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Polynomial Indexing of Integer Lattice-Points I. General Concepts and Quadratic Polynomials*

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Denoting the nonnegative integers by N and the signed integers by Z, we let S be a subset of Z^m for $m = 1, 2,...$ and f be a mapping from S into N. We call f a *storing function* on S if it is injective into N, and a *packing function* on S if it is bijective onto N. Motivation for these concepts includes extendible storage schemes for multidimensional arrays, *pairing functions* from recursive function theory, and, historically earliest, diagonal enumeration of Cartesian products. Indeed, Cantor's 1878 denumerability proof for the product N^2 exhibits the equivalent packing functions $f_{\mathrm{Cantor}}(x, y) = \{\text{either } x \text{ or } y\} + (x + y)(x + y + 1)/2$ on the domain N^2, and a 1923 Fueter–Pólya result, in our terminology, shows f_{Cantor} the only *quadratic* packing function on N^2. This paper extends the preceding result. For any real-valued function f on S we define a density $S \div f = \lim_{n\to\infty} (1/n)\#\{S \cap f^{-1}([-n, +n])\}$, and for any packing function f on S we observe the fact $S \div f = 1$. Using properties of this density, and invoking Davenport's lemma from geometric number theory, we find all polynomial storing functions with unit density on N, and exclude any polynomials with these properties on Z, then find all *quadratic* storing functions with unit density on N^2, and exclude any quadratics with these properties on $Z \times N$, Z^2. The admissible quadratics on N^2 are all nonnegative translates of f_{Cantor}. An immediate sequel to this paper excludes some higher-degree polynomials on subsets of Z^2.

1. Introduction and Synopsis

Cauchy [5, pp. 140–158]. in his "Cours d'analyse," presents a diagonal enumeration of arrays $\{a_{ij} : i, j = 1, 2,...\}$ to interpret a double series as a single summation. Specifically, he visits successive diagonals $\{a_{ij} : i + j = n\}$ for $n = 1, 2,...$ and proceeds left-to-right down each line. Cantor [2, 3], in his cardinality researches, uses this same pattern in a more fundamental way to prove the denumerability of the positive rational numbers. Indeed he applies this enumeration scheme to the lattice $\{(i, j): i, j = 1, 2,...\}$ and

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extends his result, by induction, to all finite Cartesian products of denumerable sets. Also Cantor associates the polynomial

$$f(i,j) = i + (i + j - 1)(i + j - 2)/2 \tag{1.1}$$

with the indicated pattern, since it gives the resulting integer location of the pair (i, j). Moreover Chowla [6] exhibits an m-variable generalization of the polynomial (1.1) which effects a direct polynomial enumeration of the m-dimensional positive lattice.

Some expositors of set theory have preferred other schemes for two-dimensional enumeration, but then have found no polynomial formula for the induced sequential locations. Indeed Fueter and Pólya [12] concatenate two individual notes as a joint paper which proves (1.1) and the corresponding $f(j, i)$ the only *quadratic* polynomials associated with any such pattern; while Pólya and Szegö [20, Vol. 2, Problem 243] show the inadmissibility of any nonquadratic polynomial which has highest-degree homogeneous part vanishing nowhere on the first quadrant. Our paper, together with its sequel (Lew and Rosenberg [16]), introduces an "almost complete" enumeration of denumerable sets yielding nontrivial extensions of these uniqueness results. We discuss a larger collection of domains, and exclude a larger family of polynomials.

Here we prefer the origin at zero, and thus define the set $N = \{0, 1, 2,...\}$, but we meet negative integers in this work, hence also recall the set $Z = \{0, \pm 1, \pm 2,...\}$. The symbol R, as usual, denotes the set of real numbers. We now consider a mapping $f: N^m \to N$ or, more generally, suppose an arbitrary subset S of Z^m and then consider a mapping $f: S \to N$. We call f a *storing function* on S if it is an injective map into N, and a *packing function* on S if it is a bijective map onto N. (Descriptive convenience at certain points motivates broader definitions in our formal development.) Two such maps will be called equivalent, in the context of this discussion, if they differ only by a permutation of their arguments. A natural problem, on appropriate domains S, is then to find all polynomial packing functions up to equivalence. Section 2, for illustrative purposes, contains a thorough treatment in one dimension ($m = 1$), and the remainder of this paper offers a partial analysis in two dimensions ($m = 2$), while an immediate sequel (Lew and Rosenberg [16]), via more intensive methods, obtains stronger theorems for the two-dimensional case. Further, unpublished work includes some m-dimensional results.

Storing functions in two variables play an important role in recursive function theory (Davis [8, p. 43]; Minsky [18, p. 182]), and some works in this field describe such maps as *pairing functions* (Rogers [21, p. 64]). That theory requires only one such function, in principle, but many authors provide several examples, with different ranges. These functions extend various definitions from one positive integer variable to corresponding pairs, and thus generalize important concepts, by induction, to m such variables. Thus Hammer, in unpublished correspondence (Barlaz [1]), independently questions the uniqueness of (1.1) in this context, and analogously, in a still-unsolved problem (Hammer [13]), seeks also any polynomial bijections from Z^2 onto Z. However, the principal motivation for our terms, and this investigation, is the flexible computer storage of multidimensional arrays, and, specifically, planar arrays. First Rosenberg [22–25] and then Stockmeyer [27] have studied *extendible* schemes for such storage, which allocate array positions uniquely to numbered memory cells, and permit array expansion positively along any axis, but entail no storage reshuffling whatever at any time. The common alternative of array storage by rows (respectively, columns) demands such reshuffling on adjunction of further columns (respectively, rows).

If we restrict our attention to a large array in one plane quadrant, then we may label an arbitrary point by coordinates (x, y), both in N, and we may designate its storage location by $f(x, y)$, also in N. An extendible scheme for such allocations requires f to be a storing function; and full utilization of memory cells requires f to be a packing function. Rapid computation of storage locations requires f to have a simple formula; hence the program for this investigation assumes f to be a polynomial. An integer shift of the polynomial (1.1) provides two equivalent possibilities as packing functions:

$$f_{\text{Cantor}}(x, y) = \{\text{either } x \text{ or } y\} + (x + y)(x + y + 1)/2. \qquad (1.2)$$

Our results for the domain N^2 support the implicit conjecture of the Fueter–Pólya [12] paper:

No other polynomials represent packing functions. (1.3)

Our results for the domains $Z \times N$, Z^2 suggest the impossibility of *any* polynomial packing functions. Even our partial proof of these conjectures, given the existence of more efficient extendible schemes (Rosenberg [25]), strongly recommends the use of more diverse formulas in devising storage algorithms for two-dimensional arrays. The much larger class of all polynomial storing functions prompts no comparably neat conjecture about its form. Indeed $(x + y)^m + x$ on N^2 represents such a function for any $m \geqslant 2$; while the mth Chowla [6] polynomial, by a shift of origin, becomes a packing function $f(x_1, \ldots, x_m)$ of degree m on N^m, hence yields a storing function $f(x, y, 0, \ldots, 0)$ of degree m by restriction.

A synopsis of our results demands the mention of one further concept. Given any subset $S \subset Z^m$ and any map $f\colon Z^m \to R$, we introduce the level sets $f^{-1}([-n, +n])$ and the *density*

$$S \div f = \lim_{n \to \infty} (1/n) \, \#\{S \cap f^{-1}([-n, +n])\}. \qquad (1.4)$$

Since we cannot always assume the existence of this limit, we write, more generally, $S \underline{\div} f$ for the lim inf and $S \overline{\div} f$ for the lim sup. Fueter's half of the Fueter–Pólya [12] paper supposes a quadratic injection $f(x, y)$ into $\{1, 2, \ldots\}$, generalizes the Riemann zeta function to a Dirichlet series

$$L(s) = \sum \{f(x, y)^{-s}: (x, y) \in N^2\}, \tag{1.5}$$

and considers the associated residue at $s = 1$. However, Fueter's residue, for any injective map f, equals $N^2 \div f$ by Ikehara's theorem (Widder [28, pp. 233–236]); and the hypothetical alternative

$$\lim_{t \to 0+} t \sum \{e^{-tf(x,y)}: (x, y) \in N^2\}, \tag{1.6}$$

for any negatively bounded function f, is also $N^2 \div f$ by Karamata's theorem (Widder [28, p. 192]). Pólya's half of the same paper supposes a polynomial mapping $f(x, y)$ of degree d, isolates its homogeneous part $f_d(x, y)$ of highest degree, and considers the associated area of the region $\{(x, y): 0 \leqslant x, 0 \leqslant y, f_d(x, y) \leqslant 1\}$. However, Pólya's area, for any quadratic injection f, equals $N^2 \div f$ by our Proposition 4.3, and the density (1.4), for an arbitrary function f, is the more serviceable concept in noncoincident situations. Indeed, specialization of our $S \underline{\div} f$ yields the standard asymptotic density of number theory (Niven and Zuckerman [19, p. 240]). Moreover $S \div f = 1$ for packing functions on S, and $S \overline{\div} f \leqslant 1$ for storing functions on S. Thus $S \div f \neq 1$ implies that f is *not* a packing function. Hence we broaden our investigation from its previous goal to seek all polynomial storing functions with unit density.

Section 2 extends these densities to arbitrary domains and derives some basic properties of these constructs; it defines storing and packing functions in the same generality and proves model results in one dimension. Proposition 2.3, in particular, establishes zero density for all nonlinear polynomials which represent storing functions on N or Z. Theorem 2.4, for domain N, considers polynomial storing functions with unit density, and obtains the form $f(x) = x + c$ with nonnegative integral c. Our definitions preclude linear storing functions on Z, but Theorem 2.5 describes the storing functions of least degree. Section 3 introduces some concepts of plane affine geometry which facilitate the study of two-dimensional problems; it proves structural lemmas for later use which delimit storing functions on arbitrary sectors. Indeed, Proposition 3.4 avoids some case analysis of Fueter, and Corollary 3.5 excludes all linear polynomials in two variables.

Section 4 treats quadratic storing functions on N^2. Lemma 4.2 calculates Pólya's area for these f, and Proposition 4.3 relates this area to $N^2 \div f$. Quadratic polynomials of nonparabolic type have transcendental density by Corollary 4.4, whence quadratic storing functions with unit density have form

$$f(x, y) = f_{\text{Cantor}}(x, y) + a(x + y) + b, \qquad \text{with} \quad a, b \in N, \tag{1.7}$$

by Theorem 4.5. Thus the only *quadratic* packing functions on N^2 are the Cantor polynomials, as before. Finally, Section 5 presents other two-dimensional results. Theorem 5.1, again for domain N^2, obtains the quadratic storing functions with density $\frac{1}{2}$. The domain Z^2, by Theorem 5.3, admits no polynomial storing functions below fourth degree. The domain $Z \times N$, by Corollary 5.4, admits no quadratic storing functions with unit density.

Hence this paper characterizes all polynomial packing functions of one variable, but essentially describes only quadratic packing functions of two variables. To complete an analysis in two dimensions requires some counterpart of Proposition 2.3. Our immediate sequel (Lew and Rosenberg [16]) to the present work contains a partial result of this kind: a nonquadratic polynomial $f(x, y)$ on N^2, $Z \times N$, or Z^2 cannot be a storing function with unit density either when f is sectorially increasing or when $\deg(f) \leqslant 4$. Either requirement for this nonexistence assertion is a nontrivial extension of the Pólya–Szegö [20, Vol. 2, Problem 243] result. Thus the conjecture (1.3) becomes more plausible, but a stronger two-dimensional statement is not available, though the one-dimensional analog is quite elementary. The lattice-point enumerations in the present work apply a fundamental lemma of Davenport [7], but the density calculations in our sequel invoke many other results of geometric number theory, wherefore the introductory remarks in this sequel review some relevant literature in that field. Our estimates in these papers involve the O and o notation of Bachmann and Landau (Landau [14, pp. 59–65, 883]; Erdélyi [11, Chap. 1]).

2. General Theory in One Dimension

Here we consider real-valued functions with arbitrary domains and define our various densities in this context, then introduce storing and packing functions in the same generality and obtain certain basic properties for future use. Also we treat polynomial storing functions in a single real variable and prove model results for the domains N and Z, to develop the simplest case of our analysis and illustrate our goals for higher dimensions. If S is an arbitrary set then $\#(S)$ denotes the cardinality of S; if f is an arbitrary function then $f \mid S$ denotes its restriction to S. Let X be an arbitrary set of points, and X_0 be a countable subset of X. If S is an arbitrary subset of X then

$$(\# \mid X_0)(S) = \#(X_0 \cap S) \tag{2.1}$$

in the stated notation. Moreover $\# \mid X_0$ is a positive measure and all subsets S are measurable sets. If f, in particular, is a real-valued (hence measurable) function on X, then $f \mid S$, by composition, induces a measure $(\# \mid X_0) \circ (f \mid S)^{-1}$ on R; that is, any Borel subset $B \subset R$ has measure

$$[(\# \mid X_0) \circ (f \mid S)^{-1}](B) = \#[X_0 \cap S \cap f^{-1}(B)]. \tag{2.2}$$

The conceptual framework of measure theory motivates seeking an "average density" for this set function. We suppress X_0 in our notation because we fix X_0 in each application, and we define upper and lower densities by

$$\begin{aligned} S \mathbin{\overline{\div}} f &= \varlimsup_{n\to\infty} (1/n)\, \#\{X_0 \cap S \cap f^{-1}([-n, +n])\}, \\ S \mathbin{\underline{\div}} f &= \varliminf_{n\to\infty} (1/n)\, \#\{X_0 \cap S \cap f^{-1}([-n, +n])\}. \end{aligned} \tag{2.3}$$

Clearly these limits are well defined in $[0, +\infty]$; indeed

$$0 \leqslant S \mathbin{\underline{\div}} f \leqslant S \mathbin{\overline{\div}} f \leqslant +\infty. \tag{2.4}$$

If also the two limits are equal then the set S will be called *f-amenable*, and the common limiting value will be written $S \div f$. The motivation for this language is the theory of semigroup means (Day [9]). Thus $S \mathbin{\overline{\div}} f = 0$ implies $S \div f = 0$, and $S \mathbin{\underline{\div}} f = +\infty$ implies $S \div f = +\infty$. For the identity mapping *id* on simple real subsets S we may see that $S \div (c \circ id) = c^{-1}(S \div id)$ with any positive c; for certain functions f with integer values we shall find that $S \mathbin{\underline{\div}} f$ is a standard asymptotic density from number theory (Niven and Zuckerman [19, p. 240]). Thus we shall describe $S \div f$ as a density, but represent it as a quotient, because we must eventually treat both S and f as variables, hence cannot conveniently write either as a subscript.

The following two lemmas further validate this notation, and have many later uses.

LEMMA 2.1. *Let f: $X \to R$ and $S_1, S_2 \subset X$. Then for arbitrary S_1, S_2*

$$(S_1 \cup S_2) \mathbin{\overline{\div}} f \leqslant (S_1 \mathbin{\overline{\div}} f) + (S_2 \mathbin{\overline{\div}} f), \tag{2.5}$$

and for any disjoint such subsets

$$(S_1 \cup S_2) \mathbin{\underline{\div}} f \geqslant (S_1 \mathbin{\underline{\div}} f) + (S_2 \mathbin{\underline{\div}} f). \tag{2.6}$$

while for f-amenable disjoint subsets

$$(S_1 \cup S_2) \div f = (S_1 \div f) + (S_2 \div f). \tag{2.7}$$

Proof. Standard limit theorems yield the two inequalities. □

LEMMA 2.2. *Let $f \mid (X_0 \cap S)$ be an injective map into $-k + N$ for some nonnegative integer k (where $r + N = \{r, r+1, r+2, \ldots\}$ for any real number r).*

(1) *Then $S \underline{\div} f$ is the asymptotic density of $(1 + N) \cap f(X_0 \cap S)$, and*

$$0 \leqslant S \underline{\div} f \leqslant S \overline{\div} f \leqslant 1 \tag{2.8}$$

for any such f; hence $S \underline{\div} f = 1$ implies $S \div f = 1$.

(2) *If also $N \subset f(X_0 \cap S)$ then indeed $S \div f = 1$.*

Proof. (1) The initial hypothesis yields the two inequalities

$$\begin{aligned} &\#\{x \in X_0 \cap S : f(x) \leqslant 0\} \leqslant k + 1, \\ &\#\{x \in X_0 \cap S : 0 < f(x) < n\} \leqslant n. \end{aligned} \tag{2.9}$$

To find, respectively, $S \underline{\div} f$ or the asymptotic density, we take the sum or the second cardinality, then divide by n and evaluate the lim inf (Niven and Zuckerman [19, p. 240]).

(2) The set $1 + N$ has unit asymptotic density. □

The setting of these initial remarks permits the definition of other basic concepts. Let X be an arbitrary set of points, and again X_0 a countable subset of X. Let S be an arbitrary subset of S, so that $X_0 \cap S$ is always countable, and let f be a real-valued function on X. Then f will be called a *storing function* (on S) if $f \mid (X_0 \cap S)$ is an injective map into N, and f will be called a *packing function* (on S) if $f \mid (X_0 \cap S)$ is a bijective map onto N. Clearly if f is a storing function on S then also f is a storing function on any included set. Moreover if f is a storing function on S then $S \overline{\div} f \leqslant 1$, by Lemma 2.2.1, while if f is a packing function on S then $S \div f = 1$, by Lemma 2.2.2. Therefore if $S \div f \neq 1$ for some function f, then f cannot be a packing function on S. Thus we shall seek systematically all storing functions with unit density in order to find conveniently all packing functions within some interesting family.

Polynomial storing and packing functions on certain grids are the principal subject of this paper and its sequel. Later we take $X = R^2$ and specify $X_0 = Z^2$, so that a storing function on any plane set is a storing function on the contained lattice points; but we can obtain suitably complete results in one dimension, which will foreshadow our less exhaustive conclusions in other cases. Hence we let $X = R$, $X_0 = Z$ for the remainder of this section, and we let f be a polynomial function on $X = R$. Certain arguments involve the finite-difference operators

$$\begin{aligned} \Delta^1 f(x) &= \Delta f(x) = f(x+1) - f(x), \\ \Delta^{n+1} f(x) &= \Delta\Delta^n f(x) \qquad \text{for} \quad n = 1, 2, \ldots . \end{aligned} \tag{2.10}$$

We characterize the polynomial packing functions on N, and describe the simplest storing functions on Z. Our first, simple proposition is noteworthy because its multidimensional counterparts are still incomplete.

PROPOSITION 2.3. *If* $\deg(f) \geqslant 2$, *where* f *is a polynomial storing function on* $S = N$ *or* Z, *then* $S \div f = 0$, *whence* f *is not a packing function on either* N *or* Z.

Proof. Degree 2 or more implies increasingly large gaps between successive values. $\square$

THEOREM 2.4. (1) *The most general polynomial storing function with unit density on* N *has the form* $f(x) = x + c$ *with arbitrary* c *in* N. (2) *The only polynomial packing function on* N *is the identity map on* N.

Proof. Proposition 2.3 excludes nonlinear polynomials. The rest is obvious. $\square$

A linear polynomial cannot be a storing function on Z, since it assumes negative values for some arguments; a nonlinear polynomial cannot be a packing function on Z, since it has zero density by Proposition 2.3. Hence there are no polynomial packing functions on Z, although there are infinitely many storing functions with zero density. However, any polynomial storing function on Z must have even degree, by its nonnegativity, whence the final result of this section will classify the storing functions of minimal degree. Specifically, for a, b in Z we let

$$g(a, b; x) = (a/2)\, x(x-1) + bx, \tag{2.11}$$

and for polynomials of this kind we let

$$G = \{ g(a, b; \cdot) \colon 3 \leqslant a; 0 < b < a/2 \}. \tag{2.12}$$

THEOREM 2.5. (1) *If* $f(x) = g(\pm(x - x_0)) + c_0$, where $g \in G$, $x_0 \in Z$, $c_0 \in N$, *then* f *is a quadratic storing function on* Z, *and conversely.* (2) *If* $f(x) = g(\pm x)$, *where* $g \in G$, *then also* $\min f(Z) = f(0) = 0$, *and conversely.* (3) *If* $f(x) = g(x)$, *where* $g \in G$, *then furthermore* $g(-1) > g(1) > g(0) = 0$, *and conversely.*

Proof. If g is any polynomial (2.11) in G, then $g \mid R$ assumes its minimum at $x = \frac{1}{2} - (b/a)$. Thus $g \mid R$ has its minimum in $(0, \frac{1}{2})$, by (2.12), and $g \mid Z$ assumes its minimum at 0, by

$$g(-1) = a - b > g(1) = b > g(0) = 0. \tag{2.13}$$

However $g \mid Z$ takes integer values, and so $g(Z) \subset N$. If $g \mid Z$ takes equal values at any distinct integers p, q, then $g \mid R$ would assume its minimum at the midpoint $(p + q)/2$. But $(p + q)/2$ cannot be a point of $(0, \frac{1}{2})$, whence g must be a storing function on Z, and $g(-x)$, $g(x + x_0)$, $g(x) + c_0$ are also storing functions on Z. Hence our three direct assertions are proved, and only their converses need be established. However, if the quadratic polynomial f defines an injective map from Z into N, then $f \mid Z$ assumes its minimum c_0 at a unique x_0 in Z, and if

$$f(x + x_0) - c_0 = g(x) = (a/2)\, x(x - 1) + bx + c \tag{2.14}$$

for some constants a, b, c, then g has the properties of the original f. Moreover $c = g(0) = 0$, and $b = \Delta g(0)$ is an integer, and $a = \Delta^2 f(0)$ is a positive integer. Finally, if $g(0) < g(1) < g(-1)$ then we obtain (2.13) on substitution, and $g(x) \in G$; while if $g(0) < g(-1) < g(1)$ then we replace x by $-x$, and $g(-x) \in G$. □

3. Auxiliary Results in Two Dimensions

We review several affine concepts for the plane R^2, then discuss polynomial storing functions of two variables, and obtain some preliminary facts in this setting. This paper characterizes only quadratic polynomials, but these results anticipate also future applications. Specifically, we introduce *rays* emanating from the origin, and we consider *sectors* lying between such rays. Integer unimodular matrices, as affine mappings, transform these geometric entities in obvious ways, and preserve the set Z^2 of plane lattice points. We restrict polynomials to single rays and observe their properties on these simpler domains. A polynomial storing function on a nontrivial sector, immediately via our analysis, cannot be a linear function on any included rational ray or, therefore, on R^2.

To adapt preceding definitions and study two-dimensional problems, we put $X = R^2$, $X_0 = Z^2$ and suppose $f\colon X \to R$. If U denotes any real 2×2 matrix $\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$ then U defines a mapping

$$(x, y) \to (x, y)\, U = (px + ry, qx + sy), \tag{3.1}$$

which carries R^2 into R^2, hence takes any subset S into

$$SU = \{(x, y)\, U\colon (x, y) \in S\}. \tag{3.2}$$

We let Uf be the resultant $f \circ U$; that is,

$$[Uf](x, y) = f((x, y)\, U). \tag{3.3}$$

If U, in particular, is an invertible matrix, then

$$Uf((x, y)\, U^{-1}) = f(x, y) \qquad \text{on} \quad R^2. \tag{3.4}$$

Any real pair (a, b) with $a^2 + b^2 \neq 0$ generates a *ray* $\omega(a, b)$ via

$$\omega(a, b) = \{(ta, tb): t > 0\}, \tag{3.5}$$

and any point in this set provides another generator for the same ray. Such a ray will be called, respectively, *rational* or *algebraic* if the coordinates of some generator are rational or algebraic. Indeed a rational ray, by this definition, contains a point with integer coordinates. If a ray is, respectively, not rational or not algebraic then it will be called *irrational* or *transcendental*. Clearly a ray, in more intuitive language, is a half line from the origin, and our subdivision of all rays is a classification by their slopes. Any two rays ω_1 and ω_2 in R^2 determine the closed *sector* $S(\omega_1, \omega_2)$ between them: this consists of the origin together with all rays which lie counterclockwise between ω_1 and ω_2. Hence $S(\omega, \omega) = \omega \cup \{(0, 0)\}$ but otherwise $S(\omega_1, \omega_2) \neq S(\omega_2, \omega_1)$. If ω_1 and ω_2 are rational rays then $S(\omega_1, \omega_2)$ will be called a *rational* sector; also the plane R^2, by convention, will be considered a rational sector.

The next two lemmas about integer unimodular matrices explore their relationship with our other concepts.

LEMMA 3.1. *If* $U = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$ *such that* $ps - qr = \det(U) = \pm 1$, *then*

$$U^{-1} = \det(U)^{-1} \begin{pmatrix} s & -q \\ -r & p \end{pmatrix} \tag{3.6}$$

and U induces a bijection from R^2 onto R^2. If U is an integer unimodular matrix, so that also p, q, r, s are integers, then U^{-1} is an integer unimodular matrix and U produces a bijection from Z^2 onto Z^2. If U is a proper integer unimodular matrix, so that also $\det(U) = 1$, *then the mapping*

$$(u, v) \to (x, y) = (u, v)\, U = u(p, q) + v(r, s) \tag{3.7}$$

sends the first quadrant onto the sector $S(\omega(p, q), \omega(r, s))$ and sends N^2 onto the lattice points of this sector; while the mapping $(x, y) \to (u, v) = (x, y)\, U^{-1}$ sends this sector onto the first quadrant, and the sector lattice-points onto all lattice points of this quadrant.

Remark. These results are obtained easily by direct calculation, and are collected here for convenient reference.

LEMMA 3.2. *If S is an f-amenable subset of R^2 and U is an integer unimodular matrix, then*

$$S \div f = SU^{-1} \div Uf. \tag{3.8}$$

If S is an arbitrary subset and $\div$ becomes either $\overline{\div}$ or $\underline{\div}$, then the corresponding relation is also true.

Proof. The set $X_0 \cap S \cap f^{-1}([-n, +n])$ in definition (2.3) has the same cardinality, by Lemma 3.1, as the set

$$X_0 \cap SU^{-1} \cap [Uf]^{-1}([-n, +n]) = \{X_0 \cap S \cap f^{-1}([-n, +n])\}\, U^{-1}. \quad \square \tag{3.9}$$

Let $f(x, y)$ be a real polynomial in the real variables x, y; that is, let

$$f(x, y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} x^i y^j \tag{3.10}$$

with real coefficients a_{ij}, and assume $a_{ij} = 0$ except for a finite set of pairs (i, j). Also let $d(f)$ be the total degree of f, and specifically put $d(f) = -\infty$ for identically zero f. Let f_k be the homogeneous part of f with degree k, so that

$$f(x, y) = \sum_{k=0}^{d(f)} f_k(x, y) \tag{3.11}$$

in this notation. If ω is any ray in R^2, and (a, b) is any generator of ω, then

$$f(ta, tb) = \sum_{k=0}^{d(f)} t^k f_k(a, b) \tag{3.12}$$

is a polynomial in t. If (c, d) is any other generator of ω, then $(c, d) = (wa, wb)$ for some positive w; thus $f_k(a, b)$ has the same sign for all generators (a, b), and $f_k(\omega) > 0$ (or $= 0$, or < 0) is a meaningful statement for any such ray. Moreover (3.12) has the same degree $d(\omega, f)$ for all generators (a, b), and

$$d(\omega, f) \leqslant d(f) \tag{3.13}$$

for arbitrary rays, while $d(\omega, f) = d(f)$ precisely when $f_{d(f)}(\omega) \neq 0$. In addition, if U is any real invertible 2×2 matrix, then, by (3.4),

$$d(\omega U^{-1}, Uf) = d(\omega, f). \tag{3.14}$$

We now use these concepts to prove some auxiliary results which further our attempts to describe polynomial packing functions. Our arguments involve the partial difference operators

$$\Delta_x f(x, y) = f(x+1, y) - f(x, y), \qquad \Delta_y f(x, y) = f(x, y+1) - f(x, y). \tag{3.15}$$

Proposition 3.4 is our main conclusion; Corollary 3.5 is its first application.

LEMMA 3.3. *Let $f(x, y)$ be a real polynomial of form* (3.10), *and let f take integer values on some set $G(p, q) = \{(x, y) \in Z^2 : p \leqslant x \leqslant p + d(f);\ q \leqslant y \leqslant q + d(f)\}$ with p, q integers.* (1) *Then the coefficients a_{ij} are all rational numbers.* (2) *The zero set for any form f_k with* some *nonvanishing coefficients is a finite union of algebraic rays together with the origin.*

Proof. (1) The differences $\{\Delta_x{}^i \Delta_y{}^j f(p, q) : i, j \leqslant d(f)\}$ take integer values by our assumptions, and are integer linear combinations of the a_{ij} which form a triangular system for these coefficients. (2) Any nonzero f_k, by part (1), is a polynomial with rational coefficients. □

PROPOSITION 3.4. *Let $S(\omega_1, \omega_2)$ be a plane sector with nonvoid interior, and let $f(x, y)$ be a polynomial storing function on this sector.* (1) *Then the coefficients of $f(x, y)$ are rational numbers not all equal to zero.* (2) *Moreover $f_{d(f)}(x, y) \geqslant 0$ at all points (x, y) in this sector; and $d(\omega, f) \geqslant 2, f_{d(\omega, f)}(\omega) > 0$ on all rational rays ω in this sector.*

Proof. (1) Any such sector includes a grid $G(p, q)$, and any storing function takes some nonzero values. Hence $f_{d(f)}(x, y)$ has some nonvanishing coefficients. (2) Any rational ray ω in the sector has a generator (p, q) with p, q relatively prime integers; and the values $f(mp, mq)$, by hypothesis, are distinct nonnegative integers for $m = 0, 1, 2, \ldots$. Thus $d(\omega, f) \geqslant 1$ by (3.12) and $f_{d(\omega, f)}(\omega) > 0$ by definition. Those rational rays in the sector which contain no zeros of $f_{d(f)}(x, y)$ must form a dense subset, by Lemma 3.3.2. Moreover on such rational rays, by our previous result,

$$f_{d(f)}(x, y) = f_{d(\omega, f)}(x, y) > 0, \tag{3.16}$$

whence for any sector point, by continuity, $f_{d(f)}(x, y) \geqslant 0$. These remarks prove the first and third inequalities; a preliminary normalization simplifies the remaining arguments.

We can find integers (r_0, s_0) for which $ps_0 - qr_0 = 1$, then choose integers (r, s) of form $(r_0 + mp, s_0 + mq)$ so that $S(\omega(p, q), \omega(r, s))$ has an arbitrarily narrow aperture while still $ps - qr = 1$. Interchanging x and y does not affect the desired result; whence we can assume $\omega \neq \omega_2$ and we can require $\omega(r, s) \subset S(\omega_1, \omega_2)$. Moreover $U = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$, by (3.14), yields an admissible transformation of the result, and the corresponding U^{-1}, by Lemma 3.1, maps $S(\omega(p, q), \omega(r, s))$ onto the first quadrant. Hence we may specialize $S(\omega_1, \omega_2)$ to the first quadrant, and also take ω as the positive x axis. Now the desired inequality asserts $\deg[f(x, 0)] \geqslant 2$ while the preceding paragraph implies $\deg[f(x, 0)] \geqslant 1$. Hence we may assume $f(x, 0) = a + bx$ with nonzero b, and we need only obtain a contradiction in this special case.

Clearly $f(x, y)$ has form (3.10) with rational a_{ij}, and $0 = a_{20} = a_{30} = \cdots$ by our assumption. However, $a = a_{00} = f(0, 0) \in N$ and $b = a_{10} = \Delta_x f(0, 0) \in Z$; moreover $f(N, 0) \geqslant 0$, whence $b > 0$. If we let k be the least common denominator of the nonzero a_{ij}, and we define

$$x_0(x, t) = x + kt \cdot \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} a_{ij} x^i (bkt)^{j-1}, \tag{3.17}$$

then we have $x_0(x, t)$ an integer for any x, t in N, and we note

$$\begin{aligned} f(x, bkt) &= a + bx + bkt \cdot \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} a_{ij} x^i (bkt)^{j-1} \\ &= a + bx_0(x, t) = f(x_0(x, t), 0). \end{aligned} \tag{3.18}$$

But $f \mid N^2$ is an injective mapping, whence $\{(x, y) \in N^2 : f(x, y) \leqslant a\}$ is a finite set $\{(x_i, y_i) : i = 1, \ldots, n\}$. Thus if the integer $bkt > y_0 = \max(y_1, \ldots, y_n)$ then $f(x, bkt) > a$, whence if the integer $t > y_0/bk$ then $x_0(x, t) > 0$. Hence (3.18) with any nonnegative integer x contradicts the injectiveness of $f \mid N^2$. $\square$

COROLLARY 3.5. *Let $f(x, y)$ be a polynomial storing function on a plane set S, where S includes a closed sector with nonvoid interior. Then $d(f) \geqslant 2$, so that f cannot be linear.*

Proof. By hypothesis, S includes a rational ray ω, and by (3.13), $d(f) \geqslant d(\omega, f) \geqslant 2$. $\square$

4. QUADRATIC FUNCTIONS ON THE FIRST QUADRANT

Fueter and Pólya [12] have established that the Cantor polynomials of (1.3) are the only quadratic packing functions on N^2. This section further shows that positive translations of (1.3) are the only such quadratic storing functions with unit density. We write an arbitrary quadratic polynomial in the computationally more tractable form

$$f(x, y) = (b_{20}/2)\, x(x - 1) + b_{11} xy + (b_{02}/2)\, y(y - 1) + b_{10} x + b_{01} y + b_{00} \tag{4.1}$$

and determine systematically the most general b_{ij} under our broader assumptions, using a well-known estimate from geometric number theory to replace rigorously our lattice-point enumeration by Pólya's area. This area, or equivalently $N^2 \div f$, is a transcendental number unless f, as in the earlier argument, has parabolic type; and the injectiveness of f yields the detailed form of the b_{ij}. We have just noted a result, in Corollary 3.5, which eliminates polynomials of first degree, but have proved so far no analog of Proposition 2.3, which might exclude polynomials of higher degree. Our immediate sequel (Lew and Rosenberg [16]) obtains a partial such conclusion. The following lemma avoids several Fueter–Pólya cases.

LEMMA 4.1. (1) *If in* (4.1) *all* b_{ij} *are integers then* f *maps* Z^2 *into* Z; *if in addition*

$$b_{20} > 0, \qquad b_{02} > 0, \qquad b_{11} > -(b_{20}b_{02})^{1/2}, \tag{4.2}$$

then f *has a lower bound on* N^2. (2) *Conversely if* f *is a storing function on* N^2, *then the* b_{ij} *are integers satisfying* (4.2).

Proof. (1) Clearly (4.1) with integers b_{ij} takes integer values on Z^2, and (4.2), except at the origin, implies

$$0 < 2f_2(x, y) = [xb_{20}^{1/2} - yb_{02}^{1/2}]^2 + 2xy \cdot [b_{11} + (b_{20}b_{02})^{1/2}] \tag{4.3}$$

on the first quadrant. Thus $f(x, y) > 0$ for any (x, y) in the first quadrant with sufficiently large $x^2 + y^2$. (2) Conversely for $0 \leqslant i + j \leqslant 2$ we see that $b_{ij} = \Delta_x{}^i\Delta_y{}^j f(0, 0) \in Z$, and by Proposition 3.4.2 we note that $b_{20} = 2f_2(1, 0) > 0$, $b_{02} = 2f_2(0, 1) > 0$. If $\omega(p, q)$ is any ray strictly within the first quadrant, and $f_2(x, y) = 0$ on this ray, then $f_2(x, y) = \text{const} \cdot (py - qx)^2$ by Proposition 3.4.2, and q/p is an irrational number as well. However the b_{ij} are rational, so that this is impossible. Hence (4.3) holds for all nontrivial (x, y) in the first quadrant, although the first bracket vanishes for some (x_0, y_0) in this quadrant. Substituting the latter values and canceling the positive quantity $2x_0y_0$ we obtain $b_{11} + (b_{20}b_{02})^{1/2} > 0$. $\square$

Any quadratic storing function (4.1) on N^2 has a well-defined *shape number*

$$\gamma = b_{11}(b_{20}b_{02})^{-1/2} > -1 \tag{4.4}$$

by Lemma 4.1. By standard results from analytic geometry, the level curves $f(x, y) = c$ are ellipses, parabolas, or hyperbolas according as $|\gamma|$ is, respectively, <1, $=1$, or >1. These three cases will require slightly different arguments.

LEMMA 4.2. *Let the quadratic polynomial* (4.1) *satisfy relations* (4.2), *and let the region*

$$D_0(f, n) = \{(x, y): 0 \leqslant x, 0 \leqslant y, f_2(x, y) \leqslant n\} \tag{4.5}$$

have area $A_0(f, n)$. Then $A_0(f, n)/n = \rho(b_{11}, \gamma)$, where $\rho(\cdot, \cdot)$ is independent of n. If $\gamma = 1$, in particular, then

$$\rho(b_{11}, \gamma) = b_{11}^{-1} = (b_{20}b_{02})^{-1/2}. \tag{4.6}$$

If $|\gamma| < 1$, so that $\gamma = \cos \alpha$ with $0 < \alpha < \pi$, then

$$\rho(b_{11}, \gamma) = b_{11}^{-1}\alpha \operatorname{ctn} \alpha = (b_{20}b_{02})^{-1/2}\, \alpha \csc \alpha. \tag{4.7}$$

If $\gamma > 1$, so that $\gamma = \cosh \beta$ with $0 < \beta < +\infty$, then

$$\rho(b_{11}, \gamma) = b_{11}^{-1}\beta \operatorname{ctnh} \beta = (b_{20}b_{02})^{-1/2}\, \beta \operatorname{csch} \beta. \tag{4.8}$$

Proof. If $\gamma = 1$ then $D_0(f, n)$ is the right triangle $\{(x, y): 0 \leqslant x,\ 0 \leqslant y,\ xb_{20}^{1/2} + yb_{02}^{1/2} \leqslant (2n)^{1/2}\}$ and (4.6) is an immediate consequence. If $\gamma \neq 1$ and r, θ denote polar coordinates then

$$A_0(f, n)/n = \int_0^{\pi/2} d\theta/2f_2(\cos \theta, \sin \theta). \tag{4.9}$$

If $t = (b_{02}/b_{20})^{1/2} \cdot \tan \theta$ and $Q(t) = 1 + 2\gamma t + t^2$ then

$$A_0(f, n)/n = (\gamma/b_{11}) \int_0^{\infty} dt/Q(t). \tag{4.10}$$

We might evaluate the last integral by elementary methods, but instead use a device from complex variable theory (Carrier *et al.* [4, pp. 79–80]). We recall that $\log t$ is a multivalued function for complex t, and assert that

$$(b_{11}/\gamma)\, \rho(b_{11}, \gamma) = -(2\pi i)^{-1} \int_C \log t\, dt/Q(t) \tag{4.11}$$

for a suitable contour C. For small enough positive ϵ and large enough positive λ, this C runs just above the real axis from ϵ to λ, circles once about the origin at radius λ in the counterclockwise direction, runs just below the real axis from λ to ϵ, circles once about the origin at radius ϵ in the clockwise direction. However if $\gamma \neq \pm 1$ then $Q(t)$ has distinct roots t_1 and t_2; indeed if $\gamma = \cos \alpha$ then the roots are $-\exp(\pm i\alpha)$, while if $\gamma = \cosh \beta$ then the roots are $-\exp(\pm \beta)$. We take a branch of $\log t$ which has a cut along the positive real axis, and we evaluate (4.11) by residues to obtain

$$(b_{11}/\gamma)\, \rho(b_{11}, \gamma) = -(\log t_1 - \log t_2)/(t_1 - t_2). \tag{4.12}$$

The appropriate values of the logarithms yield (4.7) and (4.8) in these two cases. □

PROPOSITION 4.3. *If f is a quadratic polynomial* (4.1) *with the properties* (4.2), *hence if f is a quadratic storing function on N^2, then $N^2 \div f = \rho(b_{11}, \gamma)$ for the defined $\rho(\cdot, \cdot)$.*

Proof. First, if $\gamma = 1$ then we can introduce $u = xb_{20}^{1/2} + yb_{02}^{1/2}$, so that $f(x, y) = (u^2/2) + (\text{linear terms}) + b_{00}$; and if $x, y \geqslant 0$ then we can find real numbers $b_\pm$, n_0, so that

$$(u + b_-)^2 \leqslant 2f(x, y) + 2n_0 \leqslant (u + b_+)^2. \tag{4.13}$$

Thus we derive the inclusion $T_+ \subset \{\text{first quadrant}\} \cap f^{-1}([-n, +n]) \subset T_-$, where we define the triangles

$$T_\pm = \{(x, y) \colon 0 \leqslant x, 0 \leqslant y, u \leqslant -b_\pm + (2n + 2n_0)^{1/2}\}. \tag{4.14}$$

The corresponding subsets of lattice points form an ordered triple in the same way. However the area of $T_\pm$ approximates the contained number of lattice points, and a bound for the error is $1 + |P_x(T_\pm)| + |P_y(T_\pm)|$ by Davenport's lemma [7]. Here $P_x(\cdot)$ denotes the projection onto the x axis, and $|P_x(\cdot)|$ denotes the length of this projection. Hence the cardinality $\#\{N^2 \cap f^{-1}([-n, +n])\}$ of (2.3) has upper and lower bounds which are both $(n/b_{11}) + O(n^{1/2})$ for large n.

For $\gamma \neq 1$ the symbol $A(f, n)$ will denote the area of $\{\text{first quadrant}\} \cap f^{-1}([-n, +n])$ and the coefficients b_{ij} will determine real numbers x_0, y_0, c with

$$2f(x, y) = b_{20}(x - x_0)^2 + 2b_{11}(x - x_0)(y - y_0) + b_{02}(y - y_0)^2 + 2c. \tag{4.15}$$

If $|\gamma| < 1$, indeed, then $f(x, y) = n$ is an ellipse with major axis $O(n^{1/2})$, and

$$\#\{N^2 \cap f^{-1}([-n, +n])\} = A(f, n) + O(n^{1/2}) \quad \text{as} \quad n \to +\infty \tag{4.16}$$

by Davenport's lemma [7]. Thus displacement of the origin yields

$$A(f, n) = A_0(f, n - c) + (|x_0| + |y_0|)\, O(n^{1/2}) = A_0(f, n) + O(n^{1/2}) \tag{4.17}$$

for large n. If $\gamma > 1$ instead, then $f(x, y) = n$ is a hyperbola, while

$$(y - y_0)/(x - x_0) = (b_{11}/b_{02}) \cdot [-1 \pm (1 - \gamma^{-2})^{1/2}] \tag{4.18}$$

are its asymptotes, and their slopes are both strictly negative. By (4.15) and simple algebra, the intercepts of this hyperbola with $x = x_0$ and $y = y_0$ are both $O(n^{1/2})$. By (4.18) and simple geometry, the projections of $\{\text{first quadrant}\} \cap f^{-1}([-n, +n])$ on the x and y axes are both $O(n^{1/2} + |x_0| + |y_0|)$. Thus (4.16) and (4.17) are valid as before. $\square$

COROLLARY 4.4. (1) *If* $\gamma = 0$ *then* $N^2 \div f = \pi/2(b_{20}b_{02})^{1/2}$. (2) *If* $\gamma \neq 1$ *then* $N^2 \div f$ *is a transcendental number.* (3) *If* $0 < \gamma < 1$ *then* $(b_{02}b_{02})^{-1/2} < N^2 \div f < b_{11}^{-1} \leqslant 1$. *If* $1 < \gamma < +\infty$ *then* $b_{11}^{-1} < N^2 \div f < (b_{20}b_{02})^{-1/2} \leqslant 1$.

Proof. To obtain (1) we use (4.7). To obtain (3) we note

$$\begin{aligned} \alpha^{-1}\sin\alpha < 1 < \alpha^{-1}\tan\alpha \qquad &\text{for} \quad 0 < \alpha < \pi/2, \\ \beta^{-1}\tanh\beta < 1 < \beta^{-1}\sinh\beta \qquad &\text{for} \quad 0 < \beta < +\infty. \end{aligned} \tag{4.19}$$

To prove (2) we let $\sigma = N^2 \div f$, and for $|\gamma| < 1$ we define $\beta = i\alpha$. If σ is algebraic then $0 \neq \beta = (b_{20}b_{02}\sigma)^{1/2}(\gamma^2 - 1)^{1/2}$ is algebraic, and in any case, $\cosh\beta = \gamma = b_{11}(b_{20}b_{02})^{-1/2}$ is algebraic. Thus $\exp(\beta)$, $\exp(-\beta)$, and unity are linearly dependent over the algebraic numbers, which is impossible by a theorem of Lindemann ([17]; Siegel [26, p. 23]). □

THEOREM 4.5. (1) *The most general quadratic storing function with unit density on* N^2 *is*

$$f(x, y) = \tfrac{1}{2}(x + y)(x + y + 1) + \{\textit{either } x \textit{ or } y\} + a(x + y) + b \tag{4.20}$$

with a, b *in* N. (2) *The most general quadratic packing function on* N^2 *is* (4.20) *with* $a = b = 0$.

Proof. The polynomials f of (4.20) map the diagonals $\{(x, y) \in N^2\colon x + y = n\}$ for each n into sequences of consecutive integers. The gap between 0 and $f(0, 0)$ is b integers; the gap between any two neighboring sequences is a integers. Hence f is one-to-one for all admissible a, b, and onto N for $a = b = 0$. Moreover in mapping the $(n + 1)(n + 2)/2$ pairs with $x + y \leqslant n$, we skip only $b + na$ elements of N, and we have $N^2 \div f = 1$ for all such f. Thus the proof of both assertions requires only the converse half of (1).

Conversely if $N^2 \div f = 1$ by hypothesis then $\gamma = 1$ by Corollary 4.4.2, so that $b_{11}^{-1} = (b_{20}b_{02})^{-1/2} = 1$ by Proposition 4.3, and $b_{20} = b_{11} = b_{02}$ by Lemma 4.1.2. We now have

$$f(x, y) = \tfrac{1}{2}(x + y)(x + y + 1) + a(x + y) + b + cx \tag{4.21}$$

for some integers a, b, c. We also find $b = f(0, 0) \geqslant 0$ by hypothesis, and we may assume $c \geqslant 0$ by xy symmetry. However, if $c = 0$ then f is constant on diagonals, while if $c \geqslant 2$ then

$$f(m, (c - 1)\, m - a - 1) = f(0, cm - a) \tag{4.22}$$

whence if m is large enough then f is not one-to-one. Thus $c = 1$. Moreover if $a < 0$ then

$$f(n + a, -a) = f(0, n + 1). \tag{4.23}$$

whence if $n \geqslant -a$ then f is not one-to-one. Thus $a \geqslant 0$. □

5. Further Results on Plane Sectors

To display further the range of our concepts, we apply them to some other natural questions. We find no quadratic storing functions with unit density on $Z \times N$, and exclude all polynomial storing functions below fourth degree on Z^2. The main results of our sequel (Lew and Rosenberg [16]), extending our analogy with Proposition 2.3, eliminate many higher-degree polynomials on these domains. However if $f(x, y)$ satisfies (4.1) and if $\gamma = 1$, then

$$N^2 \div f = (b_{20}b_{02})^{-1/2} = b_{11}^{-1}, \tag{5.1}$$

whence $(N^2 \div f)^{-1}$ is a positive integer. Thus, having just determined all quadratic storing functions on N^2 with unit density, we first describe here all such functions on N^2 with density $\frac{1}{2}$. All such polynomials with density 1 or $\frac{1}{2}$ are strictly increasing functions on this domain, whereas such polynomials with density $\frac{1}{3}$ may be strictly increasing functions only outside a bounded subset. Our characterization, as before, requires no prior specification of γ.

Theorem 5.1. *The most general quadratic storing function f on N^2, with $N^2 \div f = \frac{1}{2}$, has one of the following two forms, up to xy symmetry*:

$$f(x, y) = \tfrac{1}{2}(x + 2y)(x + 2y + 1 + 2a) + by + c \qquad \text{with} \quad a, c \in N, \tag{5.2}$$

where either $b = 2$, or $b = 1$, or $b = -1$, or a is positive and $b = -2$;

$$f(x, y) = (x + y)(x + y + a) + bx + c \qquad \text{with} \quad a, c \in N, \tag{5.3}$$

where either $b = 1$, or $b = 2$, or a is even and $b = 4$.

Proof. If f satisfies (4.1) and $N^2 \div f = \frac{1}{2}$, then $\gamma = 1$ by Corollary 4.4.2 and

$$(b_{20}, b_{11}, b_{02}) = \text{either } (1, 2, 4), \text{ or } (2, 2, 2), \text{ or } (4, 2, 1) \tag{5.4}$$

by (5.1). We ignore the last case by xy symmetry, examine the other two alternatives with all b_{ij} integers, and derive the expressions (5.2), (5.3) with undetermined integer coefficients. However, if f has form (5.2) then $c = f(0, 0) = f(-1 - 2a, 0)$; whence if f is a storing function then $a, c \geqslant 0 \neq b$. Moreover if $|b| \geqslant 3$ and $u = a + 1$, $v = (a + 1)(|b| - 1)$, then $0 < u \leqslant v/2$ and

$$\begin{aligned} f(v + 1, 0) &= f(v - 2u, u) && \text{for} \quad b > 0, \\ f(v, 0) &= f(v + 1 - 2u, u) && \text{for} \quad b < 0. \end{aligned} \tag{5.5}$$

Hence f cannot be a storing function unless $|b| \leqslant 2$. Alternatively if f has form (5.3) then $c = f(0, 0) = f(0, -a)$, whence $a, c \geqslant 0 \neq b$ as before and $b > 0$ by xy symmetry. However, if $b \geqslant 3$ and $u = a + 1$, $v = (a + 1)(b - 1)/2$, then $0 < u \leqslant v$ and $f(u, v - u) = f(0, v + 1)$, so that $f | N^2$ cannot be an injective map unless a and b are even. Further, if $b > 4$ and $u = 2(a + 2)$, $v = (a + 2)(b - 1)/2$, then $0 < u \leqslant v$ and $f(u, v - u) = f(0, v + 2)$, so that $f | N^2$ cannot be an injective map unless a is even, $b = 4$. Hence a storing function of form (5.3) must have $b \leqslant 2$ except in this last case.

Conversely if f has either specific form then $f | N^2$ is integer-valued and $N^2 \div f = \frac{1}{2}$. Moreover $f(0, 0) = c \geqslant 0$ and both partial differences are always positive, whence $f(N^2) \subset N$. Thus f need only be shown injective. However, if f satisfies (5.2) then $f(v - 2y, y) = \frac{1}{4}(v^2 + v + 2av) + by + c$, which is monotone, for fixed v, as y varies in $[0, v/2]$. Moreover if $0 < b \leqslant 2$ then

$$2f(v + 1, 0) - 2f(0, v/2) = 2 + 2a + (2 - b)\, v > 0, \tag{5.6}$$

while if $-2 \leqslant b < 0$ then

$$2f(0, v/2) - 2f(v - 1, 0) = 2a + (2 + b)\, v \geqslant 0. \tag{5.7}$$

Thus f, on successive diagonals $\{(x, y): x + 2y = m\}$, takes disjoint sets of integer values except when $a = 0$ and $b = -2$, in which case (5.7) becomes an equality. Alternatively if f satisfies (5.3) then $f(x, v - x) = v^2 + av + bx + c$, which is increasing, for fixed v, as x increases in $[0, v]$. Moreover if $0 < b \leqslant 2$ then

$$f(0, v + 1) - f(v, 0) = 1 + a + (2 - b)\, v > 0; \tag{5.8}$$

whence f, on successive diagonals $\{(x, y): x + y = m\}$, takes disjoint sets of integer values. Further, if a is even and $b = 4$ then

$$f(x, v - x) \equiv v^2 + c \equiv v + c (\mathrm{mod}\ 2); \tag{5.9}$$

wherefore successive diagonals must have disjoint values; and

$$f(0, v + 2) - f(v, 0) = 4 + 2a > 0, \tag{5.10}$$

so that *all* diagonals must exhibit disjoint values. □

Let S be an arbitrary subset of R^2 which is a closed system under componentwise addition, and let f be a real-valued function on S. Then f is *strictly increasing* on S if

$$f(x, y) < f(x + a, y + b) \qquad \text{whenever } (a, b), (x, y) \in S \tag{5.11}$$

with $a^2 + b^2 \neq 0$; while f is *eventually increasing* on S if it satisfies (5.11) for all (x, y) outside a bounded subset of S. The positivity of all partial differences shows the polynomials of Theorems 4.5 and 5.1 to be strictly increasing functions on N^2. However, a polynomial storing function on N^2 need not be a strictly increasing function, by the next result.

COROLLARY 5.2. (1) *If f is a quadratic storing function on N^2, and $\gamma = 1$, $N^2 \div f > \frac{1}{3}$, then f is a strictly increasing function on N^2.* (2) *However, $N^2 \div f$ can here take no smaller value, since*

$$f(x, y) = \tfrac{1}{2}(x + y)(3x + 3y - 5) + 3y + 1 \tag{5.12}$$

is also a storing function on N^2, and $\gamma = 1$, $N^2 \div f = \frac{1}{3}$, while f is eventually increasing but not strictly increasing:

Proof. (1) All such polynomials are those of Theorems 4.5 and 5.1, since $N^2 \div f$ is 1 or $\frac{1}{2}$ by (5.1). (2) Clearly $N^2 \div f = \frac{1}{3}$, and b_{20}, b_{11}, $b_{02} > 0$; whence $\Delta_x f$, $\Delta_y f > 0$ for sufficiently large $x + y$, and f is eventually increasing on N^2. Moreover $f(x, y)$ is a monotone function on each diagonal $\{(x, y): x + y = m\}$, and

$$f(1, m - 1) < f(m + 1, 0) < f(0, m) < f(m, 1) \tag{5.13}$$

by direct calculation. Thus values of f on successive diagonals will overlap, but no values of f on N^2 can coincide. Substitution and (5.13) now yield $0 = f(1, 0) = \min f(N^2)$, which shows f not strictly increasing but nevertheless a storing function. □

The integer unimodular images of the first quadrant are all rational proper subsectors of half planes, and the related matrix transformations of the polynomials (4.20) give the corresponding quadratic storing functions with unit density. However, this work now examines larger sectors, and the sequel (Lew and Rosenberg [16]) considerably strengthens our remarks, but the conclusions do not fully parallel Section 2. Indeed Proposition 2.3 implies the nonexistence of polynomial packing functions on Z, whereas the next theorem states partial results for polynomial storing functions on Z^2. The associated maps into memory cells would define extendible storage schemes for rectangular arrays which can add new elements on all four sides.

THEOREM 5.3. (1) *There exist no odd-degree polynomial storing functions on Z^2.* (2) *There exist no quadratic storing functions on Z^2 (hence no such packing functions as well).* (3) *There exists a quartic storing function on Z^2.*

Proof. (1) If f has leading terms of odd degree, then f assumes negative values for some integer arguments. (2) If $f(Z^2)$ is a subset of Z, where

$$f(x, y) = \sum_{i=0}^{2} \sum_{j=0}^{2-i} a_{ij} x^i y^j \tag{5.14}$$

for some real coefficients a_{ij}, then all $2a_{ij}$ are integers by Lemma 4.1.2. We exhibit two distinct points of Z^2 which yield the same value of f. Indeed if $a_{10} = a_{01} = 0$ then $f(x, y) = f(-x, -y)$ for all integers x, y; while if a_{10} and a_{01} are not both zero then

$$f(2a_{01}t, -2a_{10}t) = f(-2a_{01}t, 2a_{10}t) \tag{5.15}$$

for all integers t. (3) Clearly $f(x, y) = g(h(x), h(y))$ is a quartic storing function on Z^2, where g is any polynomial of Theorem 4.5, and h is any polynomial of Theorem 2.5. □

If we require surjective mappings rather than injective ones, then we generate an open problem from Theorem 5.3. A famous result of Fermat and Legendre (Dickson [10, Vol. II, Preface]) states that any nonnegative integer is the sum of three triangular numbers, hence implies that

$$f(x, y, z) = [x(x-1) + y(y-1) + z(z-1)]/2 \tag{5.16}$$

maps Z^3 onto N. However, no polynomial can map Z onto N, since a linear polynomial must take negative values at some integer points, while a nonlinear polynomial must have zero density by Proposition 2.3. These facts motivate the question, for the intermediate case, whether a polynomial can map Z^2 precisely onto N. A related problem of Hammer [13], presently also without solution, seeks a polynomial bijection from Z^2 onto Z instead.

Proposition 4.3 for the first quadrant also yields a corollary for the upper half plane which delimits storage schemes of hybrid type. Again $X = R^2$ and $X_0 = Z^2$. If g and h are the polynomials of Theorem 5.3.3, then $f(x, y) = g(h(x), y)$ is a cubic storing function on $Z \times N$. Again the stronger results of our sequel exclude further storing functions with unit density.

COROLLARY 5.4. *Let f be a quadratic polynomial of form* (4.1) *which is a storing function on $Z \times N$.* (1) *Then all b_{ij} are integers and* $|\gamma| < 1$, *whence*

$$(Z \times N) \div f = \pi/(b_{20}b_{02} - b_{11}^2)^{1/2}. \tag{5.17}$$

(2) *Thus no quadratic f can have unit density; so no such f can be a packing function.*

Proof. (1) If $f_\pm(x, y) = f(\pm x, y)$ for the given f, then both $f_\pm$ have form (4.1) with the same b_{20}, b_{02}, but these $f_\pm$ attach opposite signs to b_{11}. However, both $f_\pm$ are storing functions on N^2, whence all b_{ij} are integers by Lemma 4.1.2, and $b_{20}, b_{02} > 0$ from either case. If $\gamma_\pm$ denote the corresponding values for $f_\pm$, then $\gamma_+ = \gamma > -1$, $\gamma_- = -\gamma > -1$ by Lemma 4.1.2. If $\gamma_+ = \cos\alpha_+$ with $0 < \alpha_+ < \pi$ then $\gamma_- = \cos\alpha_-$ with

$\alpha_- = \pi - \alpha_+$. However, $f \mid (\{0\} \times N)$ has zero density by Proposition 2.3, wherefore

$$(Z \times N) \div f = (N^2 \div f_+) + (N^2 \div f_-) = (\alpha_+ + \alpha_-)/(b_{20}b_{02})^{1/2} \sin \alpha_\pm \tag{5.18}$$

by Lemma 2.1. Substitution now yields (5.17). (2) The resulting density is a transcendental number (Lindemann [17]). □

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References

1. J. Barlaz, Private communication, Dec. 1976.
2. G. Cantor, Ein Beitrag zur Mannigfaltigkeitslehre, *J. Reine Angew. Math.* **84** (1878), 242–258.
3. G. Cantor, Beiträge zur Begrundung der transfiniter Mengenlehre, *Math. Ann.* **46** (1895), 481–512; translated as "Contributions to the Founding of the Theory of Transfinite Numbers," Dover, New York, 1952.
4. G. F. Carrier, M. Krook, and C. E. Pearson, "Functions of a Complex Variable: Theory and Technique," McGraw–Hill, New York, 1966.
5. A. L. Cauchy, "Cours d'analyse de l'École Royale Polytechnique, 1re partie: Analyse algébrique," l'Imprimerie Royale, Paris, 1821; reprinted Wissenschaftliche Buchgesellschaft, Darmstadt, 1968.
6. P. Chowla, On some polynomials which represent every natural number exactly once, *Det Kongelige Norske Widenskabers Selskabs Fordhandlinger* **34** (1961), Nr. 2, 8–9.
7. H. Davenport, On a principle of Lipschitz, *J. London Math. Soc.* **26** (1951), 179–183.
8. M. Davis, "Computability and Unsolvability," McGraw–Hill, New York, 1958.
9. M. M. Day, Semigroups and amenability, *in* "Semigroups" (K. W. Folley, Ed.), 5–53, Academic Press, New York, 1969.
10. L. E. Dickson, "History of the Theory of Numbers," Chelsea, New York, 1952.
11. A. Erdélyi, "Asymptotic Expansions," Dover, New York, 1956.
12. R. Fueter and G. Pólya, Rationale Abzählung der Gitterpunkte, *Vierteljschr. Naturforsch. Ges. Zürich* **58** (1923), 380–386.
13. F. D. Hammer, Advanced problem 6028*, *Amer. Math. Monthly* **82** (1975), 410.
14. E. Landau, "Handbuch der Lehre von der Verteilung der Primzahlen," B. G. Teubner, Leipzig, 1909; Chelsea, New York, 1953.
15. J. S. Lew and A. L. Rosenberg, Asymptotic estimates in geometric number theory and extendible computer storage of planar arrays, *SIAM Rev.* **19** (1977), 388.

16. J. S. LEW AND A. L. ROSENBERG, Polynomial indexing of integer lattice-points II, *J. Number Theory* **10** (1978), 215–243.
17. F. LINDEMANN, Ueber die Zahl π, *Math. Ann.* **20** (1882), 213–225.
18. M. MINSKY, "Computation: Finite and Infinite Machines," Prentice-Hall, Englewood Cliffs, N.J., 1967.
19. I. NIVEN AND E. S. ZUCKERMAN, "An Introduction to the Theory of Numbers," 3rd ed., Wiley, New York, 1972.
20. G. PÓLYA AND G. SZEGÖ, "Aufgaben und Lehrsätze aus der Analysis," Springer–Verlag, New York, 1964; translated as "Problems and Theorems in Analysis," Springer–Verlag, New York, 1976.
21. H. ROGERS, JR., "Theory of Recursive Functions and Effective Computability," McGraw–Hill, New York, 1967.
22. A. L. ROSENBERG, Data graphs and addressing schemes, *J. Comput. System Sci.* **5** (1971), 193–238.
23. A. L. ROSENBERG, Allocating storage for extendible arrays, *J. Assoc. Comput. Mach.* **21** (1974), 652–670.
24. A. L. ROSENBERG, Managing storage for extendible arrays, *SIAM J. Comput.* **4** (1975), 287–306.
25. A. L. ROSENBERG, Storage mappings for extendible arrays, *in* "Current Trends in Programming Methodology IV: Data Structuring" (R. T. Yeh, Ed.), pp. 263–311, Prentice–Hall, Englewood Cliffs, N.J., 1978.
26. C. L. SIEGEL, "Transcendental Numbers," Princeton Univ. Press, Princeton, N. J., 1949.
27. L. J. STOCKMEYER, Extendible array realizations with additive traversal, IBM Research Report RC 4578, IBM T. J. Watson Research Center, Yorktown Heights, N. Y., October 15, 1973.
28. D. V. WIDDER, "The Laplace Transform," Princeton Univ. Press, Princeton, N. J., 1946.