Julius Petersen's theory of regular graphs

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Abstract

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In 1891 the Danish mathematician Julius Petersen (1839–1910) published a paper on the factorization of regular graphs. This was the first paper in the history of mathematics to contain fundamental results explicitly in graph theory. In this report Petersen's results are analysed and their development in subsequent decades are followed.

1. Introduction

Julius Petersen is famous in graph theory, first of all because of the Petersen graph, and secondly because of the theorem that bears his name: *a connected 3-regular graph with at most two leaves contains a 1-factor.* A 1-factor is a perfect matching, and a leaf is a bridgeless component that arises upon deletion of some bridge (a single disconnecting edge). Petersen exhibited 'his graph' in 1898 in a small note [23] in *L'Intermédiaire des Mathématiciens,* a journal devoted to the quick exchange of mathematical questions, problems and ideas. The graph served as a counterexample to Tait's 'theorem' [31] on the 4-colour problem: a bridgeless 3-regular graph is factorable into three 1-factors. Petersen's drawing lacked the beautiful symmetry with which we now usually draw the graph (see the first diagram in Fig. 1).

Incidentally, the first occurrence of the Petersen graph in the literature was in a geometric paper by Kempe [15] of 1886. In Kempe's drawing the vertices are organized in a nine-gon plus a vertex in its centre (cf. [5]). In Petersen's case the graph resulted from his earlier work in the factorization of regular graphs.

In 1891 Julius Petersen published a paper in the Acta Mathematica (volume 15, pages 193–220) entitled *'Die Theorie der regulären graphs'.* This paper is
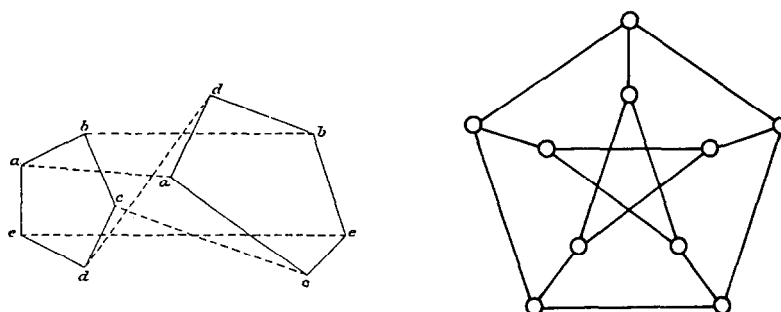
Fig. 1. The Petersen graph.

remarkable in its depth and scope. It is not just a paper with a new theorem, the nucleus of a new theory was created out of nothing. Until 1890 the most important results in graph theory were those of the German physicist Kirchhoff [16] on the exchange property of spanning trees and the cyclomatic number of a graph. However, at that time the concept of graph did not exist and his results were formulated in terms of electrical networks, although taken quite abstractly (cf. [21]). Hence one could take the view that Petersen's Acta paper is actually the first paper containing (correct) fundamental results explicitly in graph theory. The Acta paper is the main subject of this report. For information on Petersen's other mathematical work and for his biography the reader is referred to [20]. For a bibliography see [6], and for more information on the 'prehistory' of the Acta paper itself see [26] as well as [20]. Preprints of these three publications were distributed at the Julius Petersen Graph Theory Conference, 1–6 July, 1990, Hindsgavl, Denmark.

2. The origins of the Acta paper

The origins of the Acta paper are discussed more fully by Sabidussi in [26] and in [20]. Here a few observations suffice.

Petersen was involved in solving a problem in invariant theory, an important topic in the 1870's and 80's. He met J.J. Sylvester in September 1889, and the two exchanged many letters on the subject (those of Sylvester are edited in [26]). It was Sylvester, in 1878, who developed the idea of using graphs for analyzing invariants, in a rather confused way [30]. At that time he introduced the term graph [29]. In his paper in the *American Mathematical Journal* [30, p. 74] Sylvester introduced also another way of associating graphs with invariants, apparently without any motivation. It was this idea that Petersen exploited so successfully in his Acta paper. As yet we have not been able to uncover the letters which Petersen sent to Sylvester, or preliminary versions of his papers, or any other notes related to the Acta paper. Thus it is not (yet) possible to give a

complete reconstruction of how Petersen developed his results. It is clear however that his exchange of ideas with Sylvester (mentioned in the Acta paper) was essential for the development of his ideas and the successful completion of the paper.

Another person with whom Petersen discussed his ideas was his close friend F. Bing (1839–1912), mathematical director of the State Life Insurance Company and Petersen's co-author of a fundamental paper in Mathematical Economics (cf. [6]). In a footnote in the Acta paper Petersen mentioned that Bing had suggested a simplification of one of the proofs.

To develop a full understanding of how Sylvester developed the association of graphs with invariants and covariants, some knowledge of invariant theory of the 1880's is necessary. Unfortunately, since Sylvester's notes, papers, and his letters to Petersen are not always clear on this point, we still lack a full understanding of this origin of graph theory. As far as possible, Sabidussi has given a fine analysis of these matters in [26] and [20]. With respect to the Acta paper, the situation is quite different. The problem that Petersen wanted to solve is easily formulated, and no knowledge of invariant theory is needed. Therefore we omit any discussion of the notion of invariant and of how Petersen's problem is related to it.

3. A problem in invariant theory

Given variables $x_1, x_2, \ldots, x_n$, consider all products of the following form

$$\prod_{i<j} (x_i - x_j)^{a_{ij}}, \tag{*}$$

which are understood to be of the same degree α in each x_i. P. Gordan, professor at Erlangen, had proved [11] that one can single out a finite number of such products, the *primitive factors*, such that all other products can be formed as products of primitive ones. For example, for $n = 2$ or 3 there is precisely one primitive factor, viz. $(x_1 - x_2)$ and $(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$, respectively. For $n = 4$ there are precisely three primitive factors, viz.

$$(x_1 - x_2)(x_3 - x_4), \qquad (x_1 - x_3)(x_2 - x_4), \qquad (x_1 - x_4)(x_2 - x_3).$$

In the years 1888–1892, D. Hilbert proved the so-called Finite Basis Theorem in invariant theory. Thus he settled one of the major problems in late nineteenth century mathematics. Essentially, the theorem states that the ring generated by the invariants of a finite set of homogeneous polynomials in k variables is finitely generated. The binary case ($k = 2$) was already settled by Gordan in 1868. Hilbert [14] gave a new and much shorter proof for the binary case using the above theorem of Gordan on primitive factors. Hilbert was only interested in the finiteness of the generating set, and he left aside the problem of explicitly

exhibiting the generators. To accomplish this, one would need an explicit knowledge of the primitive factors of the above products (*). To provide such an explicit description of these primitive factors was precisely what Petersen set out to do in his Acta paper.

4. Hurray, graphs!

As stated above, it was Sylvester's idea to transform the problem of primitive factors into a graph theoretic problem. According to Petersen, he and Sylvester then pursued different ways to solve the problem. Apparently, in the final stages, Sylvester had difficulties in understanding Petersen's proofs. Since it took almost a year before Petersen finally submitted his paper, it is possible that Petersen himself was not completely satisfied with his work (cf. [20]). We do not know whether he changed or simplified anything in that period.

For each x_i Petersen drew a point in the plane, and, whenever the term $(x_i - x_j)$ occurred to the power k, the points x_i and x_j were joined by k distinct lines. Thus one obtains a graph G with n vertices, which is regular of degree α. If we factorize the above product (*) into two homogeneous products of lower local degrees, say β and γ with $\alpha = \beta + \gamma$, then, in graph theoretic terms, we have to partition the edges of G into two sets such that the two edge-induced subgraphs are regular spanning subgraphs of degree β and γ, respectively. A regular graph which does not admit such a partition is then called *primitive*. Hence the problem is, for any n, to determine all primitive graphs with n vertices, and for any given regular graph to find all factorizations into primitive ones.

There are some aspects here which are interesting from the historical point of view. The first is Petersen's terminology. He explicitly stated that he borrowed the word graph from English authors and it clearly remains such a borrowed word with Petersen, for throughout the paper he writes *graph* in italics without the capital of German nouns and with the English *graphs* as its plural. The rest of his terms were inspired by the problem of factorizing products or its invariant-theoretical background. The number of vertices is the *order* of the graph. A regular graph of *degree* α is primitive or factorable, and in the latter case it is the *product* of *factors*. Also it was natural to consider multiple edges, whereas there was no reason to think of loops.

A second interesting point is that Petersen introduced a graph as a figure drawn in the plane consisting of *points* (Punkte) and *lines* (Linien). We shall see below some instances showing that his graphs were not yet truly abstract graphs free from any geometric connotations.

Let us now turn to the theory that Petersen developed for the factorization of regular graphs. Petersen observed that there is a curious difference between the case of even degree and the case of odd degree. The first case allows a complete

solution and the primitive factors are quite simple. In the odd case the situation is totally different.

In what follows we will always assume the graphs to be regular (unless otherwise stated). A factor of degree k will be called *k-factor* for short. In Sections 5, 6 and 7, we discuss the even case and highlight some of Petersen's ideas.

5. Euler's theorem rediscovered?

Let us review what Petersen achieved. First a 2-regular graph consists of the disjoint union of cycles. If there is an odd cycle, then the graph is primitive. Otherwise, it can be factorized by alternatingly placing the edges along the cycles into two 1-factors. Thus, for given n, all primitive 1-regular and 2-regular graphs are easily determined.

In the case of a product graph of two arbitrary factors, Petersen had the pretty idea to colour the edges of the one factor *blue* and those of the other *red*. On the other hand, to factorize a graph of degree $\alpha + \beta$ into factors of degree α and β, it suffices that at each vertex there are α blue and β red edges. Furthermore, one can consider alternating paths and cycles. By exchanging colours on an alternating cycle one gets another factorization into two factors of degree α and β, respectively. By successive such colour exchanges one can obtain any other factorization into an α-factor and a β-factor from a given one.

Petersen established the factorability of 4-regular graphs in an elegant way. For the general case he had to fall back on another approach (see Section 7).

Theorem 1. *A connected 2k-regular graph can be drawn in one continuous stroke.*

In modern terms, the graph has an Euler tour. Petersen's proof is the most obvious one. Briefly, one starts at an arbitrary vertex u and proceeds along edges without using any edge twice. Any passage through a vertex uses two edges. Finally one will have to stop at u. If all vertices have been traversed as often as possible, then we have the required walk. Otherwise, we open the walk at a vertex where not all edges are used and extend the walk.

Petersen observed that it is easily seen that the same argument proves the theorem for the nonregular case, provided that all vertices have even degree. Did Petersen rediscover here Euler's theorem (which was fully proved only in 1873 by Hierholzer [13])? His set-up indicates that this is indeed the case: first the theorem he needs is proven, and as an aside he observes the validity for the non-regular case. In their survey on *Analysis Situs* in the *Encyklopädie der mathematischen Wissenschaften* [7] M. Dehn and P. Heegaard listed Petersen among those who rediscovered Euler's theorem (but only for the regular case, sic!). On the other hand, Petersen had a habit of only seldomly including references.
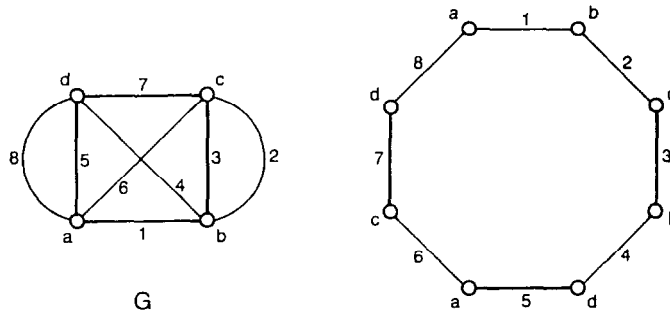
Fig. 2. A regular graph and its stretched graph.

Without the appropriate source material we cannot settle this question conclusively.

Theorem 2. *A 4-regular graph can be factorized into two 2-factors.*

Petersen's proof reflects the above mentioned geometric view on graphs. Although a connected component, say with p edges, can be drawn in one stroke, we instead draw it as a p-gon. Each vertex is then drawn twice on the p-gon. In this *stretched* graph we colour the edges alternately blue and red. This provides us with the required 2-colouring of the original graph (see Fig. 2). As an aside, Petersen noted that the same technique yields a factorization of a $2k$-regular graph into two k-factors.

6. Geometry in graphs

Using alternating cycles we may obtain all other 2-factorizations of our 4-regular graph. The question arises of when two edges will always have either the same or different colours in every factorization. This is the case when any alternating cycle contains both or none of the two edges. But apparently Petersen was not satisfied with this observation. He proceeded to develop some sufficient conditions and also a necessary and sufficient condition. It would lead too far to discuss these in detail, however a few observations should be made for here the geometric aspect is most prominent.

Consider again the stretched graph G' associated with a 4-regular connected graph G. In G' we join any two vertices representing the same vertex of G by a diagonal (see Figs. 2 and 3).

Note that G has an even number of lines (self-evident for Petersen) implying that every such diagonal cuts G' either into two even paths or into two odd paths. Accordingly, diagonals are classified as *even* or *odd*. In constructing an alternating cycle we walk along the polygon, jump along a diagonal, and proceed
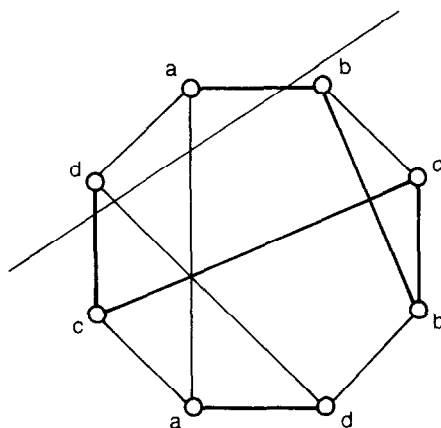
Fig. 3.

along the polygon (in the same direction if the diagonal is even, and in the opposite reversed direction otherwise). Now if we draw a straight line which cuts all the odd diagonals and no even ones, then the edges cut by this line are either of the same or of different colour in each 2-factorization (see Fig. 3). By a reduction procedure on G', Petersen was able to formulate a necessary and sufficient condition of the same type.

In another attempt to characterize such pairs of edges Petersen did some curious things. Consider the 4-regular connected graph G itself. Draw closed curves cutting the edges of G, and avoiding the vertices. Such a curve will cut the edges of any cycle of G and even number of times. Hence, if a curve cuts every edge of a cycle an odd number of times, this cycle must be of even length. Assume that we have drawn a number of closed curves such that they cut every edge of G an odd number of times (so we know that the graph is bipartite). Then we can colour the simply connected regions of the plane, cut out by the curves, black and white in a chessboard manner. Now draw all the vertices in white regions on one side of a straight line and those in black regions on the other side. Then this straight line will cut all edges. This seems to be a rather complicated proof of the fact that, whenever all cycles are even, the graph is bipartite.

7. Even degree, the general case

The factorization of a $2k$-regular graph into two factors is an easy consequence of Euler's theorem and König's theorem [17] on the 1-factorability of regular bipartite graphs (see [3]). Petersen, of course, had to start from scratch. Furthermore, because of the original problem on invariants, he was fixed on regular graphs, so that an inductive proof using edge-deletions and 'factorization'

of irregular graphs (see [18]) would not easily come to mind. But Petersen had a clever idea to overcome the problems.

Let G be a graph with edges ab and cd. If we replace ab and cd by the edges ac and bd, then we obtain G' by a *switching*. It is straightforward to prove that, if G is 2-factorable, then so is G'.

The second step is to prove that, if G and H are $2k$-regular graphs of the same order and G is 2-factorable, then H can be transformed into G by successive switchings. Petersen proved this by induction on k. Now to conclude the proof of the 2-factorability of $2k$-regular graphs, it suffices to exhibit a 2-factorable graph of order n and degree $2k$. This final step was not mentioned by Petersen. Apparently, for Petersen, the existence of such a graph is evident, which of course it is, for example, take an n-cycle and replace each edge by an edge of multiplicity k.

Using alternating cycles we can produce the other 2-factorizations. Factorizing nonprimitive 2-factors into 1-factors was covered above.

8. The odd case

The only primitive regular graphs of even degree are those of degree two containing an odd cycle. In the odd case the situation is completely different. For every k, there are primitive graphs of degree $2k + 1$. The smallest one of degree 3 is given in Fig. 4. Petersen called it *Sylvester's graph*, as Sylvester had constructed it. Petersen produced in a similar way primitive graphs for each k. Start with a vertex w adjacent to $2k + 1$ distinct vertices $x_1, \ldots, x_{2k+1}$. Each x_i is joined by an edge of multiplicity k with vertices y_i and z_i. Finally, for each i, the vertices y_i and z_i are joined by an edge of multiplicity $k + 1$. We thus obtain a primitive $(2k + 1)$-regular graph of order $6k + 4$.

Petersen's next step was to prove that, for given n, there are only finitely many primitive graphs of order $2n$. Because of the possibility of multiple edges this is not clear at first hand. Apparently for Petersen it was evident that the sum of the
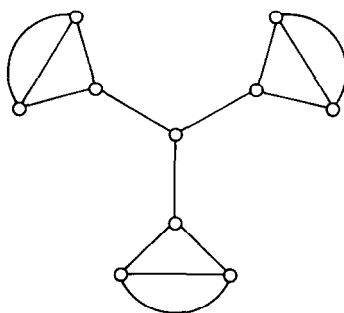

Fig. 4. Sylvester's graph.

degrees is twice the number of edges, and in particular that a $(2k + 1)$-regular graph has even order. The finiteness is a consequence of the following theorem.

Theorem 3. *Let G be a regular graph of order 2n and degree d. If $d > \frac{2}{3}n + 1$, then G is not primitive.*

Again we come across a nice idea. In modern terms, choose a maximum matching in G, say with r edges. Then the unsaturated vertices have all their neighbours among the saturated ones, giving $d(2n - 2r)$ edges between saturated and unsaturated vertices. Also, by the maximality of r, for any edge uv in the matching, u and v cannot have distinct unsaturated neighbours (otherwise we could augment the matching). Hence at most d edges go from u and v to some (unique) unsaturated vertex, and thus at least d edges (counting uv twice) go to saturated vertices. Counting the incidences between saturated vertices and edges twice, we have

$$2dr \geq dr + 2d(n - r),$$

whence

$$r \geq \tfrac{2}{3}n.$$

If $r = n$, then G contains a 1-factor. Assume that $r = n - 1$. The two unsaturated vertices a and c must have distinct neighbours, say b and d, respectively. By switching we introduce the edge ac. Thus we get a 1-factor containing ac. Then bd is in some 2-factor. The product of these factors is a 3-factor. Switching back again we obtain a 3-factor in G. If $r = n - 2$, then by switching we produce a graph with a matching of size $n - 1$ with a 3-factor containing one of the switched edges. The other edge is in a 2-factor. Switching back again we have a 5-factor in G, etc. The smallest possible value for r is $\frac{2}{3}n = n - \frac{1}{3}n$, which then forces a factor of degree at most $\frac{2}{3}n + 1$.

In 1957 C. Berge [1] proved that a matching M in a graph is of maximum size if and only if there is no M-alternating path between unsaturated vertices. Usually in the literature Berge's paper is given as the definite reference for this result (cf. [3]), and often this assertion is called Berge's theorem. Actually, we should attribute it to Petersen, for, after the above Theorem 3, he observed this condition for a matching to be maximum.

9. A modern proof of Petersen's theorem

The last ten pages of the Acta paper are devoted to a proof of what is known as Petersen's theorem.

Theorem 4. *A 3-regular primitive connected graph has at least three leaves.*
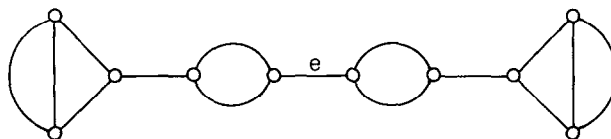
Fig. 5.

Above we defined a leaf to be a bridgeless component that arises after deleting some bridge. Note that the removal of a bridge does not always produce leaves, as is shown by the edge e in Fig. 5. Petersen had overlooked this fact. He defined a leaf to be any part of the graph that is connected to the rest by a single edge. According to this definition, the graph in Fig. 5 has six distinct leaves. Petersen clearly had in mind leaves that are mutually disjoint. This suffices to make the statement in Theorem 4 correct. But in his proof of a corollary one should really use the fact that a leaf is bridgeless. This corollary states that *a 3-regular primitive graph of order 2n with exactly three leaves contains a matching of size n − 1.* Petersen's proof of Theorem 4 is not affected, for he actually produced at least three mutually disjoint leaves in his primitive 3-regular graph.

Let us first have a look at a proof of the theorem by the Hungarian mathematician T. Schönberger [27], cf. [18]. This elegant proof is the cumulative effort of a number of mathematicians (see Section 13). A basic idea in the proof is that of splitting of an edge, which is due to Frink [10]. Following Frink, we call a bridgeless connected 3-regular graph *simple.* Let $e = uv$ be an edge in a simple graph G adjacent to the edges $1, 2, 3, 4$ as in Fig. 6. By *splitting* the edge e we mean that we delete u and v and their incident edges and add the edges 13 and 24, giving G_1, or 14 and 23, giving G_2, as in Fig. 6. Most of the work lies in the proof of the following lemma, which we omit.

Lemma 5. *If G is simple and splitting an edge produces G_1 and G_2, then G_1 or G_2 is simple.*

A *colouring* of the edges with *red* and *blue* is a factorization of G into a red 1-factor and a blue 2-factor.

Lemma 6. *A simple graph G has a colouring such that any two given edges are coloured blue.*
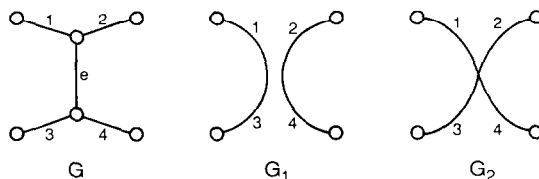

Fig. 6. Splitting an edge.

Proof. Assume the contrary, and let G be a counterexample of smallest order. Let f and g be any two edges, and let e be an edge adjacent to f and distinct from g. Splitting e produces a simple graph G' of order two less. We consider three cases depending on how g is involved in the splitting. For instance, if G' is as G_1 of Fig. 6, and $f = 1$ and $g = 4$, then we choose a colouring of G' in which 13 and 24 are coloured blue. We use this colouring to colour G, where $1, 2, 3, 4$ are then coloured blue and e is coloured red.   □

By Lemma 6, we only have to prove that a 3-regular graph G with two leaves has a proper colouring. Choose an edge in each leaf. Subdivide these edges by a new vertex, and join the two new vertices by an edge. Then we obtain a simple graph G'. Colour G' properly such that both parts of one of the subdivided edges are blue. Then the new edge is red and the other subdivided edge is again blue. This provides us with a proper colouring of G.

Petersen did not think of an inductive proof. This complicates matters considerably. Nevertheless Petersen was able to develop an ingenious technique to establish the existence of at least three leaves in a primitive 3-regular graph. Because this technique is so fascinating, we exhibit its basic ideas in the next section.

10. Petersen's own proof

In this section G is a primitive connected 3-regular graph of order $2n$. Note that G has $3n$ edges. A *chain* (Kette) is a walk containing each edge at most once.

Lemma 7. *The edge-set of G can be partitioned into n chains.*

Proof. Choose any chain of G. Extend it on both sides as far as possible. Then it will begin and end with vertices which are internal vertices of the chain. Start a new chain at some internal vertex that is not yet the beginning or end of a chain and extend the new chain as far as possible (without using edges of previous chains). Repeat this until all edges are used. Then at every vertex a chain will pass and a chain will begin or end.   □

In the sequel we always assume such a partition of G into n chains to be given. To distinguish these chains we use the term *trail* to indicate an arbitrary walk containing each edge at most once. If necessary, we adjust the partition to suit our needs. Chains are *odd* or *even* depending on the parity of their length. Colour the edges along these chains alternately *blue* and *red*. Thus an odd chain has its extreme edges blue, whereas an even chain has an extreme edge of each colour. Any passage of a chain through a vertex produces a blue and a red edge at this vertex. Hence, if all chains are odd, we have a proper colouring with a red

1-factor and a blue 2-factor. Since G is primitive, there must be even chains. Note that an even chain has precisely one end vertex incident with two red edges. Hence the number of even chains equals the number of 'red-red-blue' vertices (which is even, being the number of vertices of odd degree in the blue subgraph).

Assume that the partition into chains is such that the number of even chains is as small as possible. Because of this minimality, there are no alternating trails between red-red-blue vertices with red extreme edges, for otherwise, we could recolour the edges on this trail, thus reducing the number of red-red-blue vertices. Mistakenly, Petersen considered it obvious that this new colouring matched a partition into n chains. However, by splitting chains and combining subchains (see below for an example) we can obtain a new partition, in which the alternating trail is the first part of an even chain. This chain then starts at one red-red-blue vertex and passes through the other, at which a second even chain starts. By recolouring the trail and combining it with the second even chain, we produce an odd chain, whereas the remaining part of the first chain is odd.

Choose an even chain T starting at t. If t is an internal vertex of another chain T', then it cuts T' into two subchains, at least one of which is even (otherwise we could combine T with one of the odd subchains reducing the number of even chains by two). By combining T with an even subchain of T', we obtain a longer even chain. Hence we may assume that T begins and ends at internal vertices of T. Petersen assumed only that T ends in an internal vertex. This forced him to consider an extra case in the final stages of his proof. We may choose the colouring of T such that it starts at t with a red edge. In Fig. 7 we see an example of a partition into chains. Here T is oriented from t, while the other chains are numbered. Thin edges are blue and thick edges are red. We will illustrate some of the basic ideas using Figs. 7 and 8.

We now consider the alternating trails starting with a red edge at t, in which, besides T, only odd chains are involved. We will orient the edges along these trails from t. This produces the *oriented system* seen in Fig. 8. Some edges are doubly oriented, whereas others are not oriented at all. We build up this system
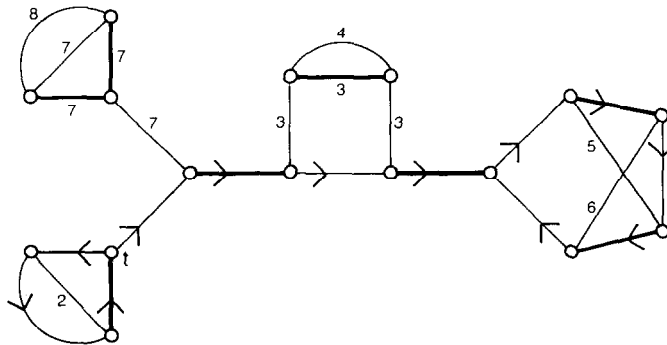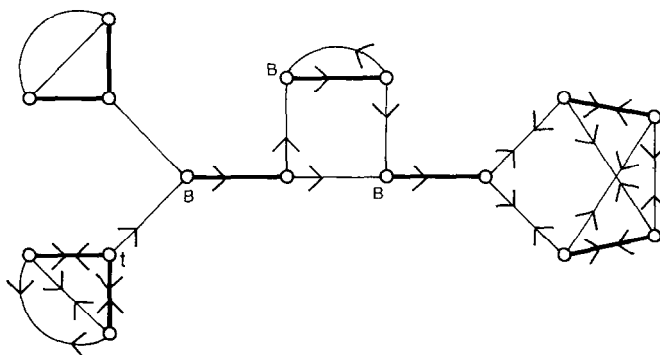

Fig. 7.

Fig. 8. The oriented system.

stepwise. At each step we add a new chain, orient it and, if necessary, doubly orient edges in the system. Then we colour a vertex *blue* (indicated by *B* in Fig. 8), whenever it can only be reached from *t* along alternating trails by a blue edge. The other vertices in the system reachable by a red edge are coloured *red*. At a later stage blue vertices may be recoloured red. Note that a red vertex can be reached from the red-red-blue vertex *t* by an alternating trail with red extreme edges. Hence, except for *t,* a red vertex must be incident with two blue edges. Vertices at which not all edges are oriented are *unsaturated*. We start by orienting the edges along *T* from *t,* as in Fig. 7. We extend the oriented system at an unsaturated vertex. Consider an unsaturated blue vertex *u*. At *u* a new chain starts with an unoriented edge. In case this edge is red, the chain is even, and we do not use this chain. Otherwise the unoriented edge at *u* is blue and we cannot extend the system at *u*. Let *v* be an unsaturated red vertex, so that the unoriented edge at *v* is necessarily blue. Then the chain starting with this edge must be odd. We may assume that this chain ends in an internal vertex of itself or in an unsaturated vertex of the oriented system obtained so far (otherwise adapt the partition). Now we orient this chain and add the necessary double orientations. Thus, at each step, we add an odd chain (starting at an unsaturated red vertex) and never an even chain. We end up with the oriented system, as in Fig. 8.

In doubly oriented subsystems all vertices are red. Petersen established the following facts about such subsystems. We do not prove these here, but they follow in a straightforward manner from the above construction. In a doubly oriented subsystem a vertex has degree two or three; it has no incoming blue edges and no outgoing red edges; and it has one incoming red edge (or none if it contains *t*) and an even number of outgoing blue edges (or odd if it contains *t*). Doubly oriented subsystems with no outgoing blue edges are leaves. If the one containing *t* has only one outgoing blue edge, then it is also a leaf.

We now contract each doubly oriented subsystem into a 'fat' vertex ignoring loops. Thus we get a graph *G'* with edges coloured red and blue and a specified (possibly fat) vertex *t.* The graph *G'* inherits the edge-orientation and vertex-

colouring from *G*. But we can recover these also in *G'* itself by applying the above process of orienting alternating trails from *t* in *G'*. Since double orientations do not occur, blue vertices will not be recoloured during the process. At each step we add a new maximal alternating trail starting with a blue edge at an unsaturated red vertex. If *t* is fat we colour *t* red.

In the process of rebuilding the oriented system of *G'*, Petersen kept count of the surplus of unsaturated blue vertices over unsaturated red vertices. After orienting the first (maximal) trail from *t*, the initial surplus is -1, -2, or -3 depending on whether *t* is a fat red vertex with one outgoing blue edge, or a fat red vertex with more outgoing blue edges, or a blue vertex, respectively. Now we check how this surplus changes when we add a new (maximal) trail and orient it. This surplus increases by 2 whenever we add a trail leading to a leaf (a fat vertex without outgoing blue edges). In all other cases it decreases or does not change. We end up with no unsaturated red vertices. Because of parity conditions (outgoing blue edges of fat vertices) we can deduce that *G'* itself contains an odd number *b* of unsaturated blue vertices. Hence, since the initial surplus was negative, we get at least two fat red vertices as leaves, unless the initial surplus is -1. In the latter case the fat vertex *t* is a leaf as well. If $b = 1$, then the unique unsaturated blue vertex produces a leaf outside the oriented system (see Fig. 8). In all other cases the increase of the surplus during the orientation process guarantees the existence of at least three fat vertices as leaves.

In the final stages Petersen considered one unnecessary case because of his choice of the initial chain *T* (see above) and overlooked one subcase. But only a few lines and no new arguments are necessary to settle this subcase.

All in all this is a complicated but very ingenious (and fascinating) proof. One can imagine that in the 1890's, when graph theory was still virtually non-existent, it took quite some time for the mathematical community to grasp Petersen's achievements.

11. The Fortschritte der Mathematik

In the *Jahrbuch über die Fortschritte der Mathematik* (the forerunner of the *Zentralblatt*) volume 23, pages 115–117 Petersen's paper was reviewed by F. Meyer in the section on Algebra, subsection Equations (Theory of forms). Meyer considered it to be a remarkable paper, because new principles from invariant theory are successfully explained and further pursued in a purely illustrative and graphic manner (in German *anschaulich*). He formulated the translation of the problem on factorizing products into a graph problem. He observed the difference between even and odd degree, gave the primitive 2-regular graphs and the switching procedure to obtain all 2-factorizations. He stated the theorem on 3-regular graphs. He closed with the remark that the nature of the methods is such that they cannot be explained further in the review, and that the reader is invited to consult the ingenious paper itself.

12. Petersen and the 4CC

After the Acta paper Petersen entered the arena of graph theory twice with contributions in *L'Intermédiaire des Mathématiciens* [23–24], the first one containing the Petersen graph (see Section 1). The second one dealt with the Four Colour Conjecture. Here he mentioned the equivalence between the 4CC and the 1-factorization of planar bridgeless 3-regular graphs. Furthermore he states the equivalence between the 4CC and an assignment of +1 and −1 to the vertices of 3-regular plane graphs such that along a face the assignments sum up to 0 (mod 3). This reformulation of the 4CC was published by Heawood [12] in the previous year, but Petersen did not mention Heawood. Again this suggests that Petersen discovered this result independently. In the same note he expressed his doubts about the possible truth of the 4CC.

In France at that time graphs usually were called *réseaux* (networks). Petersen called them *graphes*, using the French form of the English graphs. Sylvester had introduced the term graph [29]. However, I think that it is the influence of Petersen's work that it was adapted in German (and later also in French, and so forth). For example, König still considered it necessary in his paper [17] to add a footnote referring to Petersen when he used the term *Graph*.

Until well into the 1920's almost all references to Petersen's results were either in papers on the four colour problem, cf. [25], or in the framework of topology (see the next section). The notable exceptions are König's papers on matchings, for instance [17]) (cf. [18], where his other papers are listed).

13. Petersen's theorem reproved

For many years there was no follow-up to the Acta paper. It seems that the mathematicians working in graph theory in the 1890's became aware of it only in 1898, when Petersen mentioned his results in [23]. These results were also mentioned in the section on *Liniensysteme* in Dehn and Heegaard's survey of *Analysis Situs* (i.e. combinatorial topology, theory of complexes, etc.) [7]. That it was classified under topology reflects another origin of graph theory: Euler's polyhedral formula and Listing's work on topology [19], as evidenced also by the subtitle of König's book [18] (cf. [2, 21]).

Petersen's theorem was taken up in the 1910's by the American topological school of J.W. Alexander and O. Veblen. One of the students, H.R. Brahana published a new proof in 1917 [4]. The new idea was to reduce a 3-regular graph as indicated in Fig. 9 and to use induction.

To prove that every 3-regular graph with at most two leaves is factorable, assume that G is a smallest counterexample. Apply the reduction of Fig. 9, and if G has leaves, take e to be a bridge. Then G' has at most two leaves, so that G' is factorable in a red 1-factor and a blue 2-factor. If we can colour both ab and cd
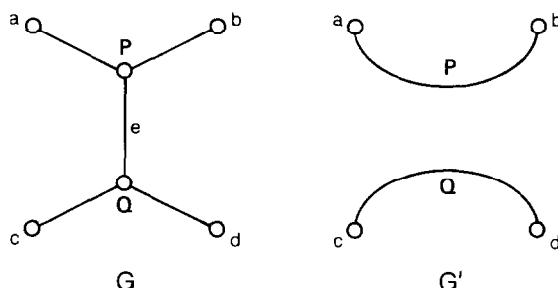
Fig. 9.

blue (using alternating cycles, if necessary), then G has a proper colouring in which e is red. Otherwise, say, ab is necessarily red. Then we search for an alternating trail from P to Q with, say, the half edges Pa and cQ as extremes. By recolouring this trail the halves aP and Pb of ab, as well as those of cd, have different colours. Hence G has a proper colouring in which e is blue. Now assume that no such trail in G' exists. Then we build up an oriented system in G' using alternating trails starting at P with a red half edge. During the process we contract any doubly oriented cycle into a single vertex. In the resulting oriented subgraph G'' every vertex has one incoming red edge, and a vertex has no outgoing blue edges whenever it corresponds to a leaf in G'. By counting the (red and blue) edges in G'' twice as well as the vertices, we can show that G' has at least three leaves, which contradicts our choice of G and G'. Note that in this approach vertices are not coloured. By using induction and this new counting procedure, Brahana is able to shorten the proof considerably, but it still lacks the elegance and transparency of Schönberger's proof above.

A few years later a Belgian mathematician A. Errera [8] published a doctoral thesis on the four colour problem. He discussed Brahana's proof and made some slight improvements. For instance, he included the possibility of loops and of an edge without ends (a closed circle) thus facilitating the induction step. Moreover, he avoided the contraction of doubly oriented edges and counted the various types of vertices in the oriented system depending on the nature of their incident edges. This makes the proof more convincing, but, with hindsight, it is still too complicated. However, Errera considered his proof suitable for a wider audience, for he published it in *Mathesis* [9], a journal aimed at mathematics teachers at grammer schools and high schools. A sketch of Errera's proof was given iñ a small elementary treatise [28] by A. Sainte-Laguë, a professor at a French Lycée. He also discussed the geometric characterization of edges in a 4-regular graph being forced in the same 2-factor, indicated above in Fig. 3.

In 1926 another American topologist O. Frink [10] had the brilliant idea of the reduction in Fig. 6. He started with Lemma 5 (see Section 9 above). His proof of this lemma was not satisfactory (actually, too much is left to the reader), but the essentials are there. The gaps were not noticed in the review in the

Fortschritte der Mathematik. Frink's next two steps were as follows. Recall that a simple graph here is a bridgeless connected 3-regular graph.

Lemma 8. *Every edge in a coloured simple graph is on an alternating cycle and hence can have its colour changed.*

This is proved by assuming the contrary and reducing, as in Fig. 6, a smallest coloured counterexample to a simple graph.

Lemma 9. *Every simple graph is colourable.*

Proof. Again assume the contrary, and let G be a smallest noncolourable simple graph. Then the reduced simple graph G', say G_1 in Fig. 6, has a proper colouring. If the new edges 13 and 24 are red, then apply Lemma 8. Thus we have that both 13 and 24 blue (colour e red) or one red and the other blue (colour e blue). □

To conclude the proof for graphs with two leaves, connect the leaves by an edge (as in Section 9). Colour this simple graph such that the new edge is red, and delete this edge.

Frink's proof of Lemma 5 was too concise (see above), but still he observed that it is *"unfortunate that such a trivial and obvious theorem requires such a long proof"*. Such a remark shows that graph theory was still in its infancy. There is not yet a fully developed intuition and appreciation of what is easy and what is difficult in graph theory. König, in his book [18], gave a full proof of Lemma 5 based on Schönberger's. He commented on Frink's remark as follows: *"Frink's theorem would not be worth all this trouble, were it not that it provides an essential step in the proof of Petersen's fundamental theorem"* (sic!). With his book König established graph theory as an independent and worthwhile subject, but apparently it was just out of its infancy.

14. Concluding remarks

In his book [19] König left aside the topics of planarity and the four colour problem, but otherwise it was a comprehensive survey of graph theory as of 1936. When we overlook his presentation, three things stand out: Kirchhoff's results on spanning trees, Petersen's theory of regular graphs, and König's own work on matchings.

It was essentially König who created the subject of 1-factorizations and matchings, at first for bipartite graphs in relation to determinants and set theory [17]. Petersen presented the nucleus of the theory of factorization. Here 1-factors

played only a marginal role, but his ideas were seminal for problems on
1-factorizations and arbitrary matchings, especially through the work of Dénes
König.

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