $F : \mathsf{Set} \to \mathsf{Set}$ is a comonad with a counit $\varepsilon : F \to Id$ and a comultiplication $\delta : F \to FF$, we get comonad coalgebras by imposing the following additional coequations to (2):

$$x_{\varepsilon_X} \cdot \sigma^X = x_{id_X},$$
$$\sigma^{FX} \cdot \sigma^X = x_{\delta_X} \cdot \sigma_X.$$

The converse statement that coequational categories with cofree coalgebras are comonadic follows from Beck's theorem and Proposition 2.9.

3 Implicit Operations

If Σ is a signature and $U : \mathsf{Coalg}(\Sigma) \to \mathsf{Set}$ the forgetful functor then each (X, Y)-ary operation symbol $\sigma \in \Sigma$ determines a natural transformation

$$\sigma : X^U \to Y^U$$

So does each (X, Y)-ary term. It leads us to define (X, Y)-ary **implicit operations**, for every concrete category $U : \mathcal{K} \to \mathsf{Set}$ as natural transformations

$$X^U \to Y^U.$$

If the functor U has a right adjoint R then (X, Y)-ary implicit operations correspond to natural transformations

$$\hom(-, RX) \to \hom(-, RY),$$

i.e., to morphisms $RX \to RY$.

Definition 3.1 (Coequations in Implicit Operations) *Let $U : \mathcal{K} \to \mathsf{Set}$ be a functor and $X, Y \in \mathsf{Set}$. Having two (X, Y)-ary implicit operations σ_1 and σ_2 in $\mathcal{K}$, we say that an object $\mathsf{K} \in \mathcal{K}$ satisfies the coequation $\sigma_1 = \sigma_2$ and write $\mathsf{K} \models \sigma_1 = \sigma_2$ iff $(\sigma_1)_\mathsf{K} = (\sigma_2)_\mathsf{K}$.*

Remark 3.2

(i) For a collection E of coequations in implicit operations, the full subcategory of all objects satisfying each coequation from E is a covariety in $\mathcal{K}$ (assuming that $\mathcal{K}$ has coproducts and that $U : \mathcal{K} \to \mathsf{Set}$ preserves them).

(ii) If U has a right adjoint and σ_1, σ_2 are represented by $s_1, s_2 : RX \to RY$, respectively, then an object K satisfies the coequation $\sigma_1 = \sigma_2$ iff every morphism $h : K \to RX$ is coequalised by σ_1 and σ_2, i.e., iff h factors through an equaliser

$$\mathsf{S} \longrightarrow RX \overset{s_1}{\underset{s_2}{\rightrightarrows}} RY$$

This notion of a coequation as a subobject S of a cofree object RX is a special case of Manes [17], Theorem 3.4, page 227. It was further investigated in [24,8,22,12,10]. Conversely, for any subobject $\mathsf{S} \to RX$, take the cokernel pair $f, g : RX \to \mathsf{A}$ and compose it with $\eta_\mathsf{A} : \mathsf{A} \to RU\mathsf{A}$ given by the unit η of the adjunction $U \dashv R$. Then the pair $\eta_\mathsf{A} \circ f, \eta_\mathsf{A} \circ g$ produces the pair of natural transformations $X^U \to Y^U$ in our sense. Thus, in the presence of cofree coalgebras, our approach is equivalent to the coequations-as-subobjects-of-cofree-objects approach.

(iii) Without cofree coalgebras, there are related concepts of a coequation in [2] and [21]. They are subsumed by coequations in implicit operations.

We already mentioned that a class definable by coequations in implicit operations is a covariety. We now show the converse which is a co-Birkhoff theorem not relying on the existence of cofree coalgebras. The proof uses certain implicit operations defined in terms of the behavioural equivalence relations $\sim_X$ (Definition 1.1). The following proposition shows that transformations $X^U \to 2^U$ that are invariant under $\sim_X$ are implicit operations.

Proposition 3.3 *Consider $U : \mathcal{K} \to \mathsf{Set}$. A transformation $\varphi : X^U \to 2^U$ is natural if for all $\mathsf{A}_i \in \mathcal{K}$, $v_i : U\mathsf{A}_i \to X$, $a_i \in U\mathsf{A}_i$, $(i = 1, 2)$*

$$(\mathsf{A}_1, v_1, a_1) \sim_X (\mathsf{A}_2, v_2, a_2) \;\Rightarrow\; \varphi_{\mathsf{A}_1}(v_1)(a_1) = \varphi_{\mathsf{A}_2}(v_2)(a_2)$$

Proof By definition, φ is natural if $\varphi_{\mathsf{A}_1}(v_2 \circ Uf)(a_1) = \varphi_{\mathsf{A}_2}(v_2)(Uf(a_1))$ for all $f : \mathsf{A}_1 \to \mathsf{A}_2$ in $\mathcal{K}$, $v_2 : U\mathsf{A}_2 \to X$, $a_1 \in U\mathsf{A}_1$. Now use that $(\mathsf{A}_1, v_2 \circ Uf, a_1) \sim_X (\mathsf{A}_2, v_2, Uf(a_1))$. $\square$

Theorem 3.4 *Let E be a coequational theory. Then every covariety $\mathcal{C}$ in* $\mathsf{Coalg}(E)$ *is definable by coequations in implicit operations.*

Proof For a set X and a E-coalgebra A, define

$$\varphi^X_{\mathsf{A}} : X^{U\mathsf{A}} \to 2^{U\mathsf{A}}$$

as follows: for $v : U\mathsf{A} \to X$ and $a \in U\mathsf{A}$, let $\varphi^X_{\mathsf{A}}(v)(a) = 1$ iff there are $\mathsf{C} \in \mathcal{C}$, $u : U\mathsf{C} \to X$, $c \in U\mathsf{C}$, such that $(\mathsf{A}, v, a) \sim_X (\mathsf{C}, u, c)$, see Definition 1.1. φ^X is an implicit operation by Proposition 3.3.

Consider coequations

$$\varphi^X = \mathit{true} \tag{3}$$

where $\mathit{true} = x_f$ is given by the constant function $f : X \to 2, x \mapsto 1$. Each $\mathsf{C} \in \mathcal{C}$ satisfies (3). Conversely, assume that A satisfies all coequations (3). Then, due to Proposition 1.3, for each $a \in U\mathsf{A}$, there are $\mathsf{B}_a \in \mathsf{Coalg}(E)$, $\mathsf{C}_a \in \mathcal{C}$, and homomorphisms f_a, g_a and mappings w_a, u_a such that

$$\begin{tikzcd}
 & U\mathsf{A} & \\
U\mathsf{A} \arrow[ur, "\mathrm{id}_{U\mathsf{A}}"] \arrow[r, "Uf_a"'] & U\mathsf{B}_a \arrow[u, "w_a"] & U\mathsf{C}_a \arrow[ul, "u_a"'] \arrow[l, "Ug_a"]
\end{tikzcd}$$

commutes and $(Uf_a)(a) \in (Ug_a)(U\mathsf{C}_a)$. Using a multiple pushout of the f_a

$$\begin{tikzcd}
 & \mathsf{B} \\
\mathsf{A} \arrow[ur, "f"] \arrow[r, "f_a"'] & \mathsf{B}_a \arrow[u, "f'_a"']
\end{tikzcd}$$

we get

$$\begin{tikzcd}
 & U\mathsf{A} & \\
U\mathsf{A} \arrow[ur, "\mathrm{id}_{U\mathsf{A}}"] \arrow[r, "Uf"'] & U\mathsf{B} \arrow[u, "w"] & U\mathsf{C}_a \arrow[ul, "u_a"'] \arrow[l, "U(f'_a \circ g_a)"]
\end{tikzcd}$$

with $(Uf)(U\mathsf{A}) \subseteq \bigcup\{U(f'_a \circ g_a)(U\mathsf{C}_a) : a \in U\mathsf{A}\}$. Note that $\bigcup\{U(f'_a \circ g_a)(U\mathsf{C}_a) : a \in U\mathsf{A}\}$ is the carrier of an E-coalgebra which is in $\mathcal{C}$ due to closure under coproducts and quotients. Since f is injective and $\mathcal{C}$ is closed under subobjects, it follows $\mathsf{A} \in \mathcal{C}$. □

Remark 3.5

(i) If $\mathsf{Coalg}(E)$ has cofree coalgebras, the implicit operation φ^X is induced by a morphism $h : RX \to R2$, $U \dashv R$, or, equivalently, by a mapping $\tilde{h} : URX \to 2$. Then, for $v : U\mathsf{A} \to X$,

$$\varphi^X_{\mathsf{A}}(v) = \tilde{h} \circ Uv^\sharp$$

with $v^\sharp : \mathsf{A} \to RX$ being the transpose of v.

(ii) Our proof works in the universe of finite sets, i.e., every covariety of finite coalgebras is given by coequations in implicit operations. This is the "Reiterman Theorem" [20] for coalgebras.

(iii) The just proved theorem does not mean that $\mathcal{C}$ is coequational. It may happen that the interpretation of the implicit operations φ^X are not forced to be the given ones. Hence the theorem does not solve Problem 2.8.

4 Davis's Theorem

We may allow signatures with (X,Y)-ary operation symbols where X and Y are classes. It leads to **meta-coequational categories**. Every meta-coequational category is co-Beck. Davis [7] proved the converse. He overstated his result by claiming that every co-Beck category is coequational, which is not true as Example 4.3 shows. The second author observed Davis's mistake in [23]; here we give a full explanation. First, we sketch an argument proving Davis's theorem.

Theorem 4.1 (Davis) *A concrete category is meta-coequational iff it is co-Beck.*

Proof Let $U : \mathcal{C} \to \mathsf{Set}$ create colimits and U-split equalisers. Let $R : \mathsf{Class} \to \mathsf{Class}$ be the density comonad of

$$\bar{U} : \mathcal{C} \xrightarrow{U} \mathsf{Set} \hookrightarrow \mathsf{Class}.$$

It means that RX is the colimit of the canonical diagram $(\bar{U}{\downarrow}X) \to \mathsf{Class}$

$$\begin{tikzcd} \bar{U}\mathsf{A} \arrow[rr, "v"] \arrow[dr, "\varepsilon"'] & & X \\ & RX \arrow[ur, "\varepsilon_X"'] & \end{tikzcd}$$

where ε_X is induced by the cone given by v's. Since $\bar{U}$ creates colimits, $(\bar{U}{\downarrow}X)$ is ∞-filtered (= every small subcategory of $(\bar{U}{\downarrow}X)$ has an upper bound). Since the (illegitimate) category $\mathsf{Coalg}(R)$ of coalgebras for the comonad R is coequational over Class, it suffices to prove that the image of the comparison functor

$$\begin{tikzcd} \mathcal{C} \arrow[rr] \arrow[dr, "\bar{U}"'] & & \mathsf{Coalg}(R) \arrow[dl, "V"] \\ & \mathsf{Class} & \end{tikzcd}$$

consists precisely of the R-coalgebras (X,ξ) with X a set. Any such coalgebra is given by a V-split equaliser

$$\begin{tikzcd}
U\mathsf{B}_i \arrow[d] & U\mathsf{A}_i \arrow[l, shift left] \arrow[l, shift right] \arrow[d] & U\mathsf{C}_i \arrow[l] \arrow[d, "c_i"'] \\
RRX & RX \arrow[l, shift right, "R\xi"'] \arrow[l, shift left, "\delta_X"] & X \arrow[l, "\xi"']
\end{tikzcd}$$

where δ is the comultiplication. Since RRX and RX are given by ∞-filtered colimits, this equaliser is an ∞-filtered colimit of U-split equalisers in $\mathcal{C}$ (we are also using that U creates U-split equalisers). Since X is a set, some R-coalgebra homomorphism $c_i : \mathsf{C}_i \to (X,\xi)$ splits, i.e., $c_i \circ s = \mathrm{id}_X$ for some $s : (X,\xi) \to \mathsf{C}_i$. Hence (X,ξ) is isomorphic to some $\mathcal{C}$-object. □

Remark 4.2 Let $\mathcal{C}$ be a concrete category and $\sigma : X^U \to Y^U$ an implicit operation where X, Y are classes. Take mappings $f : X_1 \to X$ and $g : Y \to Y_1$ where X_1, Y_1 are sets. We get an implicit operation $g^U \circ \sigma \circ f^U$ where arities are sets. If sets are codense in classes then Y is a canonical limit of the canonical diagram $(Y{\downarrow}\mathsf{Set}) \to \mathsf{Class}$ consisting of $g : Y \to Y_1$ where Y_1 is a set. Hence, the implicit operation $\sigma \circ f^U$ is determined by implicit operations $g^U \circ \sigma \circ f^U$. Moreover, since $U\mathsf{C}$ are sets, σ is determined by implicit operations $g^U \circ \sigma \circ f^U$. This is what Davis claimed. However, it does not mean that coequations of implicit operations whose arities are classes can be replaced by coequations of implicit operations whose arities are sets. There is a problem with compositions $X^U \xrightarrow{\sigma_1} Z^U \xrightarrow{\sigma_2} Y^U$ where Z is a proper class. The precise result is Proposition 5.5 in [23].

As before, sets are codense in classes iff *Ord* is not measurable, i.e., iff each *Ord*-complete ultrafilter is principal. A model of such a set theory is V_α where α is inaccessible but not measurable. On the other hand, in the theory of finite sets, i.e., in V_ω, is $\mathit{Ord} = \omega$ measurable.

Example 4.3 Let Σ consist of a single $(1, \mathit{Ord})$-ary operation symbol σ. Then Σ-coalgebras A are sets A equipped with an operation $\sigma_{\mathsf{A}} : 1^A \to \mathit{Ord}^A$, i.e., with a mapping $\alpha : A \to \mathit{Ord}$. Homomorphisms $h : (A,\alpha) \to (B,\beta)$ are mappings $h : A \to B$ such that $\beta \circ h = \alpha$. $\mathsf{Coalg}(\Sigma)$ is a (legitimate) meta-coequational category. It cannot be isomorphic to any full subcategory of $\mathsf{Coalg}(F)$ for any functor $F : \mathsf{Set} \to \mathsf{Set}$ because it contains a proper class of one-element coalgebras.

Each mapping $f : \mathit{Ord} \to m$ gives a $(1,m)$-ary term (i.e., a $(1,m)$-ary implicit operation) $x_f \cdot \sigma$). In fact, every implicit operation $\varphi : 1^U \to m^U$ is of that kind. It suffices to take $f : \mathit{Ord} \to m$ given as $f(p) = \varphi_{\mathsf{P}}(\mathrm{id}_1)$ where P is the one-element Σ-coalgebra with σ_{P} taking the value p (see [23]7.2 for the easy calculation that $\varphi = x_f \cdot \sigma$.)

Let Σ_1 be the collection of all $(1,m)$-ary operation symbols $\sigma_f, f : \mathit{Ord} \to m$. Σ_1 is not a signature because it is larger than a class. If *Ord* is not measurable then, following [23]5.5, $\mathsf{Coalg}(\Sigma)$ is described in Σ_1 by coequations $x_g \cdot \sigma_f = \sigma_{g \circ f}$, for $f : \mathit{Ord} \to m$ and $g : m \to k$. In fact, having a Σ_1-coalgebra $(A, (\sigma_f)_{\mathsf{A}})$ satisfying these equations, we get a cone $(\sigma_f)_{\mathsf{A}} : A \to m$ of the canonical diagram $(\mathit{Ord}{\downarrow}\mathsf{Set})$ and, therefore, the induced mapping $\alpha : A \to \mathit{Ord}$. (A, α) is the Σ-coalgebra determined by $(A, (\sigma_f)_{\mathsf{A}})$.

If *Ord* is measurable, then $\mathsf{Coalg}(\Sigma)$ is not coequational. In fact, it is shown in [23]7.2 that *Ord*-complete ultrafilters provide one-element coalgebras living in all coequational categories containing $\mathsf{Coalg}(\Sigma)$.

5 Modal Predicates

This section presents an explanation of implicit operations and coequations from the point of view of modal logic. For more details on modal logic and universal coalgebra see e.g. [6,24,11,13].

We show that a predicate on Kripke frames, or more generally on coalgebras, is invariant under bisimulation iff it depends naturally on Kripke frames, i.e., iff it is an implicit operation in the sense of Section 3 (Section 5.1 and 5.2). This leads us to a general notion of modal operator as a *natural* operator on predicates. The coequational logic considered in Sections 2–4 then coincides with the modal logic induced by modal predicates and operators (Section 5.3).

5.1 Kripke frames

A *Kripke frame* $\mathsf{A} = (A, \alpha)$ consists of a carrier set A and a function $\alpha : A \to \mathcal{P}A$.[8] We think of $\alpha(a)$ as the set of successors of a. Given a set I of *atomic propositions*, a *Kripke model* (A, v) consists of a frame A and a *valuation* of atomic propositions $v : A \to \prod_I 2$. The notion of a bisimulation between Kripke frames (or Kripke models) is defined in the usual way. The category KF consists of Kripke frames as objects and has as morphisms those functions whose graphs are bisimulations (also known as bounded morphisms or p-morphisms). We write $\mathcal{ML}$ for the set of *modal formulae*, formulae being built from atomic propositions using boolean operators and a unary modal operator $\Box$. The semantics of a modal formula $\varphi \in \mathcal{ML}$ is given w.r.t. to a Kripke frame A, a valuation v, and a state $a \in A$, via

$$\mathsf{A}, v, a \models p \quad \text{iff} \quad (v(a))_p = 1 \text{ for } p \in I$$

$$\mathsf{A}, v, a \models \Box\varphi \quad \text{iff} \quad \forall a' \in A \,.\, a' \in \alpha(a) \;\Rightarrow\; \mathsf{A}, v, a' \models \varphi,$$

and for boolean operators in the obvious way. One says φ holds in (A, v), written $\mathsf{A}, v \models \varphi$, iff $\mathsf{A}, v, a \models \varphi$ for all $a \in A$; φ holds in A if it holds in all

[8] $\mathcal{P}X = \{Y \mid Y \subseteq X\}$.

(A, v). φ is valid, written $\models \varphi$, iff φ holds in all Kripke frames.

We now rephrase the semantics of modal logic in terms of natural transformations. For this, let $U : \mathsf{KF} \to \mathsf{Set}$ be the functor mapping Kripke frames to their carriers and morphisms to the underlying functions. The semantics $[\![\varphi]\!]$ of a modal formula φ can then be understood as a KF-indexed class of operations

$$[\![\varphi]\!]_{\mathsf{A}} : (\prod_I 2)^{U\mathsf{A}} \to 2^{U\mathsf{A}}, \quad \mathsf{A} \in \mathsf{KF},$$

that is, each $[\![\varphi]\!]_{\mathsf{A}}$ maps valuations $v \in (\prod_I 2)^{U\mathsf{A}}$ and elements $a \in U\mathsf{A}$ to truth values $[\![\varphi]\!]_{\mathsf{A}}(v, a) \in 2 = \{0, 1\}$.

A central feature of modal logic is that formulae are invariant under bisimulation. That is, for a modal formula φ and two Kripke models (A, α, v), (B, β, w), and $a \in A$, $b \in B$, it holds

$$a, b \text{ bisimilar} \quad \Rightarrow \quad [\![\varphi]\!]_{(A,\alpha)}(v, a) = [\![\varphi]\!]_{(B,\beta)}(w, b).$$

This property can be expressed equivalently by saying that $[\![\varphi]\!]$ is a natural transformation:

Proposition 5.1 *Consider a family* $([\![\varphi]\!]_{\mathsf{A}} : (\prod_I 2)^{U\mathsf{A}} \to 2^{U\mathsf{A}})_{\mathsf{A} \in \mathsf{KF}}$. *Then* φ *is invariant under bisimulation iff* $[\![\varphi]\!]_{\mathsf{A}}$ *is natural in* A.

A proof of the proposition is given in the next subsection in a more general setting.

5.2 Modal Predicates

We now generalise the semantics of modal formulae from the previous subsection. $\mathcal{A}$ and $P_{\mathsf{A}}(v, a)$ below replace KF and $[\![\varphi]\!]_{\mathsf{A}}(v, a)$. And the behavioural equivalences $\sim_C$ (Definition 1.1) replace bisimulation.

Definition 5.2 (Modal and Behavioural Predicates) *Consider a functor* $U : \mathcal{A} \to \mathsf{Set}$. *A predicate* P *in colours from* $C \in \mathsf{Set}$ *is an operation which determines for each* $\mathsf{A} \in \mathcal{A}$, $v : U\mathsf{A} \to C$, $a \in U\mathsf{A}$ *a truth value*

$$P_{\mathsf{A}}(v, a) \in \{0, 1\}. \tag{4}$$

P *is called a* modal *predicate iff*

$$(\mathsf{A}, v, a) \sim_C (\mathsf{B}, w, b) \;\Rightarrow\; P_{\mathsf{A}}(v, a) = P_{\mathsf{B}}(w, b) \tag{5}$$

for all $w : U\mathsf{B} \to C$ *and* $a \in U\mathsf{A}$. *We also write* $\mathsf{A}, v, a \models P$ *or* $a \in P_{\mathsf{A}}(v)$ *for* $P_{\mathsf{A}}(v, a) = 1$. *As usual, we let* $\mathsf{A}, v \models P$ *iff* $\mathsf{A}, v, a \models P$ *for all* $a \in U\mathsf{A}$ *and* $\mathsf{A} \models P$ *iff* $\mathsf{A}, v \models P$ *for all* $v : U\mathsf{A} \to C$. *In case that* $C = 1$ *we call* P *a* behavioural *predicate and drop the* v *as e.g. in* $\mathsf{A}, a \models P$ *or* $P_{\mathsf{A}}(a) = 1$.

The following is immediate from the respective definitions.

Lemma 5.3 *An operation P which determines for each $\mathsf{A} \in \mathcal{A}$, $v : U\mathsf{A} \to C$, $a \in U\mathsf{A}$ a truth value $P_{\mathsf{A}}(v,a) \in \{0,1\}$ is a modal predicate iff for all morphisms $f : \mathsf{A} \to \mathsf{B}$, all valuations w for B, and all elements a of A, it holds*

$$P_{\mathsf{A}}(w \circ Uf, a) = P_{\mathsf{B}}(w, Uf(a)). \tag{6}$$

We now show that invariance of a predicate $C^U \to 2^U$ under C-behavioural equivalence is equivalent to the naturality of $C^U \to 2^U$. Recall the definition of the functor X^U from Section 1. In terms of modal logic (i.e., $\mathcal{A} = \mathsf{KF}$, $\mathcal{X} = \mathsf{Set}$, $C = \prod_I 2$), C^U maps a frame A to the set of valuations $U\mathsf{A} \to C$ and a morphism $f : \mathsf{A} \to \mathsf{B}$ to the function $C^{Uf} : C^{U\mathsf{B}} \to C^{U\mathsf{A}}$ which takes a valuation w for B and transforms it into a valuation $w \circ Uf$ for A. Also note that $2^{Uf} : 2^{U\mathsf{B}} \to 2^{U\mathsf{A}}$ is the inverse image of Uf mapping subsets $Y \subseteq U\mathsf{B}$ to $(Uf)^{-1}(Y)$.

Theorem 5.4 *Consider a functor $U : \mathcal{A} \to \mathsf{Set}$. An operation P which determines for each $\mathsf{A} \in \mathcal{A}$, $v : U\mathsf{A} \to C$, $a \in U\mathsf{A}$ a truth value $P_{\mathsf{A}}(v,a) \in \{0,1\}$ is a modal predicate iff*

$$P_{\mathsf{A}} : C^{U\mathsf{A}} \longrightarrow \{0,1\}^{U\mathsf{A}}$$

is a natural transformation.

Proof Naturality of P means that for any morphism $f : \mathsf{A} \to \mathsf{B}$

$$\begin{tikzcd} C^{U\mathsf{A}} \arrow[r, "P_{\mathsf{A}}"] & 2^{U\mathsf{A}} \\ C^{U\mathsf{B}} \arrow[u, "C^{Uf}"] \arrow[r, "P_{\mathsf{B}}"] & 2^{U\mathsf{B}} \arrow[u, "2^{Uf}"'] \end{tikzcd}$$

commutes. Given $w : U\mathsf{B} \to C$ and spelling out the definition of the vertical arrows we obtain $P_{\mathsf{A}}(w \circ Uf) = P_{\mathsf{B}}(w) \circ Uf$, i.e., $P_{\mathsf{A}}(w \circ Uf, a) = P_{\mathsf{B}}(w, Uf(a))$ for all $a \in U\mathsf{A}$, yielding condition (6) in Lemma 5.3. □

Remark 5.5 It follows that behavioural predicates are natural transformations $P_{\mathsf{A}} : 1 \longrightarrow \{0,1\}^{U\mathsf{A}}$ or also $P_{\mathsf{A}} : U\mathsf{A} \longrightarrow \{0,1\}$.

5.3 Modal Operators and the Logic of Modal Predicates

We introduce a general notion of modal operator and discuss the corresponding basic modal logic. In particular, for modal predicates in propositional variables a notion of substitution is available.

Definition 5.6 (Modal Operator) *Let $U : \mathcal{A} \to \mathsf{Set}$ be a functor and I a* *set. An I-ary modal operator is a natural transformation*

$$\mu : (\prod_I 2)^U \longrightarrow 2^U.$$

Given modal predicates $Q_i : C^U \to 2^U$, we define

$$\Box_\mu((Q_i)_{i\in I}) \quad = \quad C^U \xrightarrow{\langle Q_i\rangle_{i\in I}} \prod_I (2^U) \cong (\prod_I 2)^U \xrightarrow{\mu} 2^U.$$

We list some examples and further definitions:

(i) An I-ary **boolean operator** is a modal operator

$$f^U : (\prod_I 2)^U \longrightarrow 2^U$$

given by a function $f : \prod_I 2 \to 2$. Examples include the constant *true* : $1 \to 2$ and the 'truth-tables' $\neg : 2 \to 2$, $\to : 2 \times 2 \to 2$.

(ii) Boolean operators also include infinitary operators. For example, conjunctions over an index set I are given by $\bigwedge_I^U : (\prod_I 2)^U \longrightarrow 2^U$ where $\bigwedge_I((b_i)_{i\in I}) = 1 \Leftrightarrow \forall i \in I \,.\, b_i = 1$.

(iii) A 0-ary modal operator is called an **atomic proposition** and is given by a natural transformation $1 \to 2^U$ (or also $U \to 2$).

(iv) In case $\mathcal{A} = \mathsf{Coalg}(\mathcal{P})$, an example of a *unary modal operator* is given by $\Box$ as in Section 5.1. The corresponding natural transformation μ is[9] $\mu_{\mathsf{A}}(X, a) = 1 \Leftrightarrow \alpha(a) \subseteq X$ where $\mathsf{A} = (A, \alpha)$, $X \subseteq A$, $a \in A$. In case of $\Diamond = \neg\Box\neg$ the corresponding natural transformation is $\mu_{\mathsf{A}}(X, a) = 1 \Leftrightarrow \alpha(a) \cap X \neq \emptyset$.

(v) More generally, Pattinson's modal operators given by natural relations [18] or predicate liftings [19] are further examples.

(vi) A unary modal operator $\Box_\mu$ is called **normal** iff, for $X, Y \subseteq U\mathsf{A}$, $\mu_{\mathsf{A}}(X \cap Y) = \mu_{\mathsf{A}}(X) \cap \mu_{\mathsf{A}}(Y)$ and $\mu_{\mathsf{A}}(U\mathsf{A}) = U\mathsf{A}$. For example, in (iv), $\Box$ is a normal modal operator and $\Diamond$ is not.

(vii) In case of coalgebras $\mathsf{A} = (A, \alpha)$, $\alpha : A \to \mathcal{P}(A \times A)$, an example of a *binary modal operator* is

$$\mathsf{A}, v, a \models \Box_\mu(P, Q) \iff \exists (b, c) \in \alpha(a) \,.\, \mathsf{A}, v, b \models P \;\&\; \mathsf{A}, v, c \models Q$$

which corresponds to the natural transformation $\mu_{\mathsf{A}}(\langle X, Y\rangle, a) = 1 \Leftrightarrow \alpha(a) \cap X \times Y \neq \emptyset$ where $X, Y \subseteq A$, $a \in A$. (This binary modal operator is not a boolean combination of unary modal operators. It plays a central role in arrow logic [25].)

[9] We take the liberty to denote a mapping $v : U\mathsf{A} \to 2$ by the corresponding subset $X \subseteq U\mathsf{A}$.

(viii) Modal operators are closed under composition, i.e., for modal operators $\mu : (\prod_I 2)^U \longrightarrow 2^U$ and $\mu_i : (\prod_J 2)^U \to 2^U$, the composition $\mu \circ \langle \mu_i \rangle$ is a modal operator.

(ix) Recursively defined modalities as in dynamic logic or the μ-calculus are modal operators in our sense. Using (ii) and (viii) above, this follows from the fact that both logics embed into infinitary modal logic.

(x) Examples of modalities which are not covered by Definition 5.6 can be obtained by definitions that require a 'change of structure'. For instance, consider $\mathsf{A} = (A, \alpha) \in \mathsf{Coalg}(\mathcal{P})$ and define

$$\mathsf{A}, v, a \models \Box(\varphi, \psi) \;=\; \begin{cases} \mathsf{A}^\varphi, v^\varphi, a \models \psi & \text{if } \mathsf{A}, v, a \models \varphi \\ \mathit{false} & \text{otherwise} \end{cases}$$

where $\mathsf{A}^\varphi = (A^\varphi, \alpha^\varphi)$ is given by $A^\varphi = A \setminus \{a : \mathsf{A}, v, a \not\models \varphi\}$ and α^φ, v^φ are the restriction of α, v to A^φ. Modalities of this kind arise in epistemic logic, see [5].

The reader will have noticed that our modal operators are in fact a special case of modal predicates. More precisely they are those modal predicates that allow for a notion of substitution.

Definition 5.7 (Propositional Variables and Substitution)
Let $U : \mathcal{A} \to \mathsf{Set}$ be a functor and I a set. A modal predicate

$$P : (\prod_I 2)^U \longrightarrow 2^U$$

is called a modal predicate in propositional variables (from I) *or* an I-ary modal predicate. *Given modal predicates $Q_i : C^U \to 2^U$, the substitution $P[Q_i/i]$ is the composition*

$$C^U \xrightarrow{\langle Q_i \rangle_{i \in I}} \prod_I (2^U) \cong (\prod_I 2)^U \xrightarrow{P} 2^U.$$

Thus, in case we restrict ourselves to modal predicates in propositional variables, we have that substitution is composition. This explains why modal logic prefers to use propositional variables rather than colours.

The logic arising from a language in which formulas and connectives are interpreted as (our generalised) modal predicates and modal operators is familiar modal logic as shown by the following proposition.

Proposition 5.8 *Let $U : \mathcal{A} \to \mathsf{Set}$ be a functor and μ a normal unary modal operator. Consider a class Φ of modal predicates in propositional variables and let Q be in the closure of Φ under propositional tautologies, modus ponens, substitution, and*

$$\textit{(dist)} \quad \Box_\mu(P \to P') \to \Box_\mu P \to \Box_\mu P'$$
$$\textit{(nec)} \quad \textit{from } P \textit{ derive } \Box_\mu P$$

Then $\mathsf{A} \models \Phi \;\Rightarrow\; \mathsf{A} \models Q$.

Proof Straightforward. For (dist) and (nec) use that μ is normal. □

Remark 5.9 The basic logic of modal predicates in propositional variables consists of (possibly infinitary) propositional tautologies, modus ponens, and substitution. Additional axioms and rules as (dist) and (nec) above depend on special properties of the modal operators.

Finally, we make precise the relationship between modal predicates and the coequations of Section 3.

Proposition 5.10 *Let* $U : \mathcal{A} \to \mathsf{Set}$ *be a functor. Coequations in implicit operations (Definition 3.1) and modal predicates in propositional variables (Definition 5.7) have the same expressive power.*

Proof Each modal predicate P is logically equivalent to the coequation $P = \mathit{true}$,[10] which is to say that $\mathsf{A} \models P$ (Definition 5.2) iff $\mathsf{A} \models P = \mathit{true}$ (Definition 3.1) for all $\mathsf{A} \in \mathcal{A}$. Conversely, for a coequation $P = Q$ with $P, Q : C^U \to D^U$, we find a set I and a surjective function $e : \prod_I 2 \to C$ and a modal predicate $P \circ e \leftrightarrow Q \circ e : (\prod_I 2)^U \to 2^U$ such that $\mathsf{A} \models P = Q \;\Leftrightarrow\; \mathsf{A} \models P \circ e \leftrightarrow Q \circ e$ for all $\mathsf{A} \in \mathcal{A}$. □

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[10] *true* is the obvious natural transformation. It was denoted by true^U in (i) above.

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