Europ. J. Combinatorics (2002) **23**, 1025–1031
doi:10.1006/eujc.2002.0619
Available online at http://www.idealibrary.com on IDEAL®

Free Cumulants and Enumeration of Connected Partitions

FRANZ LEHNER

A combinatorial formula is derived which expresses free cumulants in terms of classical cumulants. As a corollary, we give a combinatorial interpretation of free cumulants of classical distributions, notably Gaussian and Poisson distributions. The latter count connected pairings and connected partitions, respectively. The proof relies on Möbius inversion on the partition lattice.

1. INTRODUCTION

A partition of $[n] = \{1, 2, \ldots, n\}$ is called *connected* if no proper subinterval of $[n]$ is a union of blocks. Connected partitions have been studied by various authors under various names. They were introduced as *irreducible diagrams* in [9] and [10] (see also earlier work of Touchard [22]), where their asymptotic enumeration properties are studied. They are reconsidered under the name of *linked diagrams* in [13, 20] and [21] and a recursion is derived there. They also appear as a basic example in the theory of decomposable combinatorial objects developed in [3]. There and in [4, 5] asymptotics of more general irreducible partitions are studied.

We reserve the term *irreducible partition* to partitions which cannot be 'factored' into subpartitions, i.e., partitions of $[n]$ for which 1 and n are in the same connected component. A partition is called *noncrossing* if its blocks do not intersect in their graphical representation, i.e., if there are no two distinct blocks B_1 and B_2 and elements $a, c \in B_1$ and $b, d \in B_2$ s.t. $a < b < c < d$. Equivalently one could say that a partition is noncrossing if each of its connected components consists of exactly one block. Typical examples of these types of partitions are shown in Figure 1.

We denote the lattice of partitions of $[n]$ by Π_n, the irreducible partitions by Π_n^{irr} and the order ideal of connected partitions by Π_n^{conn}; the lattice of noncrossing partitions will be denoted by NC_n, and the sublattice of irreducible noncrossing partitions by NC_n^{irr}. Finally, let us denote by $\mathcal{I}_n$ the lattice of interval partitions, i.e., the lattice of partitions consisting entirely of intervals.

2. INCIDENCE ALGEBRAS

Before recalling more facts about partitions, let us briefly introduce the main concepts about posets and incidence algebras which will be needed in the sequel. Rota *et al.* [8] introduced the *reduced incidence algebra* of a poset. Let $(P, \leq)$ be a finite poset. On the space $I(P)$ of complex-valued functions $f(x, y)$ defined on the pairs (x, y) s.t. $x \leq y$ ('triangular matrices') we introduce a convolution ('multiplication of triangular matrices') by

$$f * g(x, y) = \sum_{x \leq z \leq y} f(x, z) g(z, y). \tag{2.1}$$

With this operation $I(P)$ becomes a unital algebra, the *incidence algebra* of the poset P, with identity

$$\delta(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$$

It is clear by induction that a function is invertible under this convolution if and only if all the 'diagonal' entries $f(\pi, \pi)$ are nonzero. One prominent invertible function is the zeta-function

0195–6698/02/111025 + 07 $35.00/0

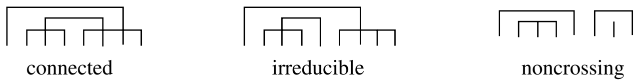


FIGURE 1. Typical partitions.

defined by

$$\zeta(x, y) = 1 \qquad \text{for } x \leq y.$$

Its inverse is called the *Möbius function* and satisfies the recursion

$$\mu(x, y) = \begin{cases} 1 & \text{if } x = y \\ -\sum_{x \leq z < y} \mu(x, z) & \text{if } x < y. \end{cases}$$

Then we have $\zeta * \mu = \mu * \zeta = \delta$ and more generally the *Möbius inversion formula* holds: for any pair of functions $f(x)$ and $g(x)$ on P we have the following equivalences

$$\forall x : f(x) = \sum_{y \geq x} g(y) \Longleftrightarrow \forall x : g(x) = \sum_{y \geq x} \mu(x, y) f(y) \tag{2.2}$$

$$\forall x : f(x) = \sum_{y \leq x} g(y) \Longleftrightarrow \forall x : g(x) = \sum_{y \leq x} f(y) \mu(y, x). \tag{2.3}$$

In our case the posets P will be the partition lattice Π_n and some of its sublattices, namely the lattice $\mathcal{I}_n$ of interval partitions and the lattice NC_n of noncrossing partitions. Denote by L_n any one of these lattices, then every segment $[\pi, \sigma]$ is canonically isomorphic to a finite direct product of full lattices $L_1^{k_1} \times L_2^{k_2} \times \cdots$. The sequence of exponents $(k_1, k_2, \ldots)$ will be called the *type* of the segment $[\pi, \sigma]$. The *reduced incidence algebra* is the algebra of functions whose values on an interval $[\pi, \sigma]$ only depend on the type of the interval. An even smaller class of functions is the set of *multiplicative functions* whose values at $[\pi, \sigma] \simeq L_1^{k_1} \times L_2^{k_2} \times \cdots$ are given by

$$f(\pi, \sigma) = f_1^{k_1} f_2^{k_2} \cdots$$

where (f_j) is a given sequence of numbers. These functions are defined on all partition lattices Π_n simultaneously. Examples of such functions are the zeta function and the Möbius function and it can be shown that the space of multiplicative functions is closed under the convolution (2.1).

Multiplicative functions on the lattice of noncrossing partitions were studied by Speicher in [17] and interval partitions in [18], see also [24] and [25].

We refer to [1] or [19, Chapter 3] for more information on incidence algebras.

3. CUMULANTS

Cumulants linearize convolution of probability measures coming from various notions of independence.

DEFINITION 3.1. *A noncommutative probability space* is pair $(\mathcal{A}, \varphi)$ of a (complex) unital algebra $\mathcal{A}$ and a unital linear functional φ. The elements of $\mathcal{A}$ are called *(noncommutative) random variables*. The collection of moments $\mu_n(a) = \varphi(a^n)$ of such a random variable $a \in \mathcal{A}$ will be called its *distribution* and denoted $\mu_a = (\mu_n(a))_n$.

Thus noncommutative probability theory follows the general 'quantum' philosophy of replacing function algebras by noncommutative algebras. We will review several notions of independence below. Convolution is defined as follows. Let a and b be 'independent' random variables. Then the convolution of the distributions of a and b is defined to be the distribution of the sum $a + b$. In all the examples below, the distribution of the sum of 'independent' random variables only depends on the individual distributions of the summands and therefore convolution is well defined and the nth moment $\mu_n(a + b)$ will be a polynomial function of the moments of a and b of order less or equal to n.

For our purposes it is sufficient to axiomatize cumulants as follows.

DEFINITION 3.2. Given a notion of independence on a noncommutative probability space $(\mathcal{A}, \varphi)$, a sequence of maps $a \mapsto k_n(a), n = 1, 2, \dots$ is called a *cumulant sequence* if it satisfies

1. $k_n(a)$ is a polynomial in the first n moments of a with leading term $\mu_n(a)$. This ensures that conversely the moments can be recovered from the cumulants.
2. Homogeneity: $k_n(\lambda a) = \lambda^n k_n(a)$.
3. Additivity: if a and b are 'independent' random variables, then $k_n(a + b) = k_n(a) + k_n(b)$.

Möbius inversion on the lattice of partitions plays a crucial role in the combinatorial approach to cumulants. We will need three kinds cumulants here, corresponding to classical, free and boolean independence, and which are connected to the three lattices of partitions considered in Section 1. Let X be a random variable with distribution ψ and moments $m_n = m_n(X) = \int x^n d\psi(x)$.

3.1. Classical cumulants. Let

$$\mathcal{F}(z) = \int e^{xz}\, d\psi(x) = \sum_{n=0}^{\infty} \frac{m_n}{n!} z^n$$

be the formal Laplace transform (or exponential moment generating function). Taking the formal logarithm we can write this series as

$$\mathcal{F}(z) = e^{K(z)}$$

where

$$K(z) = \sum_{n=1}^{\infty} \frac{\kappa_n}{n!} z^n$$

is the *cumulant generating function* and the numbers κ_n are called the *(classical) cumulants* of the random variable X.

Set partitions come in as follows. Let f and g be the multiplicative functions in the reduced incidence algebra of Π_n determined by the sequence m_n and κ_n, respectively, then $f = g * \zeta$ and $g = f * \mu$, i.e., if for a partition $\pi = \{\pi_1, \pi_2, \dots, \pi_p\}$ we put $m_\pi = m_{|\pi_1|} m_{|\pi_2|} \cdots m_{|\pi_p|}$ and $\kappa_\pi = \kappa_{|\pi_1|} \kappa_{|\pi_2|} \cdots \kappa_{|\pi_p|}$, then we can express moments and cumulants mutually as

$$m_\pi = \sum_{\sigma < \pi} \kappa_\sigma \qquad \kappa_\pi = \sum_{\sigma < \pi} m_\sigma \mu(\sigma, \pi).$$

For example, the standard Gaussian distribution $\gamma = N(0, 1)$ has cumulants

$$\kappa_n(\gamma) = \begin{cases} 1 & n = 2 \\ 0 & n \neq 2 \end{cases}$$

while the Poisson distribution P_λ (with weights $P_\lambda(\{k\}) = e^{-\lambda} \frac{\lambda^k}{k!}$) has cumulants $\kappa_n(P_\lambda) = \lambda$.

It follows that the even moments $m_{2n} = \frac{2n!}{2^n n!}$ of the standard Gaussian distribution count the number of pairings of a set with the corresponding number of elements.

On the other hand, the moments of a Poisson variable with parameter 1 are known as *Bell numbers* B_n and they are equal to the numbers of partitions of the finite sets with the corresponding cardinalities. The moment interpretation leads to *Dobinski's formula* (cf. [14])

$$B_n = e^{-1} \sum_{k=0}^{\infty} \frac{k^n}{k!}.$$

3.2. Free cumulants. Free cumulants were introduced by Speicher [17] in his combinatorial approach to Voiculescu's free probability theory [23]. Given our random variable X, let

$$M(z) = 1 + \sum_{n=1}^{\infty} m_n z^n \tag{3.1}$$

be its ordinary moment generating function. Define a formal power series

$$C(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

implicitly by the equation

$$C(z) = C(zM(z)).$$

Then the coefficients c_n are called the *free* or *noncrossing cumulants*. The latter name stems from the fact that combinatorially these cumulants are obtained by Möbius inversion on the lattice of noncrossing partitions:

$$m_\pi = \sum_{\substack{\sigma \in NC_n \\ \sigma \leq \pi}} c_\sigma \qquad c_\pi = \sum_{\substack{\sigma \in NC_n \\ \sigma \leq \pi}} m_\sigma \mu_{NC}(\sigma, \pi). \tag{3.2}$$

3.3. Boolean cumulants. Boolean cumulants linearize *boolean convolution* [18, 24, 25]. Let, again, $M(z)$ be the ordinary moment generating function of a random variable X defined by (3.1). It can be written as

$$M(z) = \frac{1}{1 - H(z)}$$

where

$$H(z) = \sum_{n=1}^{\infty} h_n z^n$$

and the coefficients are called *boolean cumulants*. Combinatorially the connection between moments and boolean cumulants is described by Möbius inversion on the lattice of interval partitions.

$$m_\pi = \sum_{\substack{\sigma \in \mathcal{I}_n \\ \sigma \leq \pi}} h_\sigma \qquad h_\pi = \sum_{\substack{\sigma \in \mathcal{I}_n \\ \sigma \leq \pi}} m_\sigma \mu_{\mathcal{I}}(\sigma, \pi). \tag{3.3}$$

The term 'boolean cumulants' is due to the fact that the lattice of interval partitions is anti-isomorphic to the boolean lattice of subsets of the same set with the first element removed. The isomorphism maps a partition to the set of first elements of its blocks, where clearly the first element of the first block is always the same and therefore redundant.

4. Enumeration of Connected Partitions

In this note we apply Möbius inversion to show that the free cumulants count connected partitions with certain weights given by the classical cumulants. The result is inspired by [3].

THEOREM 4.1. *Let* (m_n) *be a* (*formal*) *moment sequence with classical cumulants* κ_n. *Then the free cumulants of* m_n *are equal to*

$$c_n = \sum_{\pi \in \Pi_n^{\mathrm{conn}}} \kappa_\pi \tag{4.1}$$

the boolean cumulants are equal to

$$h_n = \sum_{\pi \in \Pi_n^{irr}} \kappa_\pi = \sum_{\pi \in NC_n^{irr}} c_\pi .$$

PROOF. We consider only the identity (4.1), the proof of the others being similar (and also contained in the lattice path picture of [11]). For $\sigma \in \Pi_n$ we denote by $\bar{\sigma}$ its *noncrossing closure*, that is the smallest noncrossing partition π s.t. $\sigma \leq \pi$. This noncrossing partition is obtained from σ by taking unions of all blocks which cross in the graphical representation, as in the example depicted in Figure 2.

For each $\pi \in NC_n$ define the number

$$\tilde{c}_\pi = \sum_{\substack{\sigma \in \Pi_n \\ \bar{\sigma} = \pi}} \kappa_\sigma .$$

Now note that the preimage of $\hat{1}_n = \{\{1, 2, \ldots, n\}\}$ is the set of all connected partitions, i.e.,

$$\tilde{c}_n := \tilde{c}_{\hat{1}_n} = \sum_{\sigma \in \Pi_n^{\mathrm{conn}}} \kappa_\sigma .$$

For general $\pi \in NC_n$, by considering subpartitions induced by the blocks of π, we have multiplicativity $\tilde{c}_\pi = \prod_{B \in \pi} \tilde{c}_{|B|}$. Now for $\pi \in NC_n$ we can collect terms as follows

$$\begin{aligned} m_\pi &= \sum_{\substack{\rho \in \Pi_n \\ \rho \leq \pi}} \kappa_\rho \\ &= \sum_{\substack{\sigma \in NC_n \\ \sigma \leq \pi}} \sum_{\substack{\rho \in \Pi_n \\ \bar{\rho} = \sigma}} \kappa_\rho \\ &= \sum_{\substack{\sigma \in NC_n \\ \sigma \leq \pi}} \tilde{c}_\sigma \end{aligned}$$

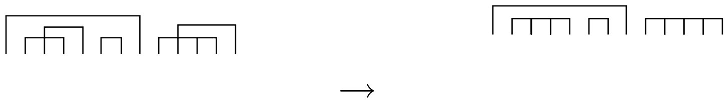
and by Möbius inversion (2.3) and (3.2) it follows that $c_\pi = \tilde{c}_\pi$. □

COROLLARY 4.2. *The free cumulants of the standard Gaussian variable are equal to the number of connected pairings.*

$$c_{2n}(\gamma) = \#\Pi_{2n}^{\mathrm{conn,pair}} .$$

COROLLARY 4.3. *The free cumulants of the Poisson distribution with parameter* 1 *are equal to the number of connected partitions*

$$c_n(P_1) = \#\Pi_n^{\mathrm{conn}} .$$



$\pi = 1,\ 8/2,\ 4/3,\ 5/6,\ 7/9,\ 11,\ 12/10,\ 13$ $\qquad$ $\bar{\pi} = 1,\ 8/2,\ 3,\ 4,\ 5/6,\ 7/9,\ 10,\ 11,\ 12,\ 13$

FIGURE 2. A partition π and its noncrossing closure.

Moreover, if we leave the formal parameter λ, the expression for the nth cumulants is the generating function of the numbers of blocks of the connected partitions.

$$c_n(P_\lambda) = \sum_{\pi \in \Pi_n^{\text{conn}}} \lambda^{|\pi|}.$$

Similar identities hold for the q-Gaussian and q-Poisson laws [7, 12] where the free cumulants provide a generating function of the number of *left-reduced crossings* of the connected partitions. Alternatively, the free cumulants of the q-Poisson laws of [2, 15, 16] count the number of *reduced crossings*, cf. [6]. In all these examples a continued fraction expansion of the moment generating function is known and the free cumulants can be expressed via Lagrange inversion. See also [11].

REFERENCES

1. M. Aigner, *Combinatorial Theory*, Springer, Berlin, 1979.
2. M. Anshelevich, Partition-dependent stochastic measures and q-deformed cumulants, *Documenta Mathematica*, **6** (2001), 343–384.
3. J. S. Beissinger, The enumeration of irreducible combinatorial objects, *J. Comb. Theory, Ser. A*, **38** (1985), 143–169.
4. E. A. Bender, A. M. Odlyzko and L. B. Richmond, The asymptotic number of irreducible partitions, *Europ. J. Combinatorics*, **6** (1985), 1–6.
5. E. A. Bender and L. B. Richmond, An asymptotic expansion for the coefficients of some power series. II. Lagrange inversion, *Discrete Math.*, **50** (1984), 135–141.
6. P. Biane, Some properties of crossings and partitions, *Discrete Math.*, **175** (1997), 41–53.
7. M. Bożejko, B. Kümmerer and R. Speicher, q-Gaussian processes: non-commutative and classical aspects, *Comm. Math. Phys.*, **185** (1997), 129–154.
8. P. Doubilet, G.-C. Rota and R. Stanley, On the foundations of combinatorial theory. VI. The idea of generating function, in: *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (University of California, Berkeley, CA, 1970/1971), Probability Theory*, Vol. II, University of California Press, Berkeley, CA, 1972, pp. 267–318.
9. W. N. Hsieh, Proportions of irreducible diagrams, *Studies Appl. Math.*, **52** (1973), 277–283.
10. D. J. Kleitman, Proportions of irreducible diagrams, *Studies Appl. Math.*, **49** (1970), 297–299.
11. F. Lehner, Cumulants, lattice paths, and orthogonal polynomials, preprint, (2001).
12. A. Nica, A one-parameter family of transforms, linearizing convolution laws for probability distributions, *Comm. Math. Phys.*, **168** (1995), 187–207.
13. A. Nijenhuis and H. S. Wilf, The enumeration of connected graphs and linked diagrams, *J. Comb. Theory, Ser. A*, **27** (1979), 356–359.
14. G.-C. Rota, The number of partitions of a set, *Am. Math. Monthly*, **71** (1964), 498–504.
15. N. Saitoh and H. Yoshida, A q-deformed Poisson distribution based on orthogonal polynomials, *J. Phys. A*, **33** (2000), 1435–1444.
16. N. Saitoh and H. Yoshida, q-deformed Poisson random variables on q-Fock space, *J. Math. Phys.*, **41** (2000), 5767–5772.

17. R. Speicher, Multiplicative functions on the lattice of noncrossing partitions and free convolution, *Math. Ann.*, **298** (1994), 611–628.
18. R. Speicher and R. Woroudi, Boolean convolution, in: *Free Probability Theory (Waterloo, ON, 1995)*, American Mathematical Society, Providence, RI, 1997, pp. 267–279.
19. R. P. Stanley, *Enumerative Combinatorics*, Vol. 1, Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, CA, 1986, With a foreword by Gian-Carlo Rota.
20. P. R. Stein and C. J. Everett, On a class of linked diagrams. II. Asymptotics, *Discrete Math.*, **21** (1978), 309–318.
21. P. R. Stein, On a class of linked diagrams. I. Enumeration, *J. Comb. Theory, Ser. A*, **24** (1978), 357–366.
22. J. Touchard, Sur un problème de configurations et sur les fractions continues, *Canadian J. Math.*, **4** (1952), 2–25.
23. D. V. Voiculescu, K. J. Dykema and A. Nica, *Free Random Variables, CRM Lecture Notes Series,* **1**, American Mathematical Society, Providence, RI, 1992.
24. W. von Waldenfels, An approach to the theory of pressure broadening of spectra lines, in: *Probability and Information Theory, II, Lecture Notes in Mathematics,* **296**, Springer, Berlin, 1973, pp. 19–69.
25. W. von Waldenfels, Interval partitions and pair interactions, in: *Séminaire de Probabilités, IX (Seconde Partie, University of Strasbourg, Strasbourg, années universitaires 1973/1974 et 1974/1975), Lecture Notes in Mathematics,* **465**, Springer, Berlin, 1975, pp. 565–588.

Received 26 November 2001 and accepted 23 July 2002

FRANZ LEHNER
Institut für Mathematik C,
Technische Universität Graz,
Steyrergasse 30,
A-8010 Graz,
Austria
E-mail: lehner@finanz.math.tu-graz.ac.at