



The disconnection number of a graph

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ABSTRACT

The disconnection number $d(X)$ is the least number of points in a connected topological graph X such that removal of $d(X)$ points will disconnect X (Nadler, 1993 [6]). Let \mathcal{D}_n denote the set of all homeomorphism classes of topological graphs with disconnection number n . The main result characterizes the members of \mathcal{D}_{n+1} in terms of four possible operations on members of \mathcal{D}_n . In addition, if X and Y are topological graphs and X is a subspace of Y with no endpoints, then $d(X) \leq d(Y)$ and Y obtains from X with exactly $d(Y) - d(X)$ operations. Some upper and lower bounds on the size of \mathcal{D}_n are discussed.

The algorithm of the main result has been implemented to construct the classes \mathcal{D}_n for $n \leq 8$, to estimate the size of \mathcal{D}_9 , and to obtain information on certain subclasses such as non-planar graphs ($n \leq 9$) and regular graphs ($n \leq 10$).

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1. Introduction

The subject of this paper, *topological graph theory*, is a rare encounter of topology with combinatorics, leading to results that *count* something. Throughout, we consider connected graphs with multiple edges and loops allowed. For all undefined notions in graph theory, see Harary [4].

The *topological realization* $|G|$ of a graph G is obtained by replacing symbolic edges by arcs and symbolic loops by circles. This results in a *topological graph*, that is: the union of finitely many topological arcs which are mutually disjoint except, possibly, for one or two common endpoints. The (local) *degree* of a point p in a connected and locally connected topological space is the supremal cardinal number of components of $N \setminus \{p\}$ where N is a connected open neighborhood of p . (The number of components of $N \setminus \{p\}$ does not decrease if N gets smaller.) With the exception of the circle, each topological graph is the realization of a graph G (henceforth referred to as *canonical*), the vertices of which are the points of $|G|$ of (local) degree $\neq 2$.

Let X be a connected topological space and let $n > 0$. The *disconnection number* $d(X)$ of X equals n provided for each subset A of X with n points the subspace $X \setminus A$ is disconnected, and n is the least number with this property. If $X = |G|$ then $d(X)$ is also called the disconnection number of G , alternatively denoted as $d(G)$. Nadler [6] showed that a metric continuum with a finite (or even countably infinite) disconnection number must be a topological graph.

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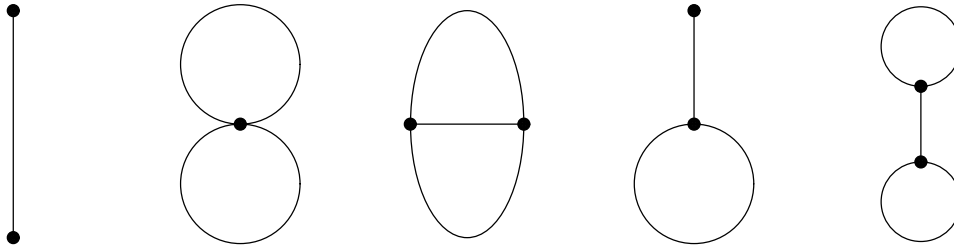


Fig. 1. The graphs in class \mathcal{D}_3 .

Let \mathcal{D}_n denote the set of all homeomorphism classes of graphs with disconnection number n . Our main result is a construction of \mathcal{D}_{n+1} from \mathcal{D}_n with four types of operations. In addition, we prove that if X and Y are topological graphs, where X is a subspace of Y and has no endpoints, then $d(X) \leq d(Y)$ and Y can be produced from X with exactly $d(Y) - d(X)$ admissible operations. This result, which we overlooked in [2], is needed to support some of the computational results at the end of this paper.

In the sequel, we use $\#A$ to denote the number of elements in a (finite) set A . Inductive application of the main theorem leads directly to an upper bound for the size of a graph in \mathcal{D}_n and to upper and lower bounds for $\#\mathcal{D}_n$. Let \mathcal{T}_n denote the set of homeomorphism classes of compact trees with n endpoints. (Note that $\mathcal{T}_n \subseteq \mathcal{D}_{n+1}$.) It appears that $\#\mathcal{D}_{n+1}$ grows proportionally at least with $n^3 \cdot \#\mathcal{T}_n$ (improving the bound in [2]) and at most with $n^2 \cdot \#\mathcal{D}_n$. The sequence of numbers $\#\mathcal{T}_n$ is known as A007827 in the *On-Line Encyclopedia of Integer Sequences*.¹

These formulae lead to an absolute upper bound for $\#\mathcal{D}_n$ in [2], which was improved by Cropper [1]. Cropper's result has been included here with a minor correction. Behind this result lies an alternative algorithm, constructing the class \mathcal{D}_{n+1} directly from \mathcal{T}_n .

The algorithm of our main theorem has been implemented to construct \mathcal{D}_n for $n \leq 8$, to find a fairly accurate estimate of $\#\mathcal{D}_9$, and to obtain some information on a few subclasses of \mathcal{D}_n for $n \leq 10$. The values of $\#\mathcal{D}_8$ and $\#\mathcal{D}_9$ were left open in [2]. We note that Cropper's algorithm, although leading to a better upper bound for \mathcal{D}_n , is less suited for an implementation. In fact, when constructing \mathcal{D}_8 from \mathcal{T}_7 , the number of operations to be performed on a single tree can be up to 200 times larger than the *total* number of operations that must be performed on \mathcal{D}_7 according to our main theorem.

The 26 members of \mathcal{D}_4 have been collected in Fig. 3 at the end of this paper.

2. Constructing the classes \mathcal{D}_n and estimating their size

In this section we describe the operations that lead from one disconnection class to the next. We also derive some bounds on the size of graphs in \mathcal{D}_n and on the size of \mathcal{D}_n itself. Up to homeomorphism, the circle is the only continuum with disconnection number 2. The following five graphs constitute the class \mathcal{D}_3 (Nadler [6]; see Fig. 1): an *arc*, a *figure eight*, a *theta curve*, a *figure six*, and a *dumbbell*. Only the first is a simple graph; the others have multiple edges or loops.

Henceforth we use the term *line* either for an edge or a loop of a graph. The following formula of Nadler [6] is fundamental. It links the disconnection number $d(G)$ of a connected graph G with the number, $v(G)$, of vertices, the number, $l(G)$, of lines, and the number, $end(G)$, of pendant vertices ("endpoints") of G .

$$d(G) = 2 - v(G) + l(G) + end(G). \quad (1)$$

For instance, a tree G has $v(G) = l(G) + 1$, whence $d(G) = 1 + end(G)$.

Eq. (1) can also be formulated in terms of the Euler characteristic $\chi(G) = v(G) - l(G)$ of G . Also, the disconnection number of a planar graph G is the sum of (i) the number of endpoints of G , and (ii) the number of components of $\mathbb{R}^2 \setminus |G|$ [6, Cor. 5.2].

Theorem 2.1. *Let $n \geq 3$. Every element of \mathcal{D}_{n+1} can be obtained from an element X of \mathcal{D}_n by applying one of the following operations to X :*

- (1) Adding an arc A to X such that $X \cap A = \{p\}$, where p is an endpoint of A and p is not an endpoint of X .
- (2) Adding an arc A to X such that $X \cap A = \{p_1, p_2\}$, where $p_1 \neq p_2$ are the endpoints of A and neither of them is an endpoint of X .
- (3) Adding a circle C to X such that $X \cap C = \{p\}$, where p is not an endpoint of X .
- (4) Adding a figure six B to X such that $X \cap B = \{p\}$, where p is the endpoint of B and is not an endpoint of X .

Conversely, for each $n \geq 2$ each of the operations (1)–(4), when applied to an element of \mathcal{D}_n , yields an element of \mathcal{D}_{n+1} .

¹ <http://www.research.att.com/~njas/sequences/>.

Proof. The last part of the theorem follows easily from the disconnection formula in Eq. (1) by observing the changes caused in the various terms by the operations (1)–(4). We will concentrate on the first part and use this fact below. Let $n \geq 3$ and let Y be an element of \mathcal{D}_{n+1} . We consider Y with its canonical (minimal) graph representation as explained in the introduction. Note that Y is not a circle (which is a member of \mathcal{D}_2).

Suppose first that Y has an endpoint y and let A be the arc realizing the edge at y . As Y is not an arc (which is a member of \mathcal{D}_3) and as Y is represented by a minimal (canonical) graph, the other endpoint x of A is of degree at least 3 in Y . Let $X := (Y \setminus A) \cup \{x\}$. We find that X is a topological graph and that x is not an endpoint of X . Therefore, Y is obtained from X by an instance of the procedure (1), whence $d(X) = d(Y) - 1 = n$.

We may henceforth assume that Y has no endpoints and is not a circle. As the canonical graph is not a tree, Y contains a topological circle S . Note that if a topological graph has endpoints, then so does the result of each of the procedures (1)–(4), applied to it. We distinguish the following cases.

Case 1. There are at least two vertices of Y in S of degree ≥ 3 in Y .

In this situation, S must be derived from a circuit in Y 's canonical graph. We may therefore assume that there is an arc A in S with endpoints x and y of degree ≥ 3 in Y , such that $\text{degree}(z) = 2$ in Y for all $z \in A \setminus \{x, y\}$. Let $X := (Y \setminus A) \cup \{x, y\}$. Then X is connected since in each path of Y that uses A , we can replace A by $(S \setminus A) \cup \{x, y\} \subseteq X$. We obtain Y from X by adding the arc A that intersects X only in the endpoints of A . The vertices x and y having degree ≥ 3 in Y , the respective degrees in X are ≥ 2 . Hence neither x nor y are an endpoint of X (they need not be vertices of X). Hence we have an instance of procedure (2) of the theorem and consequently $d(X) = d(Y) - 1 = n$.

Case 2. The topological circle S has exactly one vertex of Y and that point has degree ≥ 4 in Y .

Let $x \in S$ have degree ≥ 4 in Y , and let $X = (Y \setminus S) \cup \{x\}$. We can obtain Y from X by attaching the circle S to X at x . As $\text{degree}(x) \geq 4$ we have that x is not an endpoint of X . We have an instance of procedure (3) of the theorem, whence $d(X) = d(Y) - 1 = n$.

Case 3. The topological circle S has exactly one vertex of Y and that point has degree 3 in Y .

Let $y \in S$ have degree 3 in Y . There is an arc A of Y realizing an edge of type xy . The vertex x is not in S and $S \cup A$ is a figure six. Let $X := (Y \setminus (S \cup A)) \cup \{x\}$. Then x is the endpoint of $S \cup A$ but not an endpoint of X . We can obtain Y from X by attaching a figure six to X at the point x as described in procedure (4). Hence $d(X) = d(Y) - 1 = n$. \square

Note that the attachment rules of the theorem are somewhat restrictive. Breaking the rules can have unexpected consequences. For instance, if X is a topological graph with two endpoints, and if Y obtains from X by connecting these points with an arc, then $d(Y) = d(X) - 1$. Also, attaching a circle at an endpoint gives a proper extension with equal disconnection number. This may help to appreciate the following result, needed in Section 3.

Proposition 2.2. *Let X be a non-trivial connected topological graph without endpoints and let Y be a connected topological graph with subspace X . Then $d(X) \leq d(Y)$ and Y can be obtained from X with exactly $d(Y) - d(X)$ operations as in the main Theorem 2.1.*

Proof. As X has no endpoints, it follows from [6, Prop. 8.1] that $d(X) \leq d(Y)$. We henceforth assume that Y is properly larger than X and that Y contains no vertices of local degree 2. If X happens to be a circle, we may assume that its unique vertex is also a vertex of Y (note that the circle is the only space with $d = 2$).

Let Z be the union of all lines of Y not included in X (the “surplus” lines of Y relative to X). The case where each component of Z is a tree, touching X in one vertex, is solved directly as follows. Note that, given a component of Z , each endpoint $\notin X$ must be an endpoint of Y . The one endpoint which is in X has local degree ≥ 3 in Y . There is a well-known characterization of trees in graph theory, stating that trees are exactly the graphs that can be reduced to any pre-chosen single vertex by successively removing (“detaching”) a pending edge and its endpoint. We apply this result on each tree component of Z (choosing its vertex in X as the surviving vertex) to find a sequence of detachments of pending surplus edges of Y , taking care to clean up vertices of local degree 2 when they arise, until we are left with the subspace X . If n detachments are needed, the main Theorem 2.1 (last part) shows that $d(Y) = d(X) + n$. Having explicitly dealt with this case, we will proceed by strong induction on the number $s > 0$ of surplus lines of Y relative to X .

If some component of Z touches X in at least two points (vertices), then it has a subcontinuum A which is an arc touching X in its endpoints. In this case, we define $X' := X \cup A$. Assume all components of Z touch X in only one point. If some component of Z is not a tree, then it has a subcontinuum C which is a simple closed curve. If C touches X , we define $X' := X \cup C$. If C doesn't touch at X , there must be a subcontinuum A of Z which is a topological arc joining C with a vertex of X and meets C in exactly one point. In this case we define $X' := X \cup A \cup C$. The remaining case (all components are trees touching X at one point) being solved already, we proceed as follows. In all situations at hand, we have $d(X') = d(X) + 1$ by the last part of 2.1. Furthermore, X' has no endpoints, and the number of surplus lines in Y relative to X' is properly less than s . The result follows from the induction hypothesis. \square

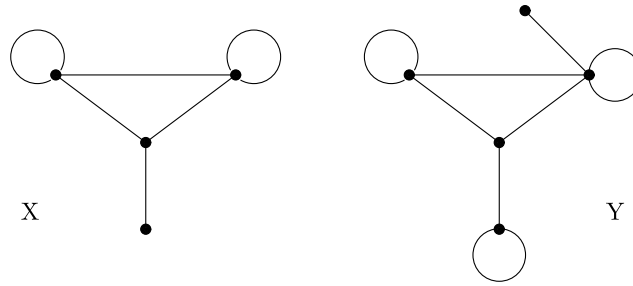


Fig. 2. A counterexample on endpoints.

In [6, Prop. 8.1] Nadler has shown that $d(X) \leq d(Y)$ if $X \subseteq Y$ are non-trivial topological graphs and $\text{end}(X) \leq \text{end}(Y)$. This may suggest an extension of Proposition 2.2, where the condition on the absence of endpoints of X is replaced by the condition $\text{end}(X) \leq \text{end}(Y)$. However, here is a counterexample.

We consider the graphs X and Y as drawn in Fig. 2. Both graphs have one endpoint, X can be embedded as a subspace of Y in essentially one way, and $d(X) = 5, d(Y) = 6$. Yet Y cannot be obtained from X by operations as described in the main theorem.

In order to determine an upper bound for $\#\mathcal{D}_{n+1}$, we only have to find out how many new graphs we can obtain from each graph in \mathcal{D}_n by applying one of the procedures (1)–(4) in Theorem 2.1. This depends on the number of vertices and lines occurring in the initial graph.

Corollary 2.3. *Let $n \geq 3$ and let $X \in \mathcal{D}_n$ be a topological graph represented with its canonical graph. Then X has at most $2n - 4$ vertices and at most $3n - 6$ lines.*

Proof. For $n = 3$, the result correctly predicts a maximum of 2 vertices and 3 lines. We proceed by induction on $n \geq 3$, assuming the result is valid for n . By Theorem 2.1, a member of the class \mathcal{D}_{n+1} can be written as $X \cup F$, where $X \in \mathcal{D}_n$, and where the union with F falls into one of the following descriptions.

- (i) F is an arc meeting X in an endpoint of F .
- (ii) F is an arc meeting X in both endpoints of F .
- (iii) F is a circle meeting X in one point.
- (iv) F is a figure six meeting X in the endpoint of F .

In addition, no point of $F \cap X$ is an endpoint of X . Operation (i) adds at most 2 new vertices to X and the number of lines of X increases by at most 2. In case (ii), the number of vertices increases by at most 2 and the number of lines increases by at most 3. In situation (iii), the number of vertices increases by at most 1 and the number of lines increases by at most 2. In case (iv), the number of vertices and the number of lines increase by at most 2 and 3, respectively. \square

Corollary 2.4. *For $n \geq 3$,*

$$\sum_{k=0}^n (-1)^{n-k} \binom{k+3}{k} \cdot \#\mathcal{T}_n \leq \#\mathcal{D}_{n+1} \leq \left(\frac{25n^2}{2} - \frac{69n}{2} + 19 \right) \cdot \#\mathcal{D}_n,$$

where \mathcal{T}_n denotes the subclass of \mathcal{D}_{n+1} , consisting of all trees with n endpoints.

Proof. Let $X \in \mathcal{D}_n$. By Corollary 2.3, there are at most $3 \cdot (2n - 4)$ possibilities to attach an arc, a circle or a figure six at a vertex of X , and there are at most $3 \cdot (3n - 6)$ possibilities to use a non-vertex point of a line for attaching one of these objects.

In order to attach an arc to X with both its endpoints, there are at most $\frac{1}{2} \cdot (2n - 4) \cdot (2n - 5)$ possibilities to select two distinct vertices of X , there are at most $(2n - 4) \cdot (3n - 6)$ possibilities to select a vertex and a line of X , and there are at most $\frac{1}{2} \cdot (3n - 6) \cdot (3n - 7) + (3n - 6)$ possibilities to choose two lines (which need not be different).

Adding things up, we find that X can be extended in at most $\frac{25}{2}n^2 - \frac{69}{2}n + 19$ ways to an object in \mathcal{D}_{n+1} .

To obtain the lower bound, we consider the following operations on an object X of \mathcal{D}_{n+1} :

- (1) Attaching a circle at an endpoint of X . This produces a circle containing exactly one vertex of X , and this vertex has degree 3 in the extension.
- (2) Attaching a double arc between two endpoints of X . This produces a circle containing exactly two vertices of X , each of degree 3 in the extension.

- (3) Replacing the edge of an endpoint by a circle (attached at the same point as the edge was). This produces a circle containing exactly one vertex of X , and this vertex has degree ≥ 4 in the extension.

By the disconnection number formula in Eq. (1), none of these operations affect the disconnection number. As $\mathcal{T}_n \subseteq \mathcal{D}_{n+1}$, this allows us to create a number of mutually disjoint subclasses of \mathcal{D}_{n+1} which are in 1–1 correspondence with \mathcal{T}_n as follows.

Let (r, s, t, u) be a decomposition of n , that is: $r + s + t + u = n$. We require $r, s, t, u \geq 0$ with r even. The number of such decompositions of n with *distinguished parts* equals

$$\sum_{k=0}^n (-1)^{n-k} \binom{k+3}{k}.$$

This can be seen with the classical *generating functions* method; see, for instance, Grimaldi [3, Ch. 9].

A decomposition (r, s, t, u) of n is used as follows. Let $T \in \mathcal{T}_n$. Choose r endpoints of T , divide them into $r/2$ pairs, and connect each pair by a double arc. Then choose another s endpoints and attach a circle to each of them. Choose yet another t endpoints and replace its edge by a circle. The remaining u endpoints are left untouched. For each decomposition (r, s, t, u) of n , we consider *one* partition of the endpoints into sets of size r, s, t and u , and *one* subdivision of the r -set into pairs.

Things being done this way, we obtain an extension of T having a collection of disjoint topological circles as subspaces. The number of vertices on each circle, together with their degree, fall under the descriptions (1)–(3) above. These are topological characteristics that allow to reconstruct T and the operations performed on it. It can now be seen that distinct decompositions of n and/or topologically distinct trees all lead to nonhomeomorphic results. \square

The lower bound of $\#\mathcal{D}_{n+1}$ improves the bound in [2] which is (roughly) $\frac{1}{4} \cdot n^2 \cdot \#\mathcal{T}_n$. The upper bound of $\#\mathcal{D}_{n+1}$ can be simplified as $\#\mathcal{D}_{n+1} < \frac{25}{2} n^2 \cdot \#\mathcal{D}_n$. Recursive application down to $\#\mathcal{D}_3 = 5$ yields

$$\#\mathcal{D}_n \leq \left(\frac{25}{2}\right)^{n-2} \cdot \frac{(n-1)!^2}{10} \quad (n \geq 3). \quad (2)$$

This bound is somewhat more efficient than the one in [2].

In computations as above, it is difficult to take into account that different operations may lead to homeomorphic graphs, even when starting from the same graph. When expressing $\#\mathcal{D}_{n+1}$ in terms of $\#\mathcal{D}_n$, the estimate appears to be a factor too large (e.g., $n = 4$: about 10 times; $n = 5$: about 15 times). The upper bound of inequality (2) accumulates these factors. An improvement was made by Cropper [1], linking $\#\mathcal{D}_{n+1}$ directly with $\#\mathcal{T}_n$ (which is a known sequence). To this end, consider the following algorithm on a topological tree T with n endpoints (where by an *inner* point is meant a non-endpoint).

Choose k endpoints of T . At each chosen point, attach an arc with one endpoint; the other endpoint of the arc is attached to an inner point of T (which may or may not be a vertex). The attached arcs are disjoint except, possibly, for the inner attachment point.

If the tree has i inner vertices ($i \leq n - 2$) then there are $n + i - 1$ edges. Taking into account the relative positions of the inner attachment points, the number of possible choices is

$$\begin{aligned} 1 + \sum_{k=1}^n \binom{n}{k} \cdot \prod_{j=0}^{k-1} (n + 2i + 2j - 1) &\leq \sum_{k=0}^n \binom{n}{k} \cdot (n + 2i + 2k - 3)^k \\ &\leq \sum_{k=0}^n \binom{n}{k} \cdot (3n + 2i - 3)^k. \end{aligned}$$

The latter equals $(3n + 2i - 2)^n \leq (5n - 6)^n$. (In Cropper's computation, part of a factor got lost; it must be $5n$, not $4n$.) The result provides a better estimate of $\#\mathcal{D}_n$ than the inequality (2) for $n \geq 8$.

Proposition 2.5. ([1, Cor. 1.3]) *The class \mathcal{D}_{n+1} consists exactly of those topological graphs that can be obtained from \mathcal{T}_n by an operation of the above kind. Consequently,*

$$\#\mathcal{D}_{n+1} \leq (5n - 6)^n \cdot \#\mathcal{T}_n \quad (n \geq 2).$$

Proof. (Sketch) Let $X \in \mathcal{D}_{n+1}$ and consider a spanning tree T_0 of X . On each arc or circle of X not in T_0 we take a point halfway and delete one of the resulting half-arc or half-circles. This can be done in a balanced way, so that no endpoint of T_0 , which is not an endpoint of X , loses all its half-arcs and half-circles. The result is a tree $T \in \mathcal{D}_{n+1}$ with (exactly n)

Table 1Size of \mathcal{D}_n .

\mathcal{D}_3	\mathcal{D}_4	\mathcal{D}_5	\mathcal{D}_6	\mathcal{D}_7	\mathcal{D}_8	\mathcal{D}_9
5	26	213	2310	32 512	555 323	$\pm 10\,612\,000$

Table 2

Descendants of Kuratowski graphs.

Class	$K_{3,3}$	K_5	Non-planar graphs
\mathcal{D}_5	1	0	1 (0.46%)
\mathcal{D}_6	13	0	13 (0.56%)
\mathcal{D}_7	354	1	355 (1.08%)
\mathcal{D}_8	12 238	12	12 247 (2.22%)
\mathcal{D}_9	493 374	410	493 558 ($\pm 4.64\%$)

endpoints among the endpoints of X and among the chosen points. The reconstruction of X from T follows exactly the description above, which explains both the statement and the formula. \square

As a simple illustration of his algorithm, Cropper [1] shows how all five members of \mathcal{D}_3 obtain from an arc (which is the only tree in \mathcal{D}_3).

3. Some numerical data

Most of the (numerical) results in this section have been obtained with the aid of a computer program based on the information of the main Theorem 2.1. As this journal is not the proper forum to discuss the fine points of the method, we only present the numerical data with some comments. For more details or the source code, the reader may contact the second author.

3.1. Determination of $\#\mathcal{D}_n$ for $n \leq 9$

Starting with the list of (five) members of \mathcal{D}_3 , and using the operations of Theorem 2.1, we can successively compute all members of the classes \mathcal{D}_n for $n = 4, 5, \dots$. The resulting quantities are presented in Table 1. Comparing with [2], we have now solved the case of $\#\mathcal{D}_8$. The approximation of $\#\mathcal{D}_9$ is also new. Basing ourselves on the largest observed difference between computations with large samples, we think the approximation is accurate up to 0.5%.

The numbers of Table 1 answer the question of how many topologically distinct graphs exist, where each subcollection of n points disconnects the space ($n \leq 8$) while some $n - 1$ points do not. Our program's output of \mathcal{D}_4 upon input of \mathcal{D}_3 has been used to draw the 26 graphs of the former class at the end of this paper.

3.2. Non-planar graphs

In his pioneering paper, Nadler used the *Kuratowski graphs* $K_{3,3}$ and K_5 to conclude that non-planar topological graphs have a disconnection number ≥ 5 [6, Cor. 8.2]. The disconnection numbers of $K_{3,3}$ and K_5 are, respectively, 5 and 7. As these two graphs have no endpoints, every connected topological graph, which properly extends one of them, must have a larger disconnection number by Proposition 2.2.

The famous Kuratowski *non-planarity theorem* [5], combined with Proposition 2.2, leads us to the following conclusion. Every connected non-planar topological graph can be obtained from $K_{3,3}$ or from K_5 with an appropriate amount of operations as in the main theorem. Conversely, every finite sequence of such operations leads to a connected non-planar topological graph. This makes non-planar graphs computable by iterated runs of our program, feeded with just the edge list of $K_{3,3}$ and of K_5 . The results are summarized in Table 2.

We already observed that $K_{3,3}$ is the only non-planar member of \mathcal{D}_5 . All non-planar graphs in \mathcal{D}_6 can be built on $K_{3,3}$ by one admissible operation of the main theorem. As $K_{3,3}$ has six points of local degree 3, it is evidently not a subspace of K_5 . Hence K_5 cannot be built on $K_{3,3}$.

Note that the total number of non-planar graphs in a class is *not* simply the sum of the numbers to its left. In fact, starting with the class \mathcal{D}_8 , there is an overlap between spaces built on $K_{3,3}$ and spaces built on K_5 .

The percentage in the right column indicates the relative amount of non-planar topological graphs in the appropriate class \mathcal{D}_n .

3.3. Simple and regular graphs

Each vertex v of a graph goes with so-called *local invariants*: the number n of neighbors of v , the number a of arcs at v , and the number c of circles at v . Given that multiple arcs may join v with a neighbor, this leads to a *partition* p [3, p. 432ff]

Table 3
Special subclasses.

	\mathcal{D}_3	\mathcal{D}_4	\mathcal{D}_5	\mathcal{D}_6	\mathcal{D}_7	\mathcal{D}_8	\mathcal{D}_9	\mathcal{D}_{10}
Simple graphs	1	2	9	49	416	4818	74974	
Regular graphs	4	5	9	14	28	107	520	4095
Simple regular graphs	1	1	2	5	20	86	511	4066

of the number a into n parts. The quadruple (n, c, a, p) gives complete numerical information on the neighborhood of a vertex.

Simple graphs are graphs without multiple edges and without loops. We define *regular graphs* somewhat stricter than usual: each vertex has the same quadruple (n, c, a, p) of local invariants. For simple graphs, this agrees with the standard notion of regularity.

Some counting results are summarized in Table 3.

By Nadler's formula for the disconnection number (our Eq. (1)), regular graphs with the same number of vertices and the same constant local degree are in the same disconnection class. Moreover, regular graphs of a fixed class \mathcal{D}_n have restricted types. For instance, the simple regular graphs of \mathcal{D}_6 are 3-regular and have 8 vertices; the simple regular graphs of \mathcal{D}_7 are either 4-regular with 5 vertices (just K_5) or 3-regular with 10 vertices. Such information can easily be obtained for each disconnection class by using a few equations involving the relevant invariants.

It is rather wasteful, however, to compute regular graphs via disconnection classes. Actually, there are specialized programs available² which rapidly compute all simple regular graphs of given size and local degree.

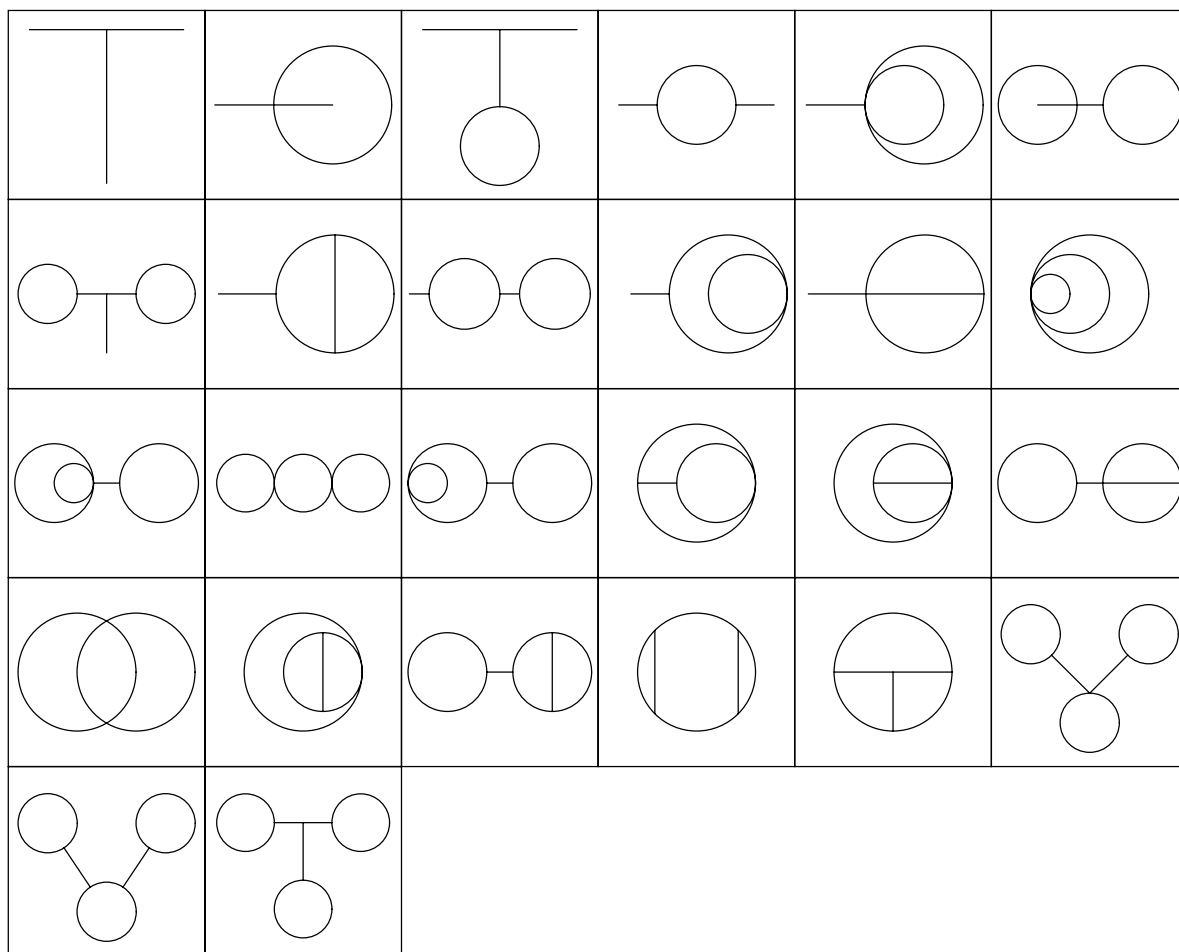


Fig. 3. The 26 graphs of \mathcal{D}_4 .

² The number in the right lower corner of Table 3 was computed with the program *genreg*: exactly 6 simple graphs with 8 points are 4-regular (which can also be seen by a combinatorial analysis), and exactly 4060 simple graphs with 16 points are 3-regular. See <http://www.mathe2.uni-bayreuth.de/markus/reggraphs.html>.

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