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Iteration of the Number-Theoretic Function $f(2n) = n$, $f(2n+1) = 3n+2$

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It is a long-standing conjecture that, under iteration of the number-theoretic function,

$$f(2n) = n, f(2n+1) = 3n+2,$$

every integer m has an iterate $f^k(m) = 1$. Since virtually nothing is known about the question, save that it seems to be true for m up to the millions, it may be of interest to know that almost every m has an iterate $f^k(m) < m$, a result proved in the present report.

I. INTRODUCTION

Under iteration of the function

$$f(m) = \begin{cases} m/2; & m \text{ even}, \\ (3m+1)/2; & m \text{ odd}, \end{cases} \tag{1}$$

every integer $m \geqslant 0$ gives rise to an infinite sequence of integers

$$m \to [m_0, m_1, m_2, \ldots], \tag{2}$$

where $m_n = f^n(m)$. Thus for $m = 7$, one finds that

$$7 \to [7, 11, 17, 26, 13, 20, 10, 5, 8, 4, 2, 1, 2, 1, 2, \ldots]$$

with $f^{11}(7) = 1$. It has been conjectured that *every* $m \geqslant 1$ has an iterate $f^k(m) = 1$. It is shown here that, at any rate, almost every integer m has an iterate $f^k(m) < m$.

II. THE PARITY SEQUENCE

The sequence (2) may be used to assign to every integer $m \geqslant 0$ a "parity sequence"

$$m \to \{x_0, x_1, x_2, \ldots\}, \tag{3}$$

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where $x_n = 0$ if $m_n = f^n(m)$ is even, and $x_n = 1$ if it is odd. For example, one sees from above that

$$7 \to \{1, 1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, \ldots\}.$$

It is trivial that $m \to \{0, 0, 0, \ldots\}$ iff $m = 0$. Similarly $m \to \{1, 0, 1, 0, \ldots\}$ iff $m = 1$, and hence the parity sequence for any m terminates in $\{x_k, x_{k+1}, \ldots\} = \{1, 0, 1, 0, \ldots\}$ iff $m_k = f^k(m) = 1$. Thus the above conjecture asserts that *every* parity sequence not the zero sequence terminates in $1, 0, 1, 0, \ldots$. If so, the list of parity sequences for $m = 1, 2, \ldots$ in (3) would be rather remarkable, in view of the following property, which it does indeed have; namely, the 2^N parity sequences for the integers $m < 2^N$ have subsequences $\{x_0, \ldots, x_{N-1}\}$ ranging over the full set of 2^N 0, 1 vectors. Thus for $N = 2$, one sees that

$$\begin{aligned} 0 &\to [0, 0, \ldots] \to \{0, 0, \ldots\} \\ 1 &\to [1, 2, \ldots] \to \{1, 0, \ldots\} \\ 2 &\to [2, 1, \ldots] \to \{0, 1, \ldots\} \\ 3 &\to [3, 5, \ldots] \to \{1, 1, \ldots\}. \end{aligned}$$

Moreover, all integers $m = a + 2^N Q$, $Q = 0, 1, 2, \ldots$ have identical parity sequences through component x_{N-1}. This is the substance of Theorem 1.

THEOREM 1. *An arbitrary diadic sequence* $\{x_0, \ldots, x_{N-1}\}$ *arises via* (3) *from a unique integer* $m < 2^N$. *Specifically, the* x_n *determine* m *and* m_N *to be of the forms*

$$\begin{aligned} m &= a_{N-1} + 2^N Q_N; \qquad 0 \leqslant a_{N-1} < 2^N. \\ m_N &= b_{N-1} + 3^X Q_N; \qquad 0 \leqslant b_{N-1} < 3^X,\ X = \sum_0^{N-1} x_n. \end{aligned} \tag{4}$$

Hence the correspondence (3) *is one to one.*

Proof by induction on $N = 1, 2, \ldots$. For $N = 1$, $x_0 = 0$ implies $m = 2Q_1$, $m_1 = Q_1$, whereas $x_0 = 1$ implies $m = 1 + 2Q_1$, $m_1 = 2 + 3Q_1$. Assuming Eq. (4) for any $N \geqslant 1$, we must consider two cases, depending on the parity of the current b_{N-1}.

Case I. b_{N-1} even.

(a) $x_N = 0$ implies $Q_N = 2Q_{N+1}$, $m = a_{N-1} + 2^{N+1}Q_{N+1}$, $m_N = b_{N-1} + 3^X \cdot 2Q_{N+1}$, $m_{N+1} = (b_{N-1}/2) + 3^X Q_{N+1}$.

(b) $x_N = 1$ implies $Q_N = 1 + 2Q_{N+1}$, $m = a_{N-1} + 2^N + 2^{N+1}Q_{N+1}$, $m_N = (b_{N-1} + 3^X) + 3^X \cdot 2Q_{N+1}$, $m_{N+1} = (\frac{1}{2})(3b_{N-1} + 3^{X+1} + 1) + 3^{X+1}Q_{N+1}$ where $(\frac{1}{2})(3b_{N-1} + 3^{X+1} + 1) \leqslant (\frac{1}{2})[3(3^X - 1) + 3^{X+1} + 1] = 3^{X+1} - 1 < 3^{X+1}$.

Case II. b_{N-1} odd.

(a) $x_N = 0$ implies $Q_N = 1 + 2Q_{N+1}$, $m = a_{N-1} + 2^N + 2^{N+1}Q_{N+1}$, $m_N = (b_{N-1} + 3^X) + 3^X \cdot 2Q_{N+1}$, $m_{N+1} = (\frac{1}{2})(b_{N-1} + 3^X) + 3^X Q_{N+1}$, where $(\frac{1}{2})(b_{N-1} + 3^X) < (\frac{1}{2})(2 \cdot 3^X) = 3^X$.

(b) $x_N = 1$ implies $Q_N = 2Q_{N+1}$, $m = a_{N-1} + 2^{N+1}Q_{N+1}$, $m_N = b_{N-1} + 3^X \cdot 2Q_{N+1}$, $m_{N+1} = (\frac{1}{2})(3b_{N-1} + 1) + 3^{X+1}Q_{N+1}$, where $(\frac{1}{2})(3b_{N-1} + 1) < (\frac{1}{2})(3^{X+1} + 1) < 3^{X+1}$. Hence Eq. (4) is true at stage $N + 1$ in all cases.

COROLLARY. *The correspondence*

$$m \to [m_0 ,\ldots, m_{N-1}] \to \{x_0 ,\ldots, x_{N-1}\}$$

induces a one to one mapping of all positive integers $m \leqslant 2^N$ *on the set of all* 2^N *diadic vectors* $\{x_0 ,\ldots, x_{N-1}\}$.

III. A DENSITY THEOREM

Let $A(M)$ denote the number of positive integers $m \leqslant M$ having some iterate $f^k(m) < m$. Our object is to show that the density $A(M)/M$ approaches 1 as $M \to \infty$.

Consider first the case $M = 2^N$, for which we have the correspondence of the corollary

$$m \to [m_0 ,\ldots, m_{N-1}] \to \{x_0 ,\ldots, x_{N-1}\}.$$

Roughly, the idea is that most diadic sequences have nearly the same number of 0's and 1's if N is large. Moreover, $x_n = 0$ implies $m_{n+1}/m_n = \frac{1}{2}$, while $x_n = 1$ implies $m_{n+1}/m_n = (3Q + 2)/(2Q + 1) \leqslant 5/3$ if $m_n > 1$. Hence most integers $m \leqslant 2^N$ should have $m_N \simeq (\frac{1}{2})^{N/2}(\frac{5}{3})^{N/2}\, m_0 < m_0$. In fact, it follows from a well-known inequality of probability (Ref. 1) that the set H_N of all diadic sequences $\{x_0 ,\ldots, x_{N-1}\}$ such that

$$\frac{1}{2} - \epsilon < \frac{X}{N} < \frac{1}{2} + \epsilon, \tag{5}$$

where $X \equiv \sum_0^{N-1} x_n$, $\epsilon \equiv L - (\frac{1}{2})$, $L \equiv \log 2/\log (10/3)$ satisfies the relation

$$\# H_N/2^N \geqslant 1 - 1/4\, \epsilon^2 N. \tag{6}$$

Since condition (5) implies the inequality

$$X/N < L, \tag{7}$$

it is clear that the set D_N of diadic sequences satisfying Eq. (7) contains the set H_N , and hence

$$\# D_N \geqslant \# H_N . \tag{8}$$

Now except for the integer $m = 1$, the integers $m \leqslant 2^N$ whose parity sequences $\{x_0, ..., x_{N-1}\}$ satisfy (7) are of two kinds, those for which some $m_n \equiv f^n(m) = 1(< m)$, $n \leqslant N - 1$, and the rest, for which no such $m_n = 1$. But for the latter, the equivalence of (7) with

$$(1/2)^{N-X}(5/3)^X < 1$$

shows that

$$m_N = m_0(m_1/m_0) \cdots (m_N/m_{N-1}) \leqslant (1/2)^{N-X}(5/3)^X m_0 < m_0 = m.$$

It follows that $A(2^N) \geqslant \# D_N - 1 \geqslant \# H_N - 1$, and hence by (6),

$$\lim_{N\to\infty} A(2^N)/2^N = 1. \tag{9}$$

It remains to consider the case $2^N < M < 2^{N+1}$. If we set $A_N = A(2^N)$, and define

$$n_N = 2^N + \{(2^{N+1} - 2^N) - (A_{N+1} - A_N)\} = A_N + (2^{N+1} - A_{N+1}) \geqslant 2^N, \tag{10}$$

then for $M = 2^N + 1, ..., n_N$, it is obvious that

$$A(M)/M \geqslant A_N/n_N. \tag{11}$$

On the other hand, for $M = n_N + k$, $k = 1, 2, ..., 2^{N+1} - 1 - n_N$, one has

$$A(M)/M \geqslant (A_N + k)/(n_N + k) \geqslant A_N/n_N, \tag{12}$$

since $n_N \geqslant 2^N \geqslant A_N$. Now by Eqs. (9) and (10)

$$A_N/n_N = \frac{A_N/2^N}{A_N/2^N + 2(1 - A_{N+1}/2^{N+1})} \to 1,$$

and hence by (9), (11) and (12),

$$\lim_{M\to\infty} A(M)/M = 1.$$

In this way we are led to Theorem 2.

THEOREM 2. *For the function f defined in Eq.* (1), *"almost every" integer m has some iterate $m_k = f^k(m) < m$, in the sense that the density $A(M)/M$ of such integers approaches unity.*

REFERENCES

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