A Continuation Algorithm for a Class of Linear Complementarity Problems Using an Extrapolation Technique

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ABSTRACT

A polynomial-time continuation algorithm is presented for a class of linear complementarity problems with positive semidefinite matrices. The linear extrapolation technique is combined with the Newton iteration in the predictor-corrector procedure of the algorithm to numerically follow the solution curve of the homotopy equations arising from the perturbed Karush-Kuhn-Tucker condition. The convergence rate of the method is proved to be $1 - 4/(7\sqrt{n})$ after each cycle consisting of one extrapolation between two Newton steps.

1. INTRODUCTION

Let M be an $n \times n$ matrix, and $q \in R^n$. The problem of finding an $(x, y) \in R^{2n}$ satisfying

$$y = Mx + q, \qquad (x, y) \geqslant 0, \qquad x^T y = 0 \tag{1}$$

is called the linear complementarity problem (abbreviated LCP). The LCP has many applications in linear and convex quadratic programs and bimatrix games [15].

A traditional method for the LCP is the pivoting algorithm [15]. Although it has been widely used over decades, the theoretical exponential-time complexity of the algorithm in the worst case still stimulates researchers to find more efficient polynomial-time methods. With the appearance of

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Karmarkar's fundamental work on a projective algorithm for linear programming [8], path-following interior-point methods have attracted tremendous attention in the field of mathematical programming. For some representative work, see [6, 7, 11, 13, 14, 16–19] and the references therein. Recently, several new algorithms belonging to the category of the interior-point method have been proposed for solving LCPs (see e.g. [2, 3, 10, 11]), and a good survey and unified approach is given in [9]. These methods have the polynomially bounded computational complexity, in contrast to the traditional pivoting method. Kojima et al. [10] first give a polynomial-time algorithm for a class of LCPs with positive semidefinite matrices, based on the theoretical formulation given by Megiddo [12]. In their method one-step Newton iteration is used for each updated equation successively. So this method may also be called a predictor-corrector algorithm with only zero-order prediction. This observation leads to the development of the first general predictor-corrector algorithm given in [2] with a one-step Euler's method as "prediction" and a one-step Newton's method as "correction." With the increase in the algebraic precision order in the prediction step, the convergence is faster even though we take account of more computational work per iteration.

In this paper we use the secant approach to the prediction instead of the tangent approach in [2], to avoid the Euler iteration. For simplicity of presentation, we use only the linear extrapolation technique between two Newton iterations. That is, within each cycle, after one Newton iteration step is performed, we linearly extrapolate the previous iterate and the current one to "predict" the next one, and then one Newton iteration from this predicted one is performed to get the next iterate. In each such cycle, we need to solve two systems of linear equations coming from the Newton iteration, but the decrease of $x^T y$, which measures how nearly x and y are complementary, is at the approximate rate of $1 - 4/(7\sqrt{n}\,)$, a better complexity result than the ones given in [10] and [2].

The next section is the description of the predictor-corrector algorithm using the extrapolation technique. We prove the polynomial convergence of the algorithm in Section 3. Some conclusions and final remarks are given in the last section.

2. THE ALGORITHM

We are given the LCP (1) with the following assumptions:

(i) $n \geqslant 3$. (The problem with $n \leqslant 2$ is trivial.)
(ii) M is positive semidefinite with no zero rows.

Throughout this paper, $\|x\| = (\sum_{i=1}^{n} x_i^2)^{1/2}$ denotes the usual Euclidean spectral norm for $x \in R^n$, and $\|A\|$ stands for the corresponding matrix norm of $A \in R^{n\times n}$. That is, $\|A\| = \sup\{\|Ax\| : \|x\| \leqslant 1\}$ is the square root of the maximal eigenvalue of A^TA. For $v \in R^m$, $v > 0$ means $v_i > 0$ for $i = 1, \dots, m$.

Let

$$S_{\text{int}} = \{(x, y) > 0 : y = Mx + q\}.$$

We assume that S_{int} is not empty in what follows. It is well known that the system of following homotopy equations

$$XYe - \mu e = 0, \tag{2}$$

$$y - Mx - q = 0 \tag{3}$$

has a unique solution $(x(\mu), y(\mu))$ for each $\mu > 0$ when the matrix M is positive semidefinite (see, e.g., [2] for a proof). Here e is the n-dimensional vector of entries 1, and for a vector $x \in R^n$, X is the corresponding diagonal matrix with the diagonal entries x_i. Our purpose is to numerically follow this path $(x(\mu), y(\mu))$ from a starting interior feasible point. For this purpose we define

$$S(\theta) = \left\{(x, y) \in S_{\text{int}} : \left\| XYe - \frac{x^Ty}{n}e \right\| \leqslant \theta \frac{x^Ty}{n} \right\}.$$

Note that a point $(x, y) \in S_{\text{int}}$ satisfies (2) if and only if $XYe - (x^Ty/n)e = 0$. Moreover, since $(x^Ty/n)e$ is the orthogonal projection of the vector XYe onto the straight line $\{\lambda e : \lambda \in R\}$,

$$\left\| XYe - \frac{x^Ty}{n}e \right\| = \min_{\lambda \in R} \|XYe - \lambda e\|.$$

Suppose an initial strictly feasible point $(x^1, y^1) \in S(\frac{1}{12})$ is known. Given a precision $\epsilon > 0$, the algorithm is the following:

Step 0: Let $\delta = 2/(7\sqrt{n})$, $k := 1$.

Step 1: Let $\mu' = (1 - \delta)(x^k)^Ty^k/n$. Solve the equations (2) and (3) using one-step Newton iteration, starting from (x^k, y^k), with $\mu = \mu'$, to get $(x', y') = (x^k - \Delta x, y^k - \Delta y)$. Here $(\Delta x, \Delta y)$ satisfies the following equations:

$$Y^k \Delta x + X^k \Delta y = X^k Y^k e - \mu' e,$$
$$\Delta y = M \Delta x.$$

Step 2: Let $\tilde{x} = x' - (1-\delta)\Delta x$, $\tilde{y} = y' - (1-\delta)\Delta y$, and $\tilde{\mu} = (1-\delta)\mu'$. Use one-step Newton iteration to solve (2) and (3) with $\mu = \tilde{\mu}$ and the starting point $(\tilde{x}, \tilde{y})$ to get $(\hat{x}, \hat{y}) = (\tilde{x} - \Delta\tilde{x}, \tilde{y} - \Delta\tilde{y})$. Here $(\Delta\tilde{x}, \Delta\tilde{y})$ satisfies the following equations:

$$\tilde{Y}\Delta\tilde{x} + \tilde{X}\Delta\tilde{y} = \tilde{X}\tilde{Y}e - \tilde{\mu}e,$$
$$\Delta\tilde{y} = M\Delta\tilde{x}.$$

Step 3: Let $x^{k+1} = \hat{x}$, $y^{k+1} = \hat{y}$, $k := k+1$.
Step 4: If $(x^k)^T y^k < \epsilon$, then stop. Otherwise, go to Step 1.

REMARK 1. The initial point $(x^1, y^1) \in S(\frac{1}{12})$ can be obtained by means of the construction of an "artificial LCP" as in [10].

REMARK 2. In Step 2, the point $(\tilde{x}, \tilde{y})$ is obtained from the extrapolation through the two points (x^k, y^k) and (x', y').

In the next section, we shall prove that after one cycle of iteration,

$$(x^{k+1})^T y^{k+1} \leqslant \left(1 - \frac{4}{7\sqrt{n}}\right)(x^k)^T y^k$$

approximately. Thus our method will generate a sequence of points $\{(x^k, y^k)\}$ with values of $(x^k)^T y^k$ decreasing at least linearly with the global convergence ratio about $1 - 4/(7\sqrt{n})$ along the sequence. This convergence rate is better than the one proposed in [10] and is an improvement on the one given in [2] which uses the one-step Euler's method as predictor and the one-step Newton's method as corrector.

3. CONVERGENCE ANALYSIS

In this section, we will give the analysis of the previous algorithm. To this purpose, we need to explore the implementation of the algorithm after each cycle of iteration.

For any $\mu > 0$ and $(x, y) \in S_{\text{int}}$, let $(\bar{x}, \bar{y}) = (x - \Delta x, y - \Delta y)$ be obtained from solving (2) and (3) by one-step Newton iteration starting at (x, y). Then $(\Delta x, \Delta y)$ satisfies

$$Y \Delta x + X \Delta y = XYe - \mu e, \tag{4}$$

$$\Delta y = M \Delta x. \tag{5}$$

The following lemma will be used frequently.

LEMMA 3.1. *Let $u = (XY)^{-1/2}(XYe - \mu e)$. Then:*

(i) $\bar{y} = M\bar{x} + q$.
(ii) $\bar{X}\bar{Y}e = \mu e + \Delta X \Delta y$.
(iii) $\|\Delta X \Delta y\| \leqslant \|u\|^2/2$.
(iv) $(\Delta x)^T \Delta y \leqslant \|u\|^2/2$.
(v) $\bar{x}^T \bar{y}/n \geqslant \mu$.

Proof. From (5), it is easy to see that $y = Mx + q$ implies $\bar{y} = M\bar{x} + q$, which gives (i).

Since $(\Delta x, \Delta y)$ satisfies (4) and (5), we have

$$\begin{aligned}
\bar{X}\bar{Y}e &= (X - \Delta X)(Y - \Delta Y)e \\
&= XYe - (Y\Delta x + X \Delta y) + \Delta X \Delta y \\
&= XYe - (XYe - \mu e) + \Delta X \Delta y \\
&= \mu e + \Delta X \Delta y.
\end{aligned}$$

Thus (ii) is true.

Let $D = (XY^{-1})^{1/2}$. Multiplying the equation (4) by the matrix $(XY)^{-1/2}$ from the left, we have

$$D^{-1} \Delta x + D \Delta y = (XY)^{-1/2}(XYe - \mu e).$$

Since M is positive semidefinite,

$$(D^{-1}\Delta x)^T D \Delta y = \Delta x^T \Delta y = \Delta x^T M \Delta x \geqslant 0.$$

By a fundamental lemma introduced in [10], $\|D^{-1}\Delta x\|\,\|D\Delta y\| \leqslant \|u\|^2/2$. Hence,

$$\|\Delta X \Delta y\| = \|D^{-1}\Delta X D \Delta y\| \leqslant \|D^{-1}\Delta X\|\,\|D\Delta y\|$$

$$\leqslant \|D^{-1}\Delta x\|\,\|D\Delta y\| \leqslant \frac{\|u\|^2}{2}.$$

This proves (iii).

(iv) follows from

$$(\Delta x)^T \Delta y = (D^{-1}\Delta x)^T D\Delta y \leqslant \|D^{-1}\Delta x\|\,\|D\Delta y\| \leqslant \frac{\|u\|^2}{2}.$$

Lastly, noting that M is positive semidefinite, from (ii) we have

$$\bar{x}^T\bar{y} = e^T\overline{X}\overline{Y}e = n\mu + (\Delta x)^T \Delta y$$

$$= n\mu + (\Delta x)^T M \Delta y \geqslant n\mu.$$

This completes the proof. ■

Now suppose $(x, y) \in S(\theta)$ for some $\theta \in (0,1)$. Let $\mu = x^T y/n$ and $\mu' = (1-\delta)\mu$ with $\delta \in (0,1)$. Then we solve (2) and (3) with μ in (2) replaced by μ', using one-step Newton iteration starting at (x, y). That is, we define

$$x' = x - \Delta x, \qquad y' = y - \Delta y,$$

where $(\Delta x, \Delta y)$ is the unique solution of the equations (4) and (5) with μ in (4) replaced by μ'.

Let $u' = (XY)^{-1/2}(XYe - \mu' e)$. We then have

$$\|u'\|^2 \leqslant \left\|(XY)^{-1/2}\right\|^2 \|XYe - \mu' e\|^2$$

$$= \left\|(XY)^{-1/2}\right\|^2 \left\{\|XYe - \mu e\|^2 + \|(\mu - \mu')e\|^2\right\}$$

$$\leqslant (1-\theta)^{-1}\mu^{-1}(\theta^2\mu^2 + n\delta^2\mu^2)$$

$$= (1-\theta)^{-1}(\theta^2 + n\delta^2)\mu = [(1-\theta)(1-\delta)]^{-1}(\theta^2 + n\delta^2)\mu'.$$

The first equality above is from the fact that μe is the orthogonal projection of the vector XYe on the straight line expanded by e, and the last inequality follows from the definition of the spectral norm of a matrix and the condition that $(x, y) \in S(\theta)$. From Lemma 3.1,

$$\|X'Y'e - \mu'e\| = \|\Delta X \Delta y\| \leqslant \frac{\|u'\|^2}{2} \leqslant \frac{(\theta^2 + n\delta^2)\mu'}{2(1-\theta)(1-\delta)}, \tag{6}$$

and

$$(\Delta x)^T \Delta y \leqslant \frac{\|u'\|^2}{2} \leqslant \frac{(\theta^2 + n\delta^2)\mu'}{2(1-\theta)(1-\delta)}. \tag{7}$$

Put

$$\theta' = \frac{\theta^2 + n\delta^2}{2(1-\theta)(1-\delta)}.$$

Then $\|X'Y'e - \mu'e\| \leqslant \theta'\mu'$. Now we prove the following proposition.

PROPOSITION 3.1. *If* $\theta + \theta' \leqslant 1$, *then* $(x', y') \in S(\theta')$.

Proof. By (i) of Lemma 3.1, $y' = Mx' + q$. Now we show that $(x', y') > 0$. By the given condition, $\theta' < 1$. It follows that $\|X'Y'e - \mu'e\| < \mu'$. Thus $x_i' y_i' > 0$ for $i = 1, 2, \dots, n$. Suppose $x_i' < 0$ and $y_i' < 0$ for some i; then $x_i < \Delta x_i$ and $y_i < \Delta y_i$. Since $(x, y) \in S(\theta)$, $|x_i y_i - \mu| < \theta\mu$. Hence,

$$(1-\theta)\mu < x_i y_i < \Delta x_i \Delta y_i \leqslant \|\Delta X \Delta y\| \leqslant \frac{\theta^2 + n\delta^2}{2(1-\theta)}\mu.$$

This means

$$1 - \theta < \frac{\theta^2 + n\delta^2}{2(1-\theta)}.$$

Therefore

$$\theta + \theta' > \theta + \frac{\theta^2 + n\delta^2}{2(1-\theta)} > 1.$$

This is a contradiction.

To finish the proof, note that

$$\left\| X'Y'e - \frac{x'^T y'}{n} e \right\| \leqslant \|X'Y'e - \mu' e\| \leqslant \theta' \mu' \leqslant \theta' \frac{x'^T y'}{n}$$

by (v) of Lemma 3.1. Hence $(x', y') \in S(\theta')$. ■

Now we assume for the moment that $(x', y') > 0$. Define

$$\tilde{x} = x' - (1-\delta)\,\Delta x, \qquad \tilde{y} = y' - (1-\delta)\,\Delta y.$$

Here we assume that θ and δ are chosen such that $(\tilde{x}, \tilde{y}) > 0$.

Let $\tilde{\mu} = (1-\delta)\mu' = (1-\delta)^2\mu$. Then we solve (2) and (3) with μ in (2) replaced by $\tilde{\mu}$, using one-step Newton's iteration starting at $(\tilde{x}, \tilde{y})$. That is, we define

$$\hat{x} = \tilde{x} - \Delta\tilde{x}, \qquad \hat{y} = \tilde{y} - \Delta\tilde{y},$$

where $(\Delta\tilde{x}, \Delta\tilde{y})$ satisfies (4) and (5) with X, Y, and μ in (4) replaced by $\tilde{X}$, $\tilde{Y}$, and $\tilde{\mu}$, respectively.

Let $\tilde{u} = (\tilde{X}\tilde{Y})^{-1/2}(\tilde{X}\tilde{Y}e - \tilde{\mu}e)$. Then $\|\Delta\tilde{X}\Delta\tilde{y}\| \leqslant \|\tilde{u}\|^2/2$ and $(\Delta\tilde{x})^T \Delta\tilde{y} \leqslant \|\tilde{u}\|^2/2$ by Lemma 3.1. Now we estimate $\|\tilde{u}\|$. Since

$$\begin{aligned}\tilde{X}\tilde{Y}e &= [X' - (1-\delta)\,\Delta X][Y' - (1-\delta)\,\Delta Y]e \\ &= X'Y'e - (1-\delta)[Y'\Delta x + X'\Delta y] + (1-\delta)^2 \Delta X \Delta y,\end{aligned}$$

taking consideration of (ii) of Lemma 3.1, we get

$$\begin{aligned}
\tilde{X}\tilde{Y}e - \tilde{\mu}e &= X'Y'e - \tilde{\mu}e - (1-\delta)(Y'\Delta x + X'\Delta y) + (1-\delta)^2 \Delta X \Delta y \\
&= X'Y'e - \tilde{\mu}e - (1-\delta)[(Y - \Delta Y)\Delta x + (X - \Delta X)\Delta y] \\
&\quad + (1-\delta^2)\Delta X \Delta y \\
&= X'Y'e - \tilde{\mu}e - (1-\delta)[Y\Delta x + X\Delta y - (\Delta Y \Delta x + \Delta X \Delta y)] \\
&\quad + (1-\delta)^2 \Delta X \Delta y \\
&= X'Y'e - \tilde{\mu}e - (1-\delta)[XYe - \mu' e] \\
&\quad + \left[2(1-\delta) + (1-\delta)^2\right] \Delta X \Delta y \\
&= (X'Y'e - \mu' e) - (1-\delta)[XYe - \mu e] \\
&\quad + \left[2(1-\delta) + (1-\delta)^2\right] \Delta X \Delta y \\
&= \left[1 + 2(1-\delta) + (1-\delta)^2\right] \Delta X \Delta y - (1-\delta)(XYe - \mu e).
\end{aligned} \tag{8}$$

Hence,

$$\begin{aligned}
\|\tilde{X}\tilde{Y}e - \tilde{\mu}e\| &\leqslant \left[1 + 2(1-\delta) + (1-\delta)^2\right] \|\Delta X \Delta y\| \\
&\quad + (1-\delta)\|XYe - \mu e\| \\
&= (2-\delta)^2 \|\Delta X \Delta y\| + (1-\delta)\|XYe - \mu e\| \\
&\leqslant \frac{(2-\delta)^2(\theta^2 + n\delta^2)}{2(1-\theta)(1-\delta)}\mu' + (1-\delta)\theta\mu \\
&= \left[\left(\frac{2-\delta}{1-\delta}\right)^2 \frac{\theta^2 + n\delta^2}{2(1-\theta)} + \frac{\theta}{1-\delta}\right]\tilde{\mu}.
\end{aligned}$$

Let

$$\tilde{\theta} = \left(\frac{2-\delta}{1-\delta}\right)^2 \frac{\theta^2 + n\delta^2}{2(1-\theta)} + \frac{\theta}{1-\delta}.$$

We thus have

$$\|\tilde{X}\tilde{Y}e - \tilde{\mu}e\| \leqslant \tilde{\theta}\tilde{\mu}. \tag{9}$$

It follows immediately that

$$\left\|\left(\tilde{X}\tilde{Y}\right)^{-1/2}\right\| \leqslant \left(1-\tilde{\theta}\right)^{-1/2}\tilde{\mu}^{-1/2}$$

if in addition $\tilde{\theta} < 1$. This together with (9) implies

$$\|\tilde{u}\| \leqslant \left\|\left(\tilde{X}\tilde{Y}\right)^{-1/2}\right\| \|\tilde{X}\tilde{Y}e - \tilde{\mu}e\| \leqslant \frac{\tilde{\theta}\tilde{\mu}^{1/2}}{\left(1-\tilde{\theta}\right)^{1/2}}.$$

Therefore,

$$\|\Delta\tilde{X}\Delta\tilde{y}\| \leqslant \frac{\|\tilde{u}\|^2}{2} \leqslant \frac{\tilde{\theta}^2\tilde{\mu}}{2(1-\tilde{\theta})},$$

and

$$(\Delta\tilde{x})^T\Delta\tilde{y} \leqslant \frac{\|\tilde{u}\|^2}{2} \leqslant \frac{\tilde{\theta}^2\tilde{\mu}}{2(1-\tilde{\theta})}.$$

Now, from (ii) of Lemma 3.1,

$$\|\hat{X}\hat{Y}e - \tilde{\mu}e\| = \|\Delta\tilde{X}\Delta\tilde{y}\| \leqslant \frac{\tilde{\theta}^2}{2(1-\tilde{\theta})}\tilde{\mu}.$$

Denote $\hat{\theta} = \tilde{\theta}^2/2(1-\tilde{\theta})$. Then

$$\|\hat{X}\hat{Y}e - \tilde{\mu}e\| \leqslant \hat{\theta}\tilde{\mu}. \tag{10}$$

Let $\hat{\mu} = \hat{x}^T\hat{y}/n$. Then from (v) of Lemma 3.1, $\hat{\mu} \geqslant \tilde{\mu}$. Moreover, since $\hat{\mu}e$ is the orthogonal projection of the vector $\hat{X}\hat{Y}e$ onto the line through e, we get

$$\|\hat{X}\hat{Y}e - \hat{\mu}e\| \leqslant \|\hat{X}\hat{Y}e - \tilde{\mu}e\| \leqslant \hat{\theta}\tilde{\mu} \leqslant \hat{\theta}\hat{\mu}. \tag{11}$$

On the other hand, from (ii) of Lemma 3.1, we obtain

$$\begin{aligned}
\hat{\mu} &= \frac{\hat{x}^T\hat{y}}{n} = \tilde{\mu} + \frac{e^T \Delta \tilde{X} \Delta \tilde{y}}{n} \\
&= \tilde{\mu} + \frac{(\Delta \tilde{x})^T \Delta \tilde{y}}{n} \leqslant \tilde{\mu} + \frac{\tilde{\theta}^2}{2n(1-\tilde{\theta})}\tilde{\mu} \\
&= (1-\delta)\left(1 + \frac{\hat{\theta}}{n}\right)\mu' = (1-\delta)^2\left(1 + \frac{\hat{\theta}}{n}\right)\mu.
\end{aligned} \tag{12}$$

In summary, we have the following assertion.

PROPOSITION 3.2. *With the same notation as above, if θ and δ are chosen such that $(x', y') > 0$, $(\tilde{x}, \tilde{y}) > 0$, and $\tilde{\theta} < 1$, then*

(i) $\|X'Y'e - \mu'e\| \leqslant \theta'\mu'$,
(ii) $\|\tilde{X}\tilde{Y}e - \tilde{\mu}e\| \leqslant \tilde{\theta}\tilde{\mu}$,
(iii) $\|\hat{X}\hat{Y}e - \hat{\mu}e\| \leqslant \hat{\theta}\hat{\mu}$,
(iv) $\hat{\mu} \leqslant (1-\delta)^2(1 + \hat{\theta}/n)\mu$.

PROPOSITION 3.3. *Suppose $\theta + \theta' \leqslant 1$. If $\tilde{\theta} < 1$, then $(\tilde{x}, \tilde{y}) \in S(\tilde{\theta})$. In addition, if $\tilde{\theta} \leqslant 2 - \sqrt{2}$, then $(\hat{x}, \hat{y}) \in S(\hat{\theta})$.*

Proof. Since $y = Mx + q$ and $y' = Mx' + q$, from

$$\tilde{x} = x' - (1-\delta)\,\Delta x, \qquad \tilde{y} = y' - (1-\delta)\,\Delta y,$$

we have

$$\begin{aligned}
M\tilde{x} &= Mx' - (1-\delta)M\,\Delta x = Mx' - (1-\delta)\,\Delta y \\
&= y' - q - (1-\delta)\,\Delta y = \tilde{y} - q.
\end{aligned}$$

That is, $\tilde{y} = M\tilde{x} + q$. Thus, $\hat{y} = M\hat{x} + q$ by (i) of Lemma 3.1. Now we prove $(\tilde{x}, \tilde{y}) > 0$. Suppose not; there would be an index i such that $\tilde{x}_i = x'_i - (1-\delta)\Delta x_i < 0$ and $\tilde{y}_i - (1-\delta)\Delta y_i < 0$. Then from (6),

$$(1-\theta')\mu' \leqslant x'_i y'_i < (1-\delta)^2 \Delta x_i \Delta y_i$$

$$\leqslant (1-\delta)^2 \|\Delta X \Delta y\| \leqslant (1-\delta)^2 \theta' \mu'.$$

Hence

$$1 < \theta' + (1-\delta)^2\theta' = \left[1 + (1-\delta)^2\right]\theta'$$

$$= \left[1 + (1-\delta)^2\right]\frac{\theta^2 + n\delta^2}{2(1-\theta)(1-\delta)}$$

$$= \left[1 + (1-\delta)^2\right](1-\delta)\frac{\theta^2 + n\delta^2}{2(1-\theta)(1-\delta)^2}$$

$$< \frac{(2-\delta)^2(\theta^2 + n\delta^2)}{2(1-\theta)(1-\delta)^2} < \tilde{\theta},$$

which is a contradiction to the assumption. From (8),

$$\frac{\tilde{x}^T\tilde{y}}{n} = \tilde{\mu} + (2-\delta)^2\frac{(\Delta x)^T M \Delta x}{n} \geqslant \tilde{\mu}.$$

Hence,

$$\left\| \tilde{X}\tilde{Y}e - \frac{\tilde{x}^T\tilde{y}}{n}e \right\| \leqslant \|\tilde{X}\tilde{Y}e - \tilde{\mu}e\| \leqslant \tilde{\theta}\tilde{\mu} \leqslant \tilde{\theta}\frac{\tilde{x}^T\tilde{y}}{n}.$$

This proves $(\tilde{x}, \tilde{y}) \in S(\tilde{\theta})$.

If in addition $\tilde{\theta} \leqslant 2 - \sqrt{2}$, then from (iii) of Proposition 3.2 we have $\hat{x}_i \hat{y}_i > 0$ for each i, since $\hat{\theta} = \tilde{\theta}^2/[2(1 - \theta)] < 1$. Suppose $\hat{x}_i < 0$ and $\hat{y}_i < 0$ for some i. Then

$$(1 - \tilde{\theta})\tilde{\mu} \leqslant \tilde{x}_i \tilde{y}_i < \Delta \tilde{x}_i \Delta \tilde{y}_i \leqslant \|\Delta \tilde{X} \Delta \tilde{y}\| \leqslant \frac{\tilde{\theta}^2}{2(1 - \tilde{\theta})}\tilde{\mu}.$$

It follows that

$$\tilde{\theta}^2 - 4\tilde{\theta} + 2 < 0.$$

Thus $\tilde{\theta} > 2 - \sqrt{2}$, a contradiction. This completes the proof. ■

By Proposition 3.2 and Proposition 3.3, if we choose the two parameters δ and θ such that $\theta + \theta' \leqslant 1$, $\tilde{\theta} \leqslant 2 - \sqrt{2}$, and $\hat{\theta} \leqslant \theta$, then $(\hat{x}, \hat{y}) \in S(\theta)$, and

$$\frac{\hat{x}^T \hat{y}}{n} \leqslant (1 - \delta)^2 \left(1 + \frac{\theta}{n}\right) \frac{x^T y}{n}.$$

The best choice of δ and θ satisfies the following optimization problem:

$$\max\{\delta : \theta + \theta' \leqslant 1, \tilde{\theta} \leqslant 2 - \sqrt{2}, \hat{\theta} \leqslant \theta\}.$$

In particular, we obtain the following result needed for our algorithm in the previous section.

THEOREM 3.1. *Choose* $\delta = 2/(7\sqrt{n})$ *and* $\theta = \frac{1}{12}$. *Then* $\theta + \theta' \leqslant 1$, $\tilde{\theta} \leqslant 2 - \sqrt{2}$, *and* $\hat{\theta} \leqslant \theta$. *Thus* $(\hat{x}, \hat{y}) \in S(\theta)$ *and*

$$\hat{\mu} < \left(1 - \frac{4}{7\sqrt{n}} + \frac{97}{588n} - \frac{1}{21n\sqrt{n}} + \frac{1}{147n^2}\right)\mu.$$

Proof. For $\delta = 2/(7\sqrt{n})$ and $\theta = \frac{1}{12}$, considering $n \geqslant 3$, we get

$$\begin{aligned}
\theta + \theta' &= \theta + \frac{\theta^2 + n\delta^2}{2(1-\theta)(1-\delta)} \\
&= \frac{1}{12} + \frac{\left(\frac{1}{2}\right)^2 + \frac{4}{49}}{2\left(1 - \frac{1}{12}\right)\left(1 - 2/(7\sqrt{n})\right)} \\
&\leqslant \frac{1}{12} + \frac{625}{12936\left(1 - 2/(7\sqrt{3})\right)} \\
&< 0.2 < 1, \\
\tilde{\theta} &= \left(\frac{2-\delta}{1-\delta}\right)^2 \frac{\theta^2 + n\delta^2}{2(1-\theta)} + \frac{\theta}{1-\delta} \\
&= \left(1 + \frac{1}{1 - 2/(7\sqrt{n})}\right)^2 \frac{\left(\frac{1}{12}\right)^2 + \frac{4}{49}}{2\left(1 - \frac{1}{12}\right)} + \frac{\frac{1}{12}}{1 - 2/(7\sqrt{n})} \\
&\leqslant \left(1 + \frac{1}{1 - 2/(7\sqrt{3})}\right)^2 \frac{625}{12936} + \frac{1}{12\left(1 - 2/(7\sqrt{3})\right)} \\
&< 0.33312 < 2 - \sqrt{2},
\end{aligned}$$

and

$$\hat{\theta} = \frac{\tilde{\theta}^2}{2(1-\tilde{\theta})} < \frac{0.33312^2}{2(1 - 0.33312)} < \frac{1}{12} = \theta.$$

Hence from Proposition 3.3, $(\hat{x}, \hat{y}) \in S(\hat{\theta}) \subset S(\theta)$. Now (iv) of Proposition 3.2 gives

$$\begin{aligned}
\hat{\mu} &\leqslant (1-\delta)^2\left(1 + \frac{\hat{\theta}}{n}\right)\mu < \left(1 - \frac{2}{(7\sqrt{n})}\right)^2\left(1 + \frac{1}{12n}\right)\mu \\
&= \left(1 - \frac{4}{7\sqrt{n}} + \frac{4}{49n}\right)\left(1 + \frac{1}{12n}\right)\mu \\
&= \left(1 - \frac{4}{7\sqrt{n}} + \frac{97}{588n} - \frac{1}{21n\sqrt{n}} + \frac{1}{147n^2}\right)\mu. \qquad \blacksquare
\end{aligned}$$

In [10], the complexity bound of the $O(n^{3.5}L)$ algorithm for the LCP is $1 - 1/(8\sqrt{n})$ after one Newton iteration. Thus after two iterations, with the same notation in their algorithm,

$$\hat{\mu} \leqslant \left(1 - \frac{1}{8\sqrt{n}}\right)^2 \mu = \left(1 - \frac{1}{4\sqrt{n}} + \frac{1}{64n}\right)\mu.$$

The predictor-corrector algorithm proposed in [2] has the complexity bound $1 - 2/(5\sqrt{n})$ after one cycle of one Euler iteration and one Newton iteration. Therefore our algorithm has a better complexity.

4. CONCLUSIONS

In this paper, a polynomial-time predictor-corrector algorithm is presented for a class of LCPs with positive semidefinite matrices, based on a simple linear extrapolation technique combined with Newton's iteration. The corresponding complexity bound is shown to be much better than the one given in [10] and is better than the one in [2] using Euler's method as prediction. Also, now we don't need to solve systems of linear equations related to Euler's iteration. Instead we always solve systems of linear equations of the same type related to Newton's iteration, which makes the algorithm easier to implement.

For the LCPs, the linear-extrapolation approach for the prediction step has some advantage over the higher-order ones in that we can always keep the iterates feasible, that is, $y^k = Mx^k + q$ for all k. We expect that if the linear extrapolation is used before each Newton iteration instead of every two Newton iterations as given here, the complexity bound will be better. On the other hand, we may apply the quadratic polynomial extrapolation technique to the algorithm to speed up the convergence, since the system of nonlinear equations (2) and (3) is actually a quadratic polynomial one.

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