4

Cubic polynomials

4.1 Kick off with CAS

Cubic transformations

1 Using technology, sketch the following cubic functions.

a $y = x^3$ b $y = -x^3$ c $y = -3x^3$ d $y = \frac{1}{2}x^3$ e $y = -\frac{2}{5}x^3$

2 Using technology, enter $y = ax^3$ into the function entry line and use a slider to change the values of a.

3 Complete the following sentences.

a When sketching a cubic function, a negative sign in front of the x^3 term ________ the graph of $y = x^3$.

b When sketching a cubic function, $y = ax^3$, for values of $a < -1$ and $a > 1$, the graph of $y = x^3$ becomes ___________.

c When sketching a cubic function, $y = ax^3$, for values of $-1 < a < 1$, $a \neq 0$, the graph of $y = x^3$ becomes ___________.

4 Using technology, sketch the following functions.

a $y = x^3$ b $y = (x + 1)^3$ c $y = -(x - 2)^3$

d $y = x^3 - 1$ e $y = -x^3 + 2$ f $y = 3 - x^3$

5 Using technology, enter $y = (x - h)^3$ into the function entry line and use a slider to change the values of h.

6 Using technology, enter $y = x^3 + k$ into the function entry line and use a slider to change the values of k.

7 Complete the following sentences.

a When sketching a cubic function, $y = (x - h)^3$, the graph of $y = x^3$ is _________.

b When sketching a cubic function, $y = x^3 + k$, the graph of $y = x^3$ is _________.

8 Use technology and your answers to questions 1–7 to determine the equation that could model the shape of the Bridge of Peace in Georgia. If the technology permits, upload a photo of the bridge to make this easier.

Please refer to the Resources tab in the Prelims section of your **eBookPLUS** for a comprehensive step-by-step guide on how to use your CAS technology.

4.2 Polynomials

A **polynomial** is an algebraic expression in which the power of the variable is a positive whole number. For example, $3x^2 + 5x - 1$ is a quadratic polynomial in the variable x but $\frac{3}{x^2} + 5x - 1$ i.e. $3x^{-2} + 5x - 1$ is not a polynomial because of the $3x^{-2}$ term. Similarly, $\sqrt{3}x + 5$ is a linear polynomial but $3\sqrt{x} + 5$ i.e. $3x^{\frac{1}{2}} + 5$ is not a polynomial because the power of the variable x is not a whole number. Note that the coefficients of x can be positive or negative integers, rational or irrational real numbers.

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AOS 1

Topic 3

Concept 1

Polynomials
Concept summary

Classification of polynomials

- The **degree of a polynomial** is the highest power of the variable. For example, linear polynomials have degree 1, quadratic polynomials have degree 2 and cubic polynomials have degree 3.
- The **leading term** is the term containing the highest power of the variable.
- If the coefficient of the leading term is 1 then the polynomial is said to be monic.
- The **constant term** is the term that does not contain the variable.

A polynomial of degree n has the form $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ where $n \in N$ and the coefficients $a_n, a_{n-1}, \ldots a_1, a_0 \in R$. The leading term is $a_n x^n$ and the constant term is a_0.

WORKED EXAMPLE 1

Select the polynomials from the following list of algebraic expressions and for these polynomials, state the degree, the coefficient of the leading term, the constant term and the type of coefficients.

A $5x^3 + 2x^2 - 3x + 4$ **B** $5x - x^3 + \frac{x^4}{2}$ **C** $4x^5 + 2x^2 + 7x^{-3} + 8$

THINK	WRITE
1 Check the powers of the variable x in each algebraic expression.	A and B are polynomials since all the powers of x are positive integers. C is not a polynomial due to the $7x^{-3}$ term.
2 For polynomial A, state the degree, the coefficient of the leading term and the constant term.	Polynomial A: the leading term of $5x^3 + 2x^2 - 3x + 4$ is $5x^3$. Therefore, the degree is 3 and the coefficient of the leading term is 5. The constant term is 4.
3 Classify the coefficients of polynomial A as elements of a subset of R.	The coefficients in polynomial A are integers. Therefore, A is a polynomial over Z.
4 For polynomial B, state the degree, the coefficient of the leading term and the constant term.	Polynomial B: the leading term of $5x - x^3 + \frac{x^4}{2}$ is $\frac{x^4}{2}$. Therefore, the degree is 4 and the coefficient of the leading term is $\frac{1}{2}$. The constant term is 0.
5 Classify the coefficients of polynomial B as elements of a subset of R.	The coefficients in polynomial B are rational numbers. Therefore, B is a polynomial over Q.

Polynomial notation

- The polynomial in variable x is often referred to as $P(x)$.
- The value of the polynomial $P(x)$ when $x = a$ is written as $P(a)$.
- $P(a)$ is evaluated by substituting a in place of x in the $P(x)$ expression.

WORKED EXAMPLE 2

a If $P(x) = 5x^3 + 2x^2 - 3x + 4$ calculate $P(-1)$.

b If $P(x) = ax^2 - 2x + 7$ and $P(4) = 31$, obtain the value of a.

THINK	WRITE
a Substitute −1 in place of x and evaluate.	**a** $P(x) = 5x^3 + 2x^2 - 3x + 4$ $P(-1) = 5(-1)^3 + 2(-1)^2 - 3(-1) + 4$ $= -5 + 2 + 3 + 4$ $= 4$
b 1 Find an expression for $P(4)$ by substituting 4 in place of x, and then simplify.	**b** $P(x) = ax^2 - 2x + 7$ $P(4) = a(4)^2 - 2(4) + 7$ $= 16a - 1$
2 Equate the expression for $P(4)$ with 31.	$P(4) = 31$ $\Rightarrow 16a - 1 = 31$
3 Solve for a.	$16a = 32$ $a = 2$

Identity of polynomials

If two polynomials are **identically equal** then the coefficients of like terms are equal. **Equating co-efficients** means that if $ax^2 + bx + c \equiv 2x^2 + 5x + 7$ then $a = 2$, $b = 5$ and $c = 7$. The identically equal symbol '≡' means the equality holds for all values of x. For convenience, however, we shall replace this symbol with the equality symbol '=' in working steps.

WORKED EXAMPLE 3

Calculate the values of a, b and c so that $x(x - 7) \equiv a(x - 1)^2 + b(x - 1) + c$.

THINK	WRITE
1 Expand each bracket and express both sides of the equality in expanded polynomial form.	$x(x - 7) \equiv a(x - 1)^2 + b(x - 1) + c$ $\therefore x^2 - 7x = a(x^2 - 2x + 1) + bx - b + c$ $\therefore x^2 - 7x = ax^2 + (-2a + b)x + (a - b + c)$
2 Equate the coefficients of like terms.	Equate the coefficients. $x^2 : 1 = a$ ....(1) $x : -7 = -2a + b$ ....(2) Constant: $0 = a - b + c$ ....(3)

3 Solve the system of simultaneous equations.	Since $a = 1$, substitute $a = 1$ into equation (2). $-7 = -2(1) + b$ $b = -5$ Substitute $a = 1$ and $b = -5$ into equation (3). $0 = 1 - (-5) + c$ $c = -6$
4 State the answer.	$\therefore a = 1, b = -5, c = -6$

Operations on polynomials

The addition, subtraction and multiplication of two or more polynomials results in another polynomial. For example, if $P(x) = x^2$ and $Q(x) = x^3 + x^2 - 1$, then $P(x) + Q(x) = x^3 + 2x^2 - 1$, a polynomial of degree 3; $P(x) - Q(x) = -x^3 + 1$, a polynomial of degree 3; and $P(x)Q(x) = x^5 + x^4 - x^2$, a polynomial of degree 5.

The operations of addition, subtraction and multiplication on polynomials have previously been encountered.

WORKED EXAMPLE 4

Given $P(x) = 3x^3 + 4x^2 + 2x + m$ and $Q(x) = 2x^2 + kx - 5$, find the values of m and k for which $2P(x) - 3Q(x) = 6x^3 + 2x^2 + 25x - 25$.

THINK	WRITE
1 Form a polynomial expression for $2P(x) - 3Q(x)$ by collecting like terms together.	$2P(x) - 3Q(x)$ $= 2(3x^3 + 4x^2 + 2x + m) - 3(2x^2 + kx - 5)$ $= 6x^3 + 2x^2 + (4 - 3k)x + (2m + 15)$
2 Equate the two expressions for $2P(x) - 3Q(x)$.	Hence, $6x^3 + 2x^2 + (4 - 3k)x + (2m + 15)$ $= 6x^3 + 2x^2 + 25x - 25$
3 Calculate the values of m and k.	Equate the coefficients of x. $4 - 3k = 25$ $k = -7$ Equate the constant terms. $2m + 15 = -25$ $m = -20$
4 State the answer.	Therefore $m = -20,\ k = -7$

Division of polynomials

There are several methods for performing the **division of polynomials** and CAS technology computes the division readily. Here, two 'by-hand' methods will be shown.

The inspection method for division

The division of one polynomial by another polynomial of equal or lesser degree can be carried out by expressing the numerator in terms of the denominator.

To divide $(x + 3)$ by $(x - 1)$, or to find $\frac{x+3}{x-1}$, write the numerator $x + 3$ as $(x - 1) + 1 + 3 = (x - 1) + 4$.

$$\frac{x+3}{x-1} = \frac{(x-1)+4}{x-1}$$

This expression can then be split into the sum of **partial fractions** as:

$$\begin{aligned}\frac{x+3}{x-1} &= \frac{(x-1)+4}{x-1} \\ &= \frac{x-1}{x-1} + \frac{4}{x-1} \\ &= 1 + \frac{4}{x-1}\end{aligned}$$

The division is in the form: $\dfrac{\textbf{dividend}}{\textbf{divisor}} = \textbf{quotient} + \dfrac{\textbf{remainder}}{\textbf{divisor}}$

In the language of division, when the dividend $(x + 3)$ is divided by the divisor $(x - 1)$ it gives a quotient of 1 and a remainder of 4. Note that from the division statement $\frac{x+3}{x-1} = 1 + \frac{4}{x-1}$ we can write $x + 3 = 1 \times (x - 1) + 4$.

This is similar to the division of integers. For example, 7 divided by 2 gives a quotient of 3 and a remainder of 1.

$$\frac{7}{2} = 3 + \frac{1}{2}$$

$$\therefore 7 = 3 \times 2 + 1$$

This inspection process of division can be extended, with practice, to division involving non-linear polynomials. It could be used to show that $\frac{x^2+4x+1}{x-1} = \frac{x(x-1)+5(x-1)+6}{x-1}$ and therefore $\frac{x^2+4x+1}{x-1} = x + 5 + \frac{6}{x-1}$.

This result can be verified by checking that $x^2 + 4x + 1 = (x + 5)(x - 1) + 6$.

WORKED EXAMPLE 5

a **Calculate the quotient and the remainder when $(x + 7)$ is divided by $(x + 5)$.**

b **Use the inspection method to find $\frac{3x-4}{x+2}$.**

THINK	WRITE
a **1** Write the division of the two polynomials as a fraction.	**a** $\frac{x+7}{x+5} = \frac{(x+5)-5+7}{x+5}$
2 Write the numerator in terms of the denominator.	$= \frac{(x+5)+2}{x+5}$

3 Split into partial fractions.

$$= \frac{(x+5)}{x+5} + \frac{2}{x+5}$$

4 Simplify.

$$= 1 + \frac{2}{x+5}$$

5 State the answer.

The quotient is 1 and the remainder is 2.

b 1 Express the numerator in terms of the denominator.

b The denominator is $(x + 2)$.
Since $3(x + 2) = 3x + 6$, the numerator is

$$3x - 4 = 3(x + 2) - 6 - 4$$

$$\therefore 3x - 4 = 3(x + 2) - 10$$

2 Split the given fraction into its partial fractions.

$$\frac{3x-4}{x+2} = \frac{3(x+2) - 10}{x+2}$$

$$= \frac{3(x+2)}{(x+2)} - \frac{10}{x+2}$$

3 Simplify and state the answer.

$$= 3 - \frac{10}{x+2}$$

$$\therefore \frac{3x-4}{x+2} = 3 - \frac{10}{x+2}$$

Algorithm for long division of polynomials

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Interactivity
Division of polynomials
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The inspection method of division is very efficient, particularly when the division involves only linear polynomials. However, it is also possible to use the long-division **algorithm** to divide polynomials.

The steps in the long-division algorithm are:

1. Divide the leading term of the divisor into the leading term of the dividend.

2. Multiply the divisor by this quotient.

3. Subtract the product from the dividend to form a remainder of lower degree.

4. Repeat this process until the degree of the remainder is lower than that of the divisor.

To illustrate this process, consider $(x^2 + 4x + 1)$ divided by $(x - 1)$. This is written as:

$$x - 1 \overline{\smash{\big)} x^2 + 4x + 1}$$

Step 1. The leading term of the divisor $(x - 1)$ is x; the leading term of the dividend $(x^2 + 4x + 1)$ is x^2. Dividing x into x^2, we get $\frac{x^2}{x} = x$. We write this quotient x on top of the long-division symbol.

$$\begin{array}{r} x \phantom{{}+4x+1} \\ x - 1 \overline{\smash{\big)} x^2 + 4x + 1} \end{array}$$

Step 2. The divisor $(x - 1)$ is multiplied by the quotient x to give $x(x - 1) = x^2 - x$. This product is written underneath the terms of $(x^2 + 4x + 1)$; like terms are placed in the same columns.

$$\begin{array}{r} x \\ x-1\,\overline{)\,x^2+4x+1} \\ x^2-x \end{array}$$

Step 3. $x^2 - x$ is subtracted from $(x^2 + 4x + 1)$. This cancels out the x^2 leading term to give $x^2 + 4x + 1 - (x^2 - x) = 5x + 1$.

$$\begin{array}{r} x \\ x-1\,\overline{)\,x^2+4x+1} \\ \underline{x^2-x} \\ 5x+1 \end{array}$$

The division statement, so far, would be $\frac{x^2 + 4x + 1}{x - 1} = x + \frac{5x + 1}{x - 1}$. This is incomplete since the remainder $(5x + 1)$ is not of a smaller degree than the divisor $(x - 1)$. The steps in the algorithm must be repeated with the same divisor $(x - 1)$ but with $(5x + 1)$ as the new dividend.

Continue the process.

Step 4. Divide the leading term of the divisor $(x - 1)$ into the leading term of $(5x + 1)$; this gives $\frac{5x}{x} = 5$. Write this as $+5$ on the top of the long-division symbol.

$$\begin{array}{r} x+5 \\ x-1\,\overline{)\,x^2+4x+1} \\ \underline{x^2-x} \\ 5x+1 \end{array}$$

Step 5. Multiply $(x - 1)$ by 5 and write the result underneath the terms of $(5x + 1)$.

$$\begin{array}{r} x+5 \\ x-1\,\overline{)\,x^2+4x+1} \\ \underline{x^2-x} \\ 5x+1 \\ 5x-5 \end{array}$$

Step 6. Subtract $(5x - 5)$ from $(5x + 1)$.

$$\begin{array}{rl} x+5 & \leftarrow \text{Quotient} \\ x-1\,\overline{)\,x^2+4x+1} & \\ \underline{x^2-x} & \\ 5x+1 & \\ \underline{5x-5} & \\ \underline{6} & \leftarrow \text{Remainder} \end{array}$$

The remainder is of lower degree than the divisor so no further division is possible and we have reached the end of the process.

Thus: $\frac{x^2 + 4x + 1}{x - 1} = x + 5 + \frac{6}{x - 1}$

This method can be chosen instead of the inspection method, or if the inspection method becomes harder to use.

WORKED EXAMPLE 6

a Given $P(x) = 4x^3 + 6x^2 - 5x + 9$, use the long-division method to divide $P(x)$ by $(x + 3)$ and state the quotient and the remainder.

b Use the long-division method to calculate the remainder when $\left(3x^3 + \frac{5}{3}x\right)$ is divided by $(5 + 3x)$.

THINK	WRITE
a 1 Set up the long division.	**a** $x+3\overline{)4x^3+6x^2-5x+9}$
2 The first stage of the division is to divide the leading term of the divisor into the leading term of the dividend.	$\begin{array}{r} 4x^2 \\ x+3\overline{)4x^3+6x^2-5x+9} \end{array}$
3 The second stage of the division is to multiply the result of the first stage by the divisor. Write this product placing like terms in the same columns.	$\begin{array}{r} 4x^2 \\ x+3\overline{)4x^3+6x^2-5x+9} \\ 4x^3+12x^2 \end{array}$
4 The third stage of the division is to subtract the result of the second stage from the dividend. This will yield an expression of lower degree than the original dividend.	$\begin{array}{r} 4x^2 \\ x+3\overline{)4x^3+6x^2-5x+9} \\ \underline{4x^3+12x^2} \\ -6x^2-5x+9 \end{array}$
5 The algorithm needs to be repeated. Divide the leading term of the divisor into the leading term of the newly formed dividend.	$\begin{array}{r} 4x^2-6x \\ x+3\overline{)4x^3+6x^2-5x+9} \\ \underline{4x^3+12x^2} \\ -6x^2-5x+9 \end{array}$
6 Multiply the result by the divisor and write this product keeping like terms in the same columns.	$\begin{array}{r} 4x^2-6x \\ x+3\overline{)4x^3+6x^2-5x+9} \\ \underline{4x^3+12x^2} \\ -6x^2-5x+9 \\ -6x^2-18x \end{array}$
7 Subtract to yield an expression of lower degree. *Note:* The degree of the expression obtained is still not less than the degree of the divisor so the algorithm will need to be repeated again.	$\begin{array}{r} 4x^2-6x \\ x+3\overline{)4x^3+6x^2-5x+9} \\ \underline{4x^3+12x^2} \\ -6x^2-5x+9 \\ \underline{-6x^2-18x} \\ 13x+9 \end{array}$

8 Divide the leading term of the divisor into the dividend obtained in the previous step.

$$
\begin{array}{r}
4x^2 - 6x + 13 \\
x + 3\,\big)\overline{4x^3 + 6x^2 - 5x + 9} \\
\underline{4x^3 + 12x^2 } \\
- 6x^2 - 5x + 9 \\
\underline{- 6x^2 - 18x } \\
13x + 9
\end{array}
$$

9 Multiply the result by the divisor and write this product keeping like terms in the same columns.

$$
\begin{array}{r}
4x^2 - 6x + 13 \\
x + 3\,\big)\overline{4x^3 + 6x^2 - 5x + 9} \\
\underline{4x^3 + 12x^2 } \\
- 6x^2 - 5x + 9 \\
\underline{- 6x^2 - 18x } \\
13x + 9 \\
13x + 39
\end{array}
$$

10 Subtract to yield an expression of lower degree.

Note: The term reached is a constant so its degree is less than that of the divisor. The division is complete.

$$
\begin{array}{r}
4x^2 - 6x + 13 \\
x + 3\,\big)\overline{4x^3 + 6x^2 - 5x + 9} \\
\underline{4x^3 + 12x^2 } \\
- 6x^2 - 5x + 9 \\
\underline{- 6x^2 - 18x } \\
13x + 9 \\
\underline{13x + 39} \\
\underline{- 30}
\end{array}
$$

11 State the answer.

$$\frac{4x^3 + 6x^2 - 5x + 9}{x + 3} = 4x^2 - 6x + 13 - \frac{30}{x + 3}$$

The quotient is $4x^2 - 6x + 13$ and the remainder is -30.

b 1 Set up the division, expressing both the divisor and the dividend in decreasing powers of x. This creates the columns for like terms.

b $3x^3 + \frac{5}{3}x = 3x^3 + 0x^2 + \frac{5}{3}x + 0$

$5 + 3x = 3x + 5$

$$3x + 5\,\big)\overline{3x^3 + 0x^2 + \tfrac{5}{3}x + 0}$$

2 Divide the leading term of the divisor into the leading term of the dividend, multiply this result by the divisor and then subtract this product from the dividend.

$$
\begin{array}{r}
x^2 \phantom{+ \tfrac{5}{3}x + 0} \\
3x + 5\,\big)\overline{3x^3 + 0x^2 + \tfrac{5}{3}x + 0} \\
\underline{3x^3 + 5x^2 \phantom{+ \tfrac{5}{3}x + 0}} \\
- 5x^2 + \tfrac{5}{3}x + 0
\end{array}
$$

3 Repeat the three steps of the algorithm using the dividend created by the first application of the algorithm.

$$\begin{array}{r}
x^2 - \frac{5}{3}x \phantom{+ \frac{5}{3}x + 0}\\
3x+5 \overline{\big) 3x^3 + 0x^2 + \frac{5}{3}x + 0} \\
\underline{3x^3 + 5x^2 \phantom{+ \frac{5}{3}x + 0}} \\
-5x^2 + \frac{5}{3}x + 0 \\
\underline{-5x^2 - \frac{25}{3}x } \\
10x + 0
\end{array}$$

4 Repeat the algorithm using the dividend created by the second application of the algorithm.

$$\begin{array}{r}
x^2 - \frac{5}{3}x + \frac{10}{3}\\
3x+5 \overline{\big) 3x^3 + 0x^2 + \frac{5}{3}x + 0} \\
\underline{3x^3 + 5x^2 \phantom{+ \frac{5}{3}x + 0}} \\
-5x^2 + \frac{5}{3}x + 0 \\
\underline{-5x^2 - \frac{25}{3}x } \\
10x + 0 \\
\underline{10x + \frac{50}{3}} \\
\underline{-\frac{50}{3}}
\end{array}$$

5 State the answer.

$$\frac{3x^3 + \frac{5}{3}x}{3x+5} = x^2 - \frac{5}{3}x + \frac{10}{3} + \frac{-\frac{50}{3}}{3x+5}$$

$$= x^2 - \frac{5}{3}x + \frac{10}{3} - \frac{50}{3(3x+5)}$$

The remainder is $-\frac{50}{3}$.

EXERCISE 4.2 Polynomials

PRACTISE

1 WE1 Select the polynomials from the following list of algebraic expressions and state their degree, the coefficient of the leading term, the constant term and the type of coefficients.

A $30x + 4x^5 - 2x^3 + 12$ **B** $\frac{3x^2}{5} - \frac{2}{x} + 1$ **C** $5.6 + 4x - 0.2x^2$

2 Write down a monic polynomial over R in the variable y for which the degree is 7, the coefficient of the y^2 term is $-\sqrt{2}$, the constant term is 4 and the polynomial contains four terms.

3 WE2 **a** If $P(x) = 7x^3 - 8x^2 - 4x - 1$ calculate $P(2)$.

b If $P(x) = 2x^2 + kx + 12$ and $P(-3) = 0$, find k.

4 If $P(x) = -2x^3 + 9x + m$ and $P(1) = 2P(-1)$, find m.

5 WE3 Calculate the values of a, b and c so that $(2x+1)(x-5) \equiv a(x+1)^2 + b(x+1) + c$.

6 Express $(x+2)^3$ in the form $px^2(x+1)+qx(x+2)+r(x+3)+t$.

7 **WE4** Given $P(x)=4x^3-px^2+8$ and $Q(x)=3x^2+qx-7$, find the values of p and q for which $P(x)+2Q(x)=4x^3+x^2-8x-6$.

8 If $P(x)=x^4+3x^2-7x+2$ and $Q(x)=x^3+x+1$, expand the product $P(x)Q(x)$ and state its degree.

9 **WE5** a Calculate the quotient and the remainder when $(x-12)$ is divided by $(x+3)$.

b Use the inspection method to find $\dfrac{4x+7}{2x+1}$.

10 Calculate the values of a, b and c if $3x^2-6x+5\equiv ax(x+2)+b(x+2)+c$ and hence express $\dfrac{3x^2-6x+5}{x+2}$ in the form $P(x)+\dfrac{d}{x+2}$ where $P(x)$ is a polynomial and $d\in R$.

11 **WE6** a Given $P(x)=2x^3-5x^2+8x+6$, divide $P(x)$ by $(x-2)$ and state the quotient and the remainder.

b Use the long-division method to calculate the remainder when (x^3+10) is divided by $(1-2x)$.

12 Obtain the quotient and remainder when $(x^4-3x^3+6x^2-7x+3)$ is divided by $(x-1)^2$.

CONSOLIDATE

13 Consider the following list of algebraic expressions.

A $3x^5+7x^4-\dfrac{x^3}{6}+x^2-8x+12$

B $9-5x^4+7x^2-\sqrt{5}x+x^3$

C $\sqrt{4x^5}-\sqrt{5}x^3+\sqrt{3}x-1$

D $2x^2(4x-9x^2)$

E $\dfrac{x^6}{10}-\dfrac{2x^8}{7}+\dfrac{5}{3x^2}-\dfrac{7x}{5}+\dfrac{4}{9}$

F $(4x^2+3+7x^3)^2$

a Select the polynomials from the list and for each of these polynomials state:
i its degree
ii the type of coefficients
iii the leading term
iv the constant term.

b Give a reason why each of the remaining expressions is not a polynomial.

14 Given $P(x)=2x^3+3x^2+x-6$, evaluate the following.

a $P(3)$
b $P(-2)$
c $P(1)$
d $P(0)$
e $P\left(-\frac{1}{2}\right)$
f $P(0.1)$

15 If $P(x)=x^2-7x+2$, obtain expressions for the following.

a $P(a)-P(-a)$
b $P(1+h)$
c $P(x+h)-P(x)$

16 a If $P(x)=4x^3+kx^2-10x-4$ and $P(1)=15$, obtain the value of k.

b If $Q(x)=ax^2-12x+7$ and $Q(-2)=-5$, obtain the value of a.

c If $P(x)=x^3-6x^2+nx+2$ and $P(2)=3P(-1)$, obtain the value of n.

d If $Q(x)=-x^2+bx+c$ and $Q(0)=5$ and $Q(5)=0$, obtain the values of b and c.

17 a If $3x^2+4x-7\equiv a(x+1)^2+b(x+1)+c$ calculate a, b and c.

b If $x^3+mx^2+nx+p\equiv(x-2)(x+3)(x-4)$ calculate m, n and p.

c If $x^2-14x+8\equiv a(x-b)^2+c$ calculate a, b and c and hence express $x^2-14x+8$ in the form $a(x-b)^2+c$.

d Express $4x^3+2x^2-7x+1$ in the form $ax^2(x+1)+bx(x+1)+c(x+1)+d$.

18 a If $P(x) = 2x^2 - 7x - 11$ and $Q(x) = 3x^4 + 2x^2 + 1$, find, expressing the terms in descending powers of x:

i $Q(x) - P(x)$ ii $3P(x) + 2Q(x)$ iii $P(x)Q(x)$

b If $P(x)$ is a polynomial of degree m and $Q(x)$ is a polynomial of degree n where $m > n$, state the degree of:

i $P(x) + Q(x)$ ii $P(x) - Q(x)$ iii $P(x)Q(x)$

19 a Determine the values of a, b, p and q if $P(x) = x^3 - 3x^2 + px - 2$, $Q(x) = ax^3 + bx^2 + 3x - 2a$ and $2P(x) - Q(x) = 5(x^3 - x^2 + x + q)$.

b i Express $4x^4 + 12x^3 + 13x^2 + 6x + 1$ in the form $(ax^2 + bx + c)^2$ where $a > 0$.

ii Hence state a square root of $4x^4 + 12x^3 + 13x^2 + 6x + 1$.

20 $P(x) = x^4 + kx^2 + n^2$, $Q(x) = x^2 + mx + n$ and the product $P(x)Q(x) = x^6 - 5x^5 - 7x^4 + 65x^3 - 42x^2 - 180x + 216$.

a Calculate k, m and n.

b Obtain the linear factors of $P(x)Q(x) = x^6 - 5x^5 - 7x^4 + 65x^3 - 42x^2 - 180x + 216$.

21 Specify the quotient and the remainder when:

a $(x + 7)$ is divided by $(x - 2)$

b $(8x + 5)$ is divided by $(2x + 1)$

c $(x^2 + 6x - 17)$ is divided by $(x - 1)$

d $(2x^2 - 8x + 3)$ is divided by $(x + 2)$

e $(x^3 + 2x^2 - 3x + 5)$ is divided by $(x - 3)$

f $(x^3 - 8x^2 + 9x - 2)$ is divided by $(x - 1)$.

22 Perform the following divisions.

a $(8x^3 + 6x^2 - 5x + 15)$ divided by $(1 + 2x)$

b $(4x^3 + x + 5)$ divided by $(2x - 3)$

c $(x^3 + 6x^2 + 6x - 12) \div (x + 6)$

d $(2 + x^3) \div (x + 1)$

e $\dfrac{x^4 + x^3 - x^2 + 2x + 5}{x^2 - 1}$

f $\dfrac{x(7 - 2x^2)}{(x + 2)(x - 3)}$

MASTER

23 a Use CAS technology to divide $(4x^3 - 7x^2 + 5x + 2)$ by $(2x + 3)$.

b State the remainder and the quotient.

c Evaluate the dividend if $x = -\frac{3}{2}$.

d Evaluate the divisor if $x = -\frac{3}{2}$.

24 a Define $P(x) = 3x^3 + 6x^2 - 8x - 10$ and $Q(x) = -x^3 + ax - 6$.

b Evaluate $P(-4) + P(3) - P\left(\dfrac{2}{3}\right)$.

c Give an algebraic expression for $P(2n) + 24Q(n)$.

d Obtain the value of a so that $Q(-2) = -16$.

4.3 The remainder and factor theorems

The remainder obtained when dividing $P(x)$ by the linear divisor $(x - a)$ is of interest because if the remainder is zero, then the divisor must be a linear factor of the polynomial. To pursue this interest we need to be able to calculate the remainder quickly without the need to do a lengthy division.

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The remainder theorem

The actual division, as we know, will result in a quotient and a remainder. This is expressed in the division statement $\dfrac{P(x)}{x-a} = \text{quotient} + \dfrac{\text{remainder}}{x-a}$.

Since $(x - a)$ is linear, the remainder will be some constant term independent of x.

From the division statement it follows that:

$$P(x) = (x - a) \times \text{quotient} + \text{remainder}$$

If we let $x = a$, this makes $(x - a)$ equal to zero and the statement becomes:

$$P(a) = 0 \times \text{quotient} + \text{remainder}$$

Therefore:

$$P(a) = \text{remainder}$$

This result is known as the **remainder theorem**.

> **If a polynomial $P(x)$ is divided by $(x - a)$ then the remainder is $P(a)$.**

Note that:

- If $P(x)$ is divided by $(x + a)$ then the remainder would be $P(-a)$ since replacing x by $-a$ would make the $(x + a)$ term equal zero.
- If $P(x)$ is divided by $(ax + b)$ then the remainder would be $P\left(-\dfrac{b}{a}\right)$ since replacing x by $-\dfrac{b}{a}$ would make the $(ax + b)$ term equal zero.

WORKED EXAMPLE 7

Find the remainder when $P(x) = x^3 - 3x^2 - 2x + 9$ is divided by:

a $x - 2$ **b** $2x + 1$.

THINK	WRITE
a 1 What value of x will make the divisor zero?	**a** $(x - 2) = 0 \Rightarrow x = 2$
2 Write an expression for the remainder.	$P(x) = x^3 - 3x^2 - 2x + 9$ Remainder is $P(2)$.
3 Evaluate to obtain the remainder.	$P(2) = (2)^3 - 3(2)^2 - 2(2) + 9$ $= 1$ The remainder is 1.
b 1 Find the value of x which makes the divisor zero.	**b** $(2x + 1) = 0 \Rightarrow x = -\frac{1}{2}$

2 Write an expression for the remainder and evaluate it.

Remainder is $P\left(-\frac{1}{2}\right)$.

$$
\begin{aligned}
P\left(-\tfrac{1}{2}\right) &= \left(-\tfrac{1}{2}\right)^3 - 3\left(-\tfrac{1}{2}\right)^2 - 2\left(-\tfrac{1}{2}\right) + 9 \\
&= -\tfrac{1}{8} - \tfrac{3}{4} + 1 + 9 \\
&= 9\tfrac{1}{8}
\end{aligned}
$$

Remainder is $9\frac{1}{8}$.

The factor theorem

We know 4 is a factor of 12 because it divides 12 exactly, leaving no remainder. Similarly, if the division of a polynomial $P(x)$ by $(x - a)$ leaves no remainder, then the divisor $(x - a)$ must be a factor of the polynomial $P(x)$.

$$P(x) = (x - a) \times \text{quotient} + \text{remainder}$$

If the remainder is zero, then $P(x) = (x - a) \times$ quotient.

Therefore $(x - a)$ is a factor of $P(x)$.

This is known as the **factor theorem**.

If $P(x)$ is a polynomial and $P(a) = 0$ then $(x - a)$ is a factor of $P(x)$.

Conversely, if $(x - a)$ is a factor of a polynomial $P(x)$ then $P(a) = 0$.

a is a **zero** of the polynomial.

It also follows from the remainder theorem that if $P\left(-\dfrac{b}{a}\right) = 0$, then $(ax + b)$ is a factor of $P(x)$ and $\dfrac{-b}{a}$ is a zero of the polynomial.

WORKED EXAMPLE 8

a Show that $(x + 3)$ is a factor of $Q(x) = 4x^4 + 4x^3 - 25x^2 - x + 6$.

b Determine the polynomial $P(x) = ax^3 + bx + 2$ which leaves a remainder of -9 when divided by $(x - 1)$ and is exactly divisible by $(x + 2)$.

THINK	WRITE
a 1 State how the remainder can be calculated when $Q(x)$ is divided by the given linear expression.	**a** $Q(x) = 4x^4 + 4x^3 - 25x^2 - x + 6$ When $Q(x)$ is divided by $(x + 3)$, the remainder equals $Q(-3)$.
2 Evaluate the remainder.	$Q(-3) = 4(-3)^4 + 4(-3)^3 - 25(-3)^2 - (-3) + 6$ $= 324 - 108 - 225 + 3 + 6$ $= 0$
3 It is important to explain in the answer why the given linear expression is a factor.	Since $Q(-3) = 0$, $(x + 3)$ is a factor of $Q(x)$.

b **1** Express the given information in terms of the remainders.

b $P(x) = ax^3 + bx + 2$

Dividing by $(x - 1)$ leaves a remainder of -9.

$\Rightarrow P(1) = -9$

Dividing by $(x + 2)$ leaves a remainder of 0.

$\Rightarrow P(-2) = 0$

2 Set up a pair of simultaneous equations in a and b.

$$P(1) = a + b + 2$$

$$a + b + 2 = -9$$

$$\therefore \quad a + b = -11 \ldots\ldots\ldots\ldots(1)$$

$$P(-2) = -8a - 2b + 2$$

$$-8a - 2b + 2 = 0$$

$$\therefore \quad 4a + b = 1 \ldots\ldots\ldots\ldots(2)$$

3 Solve the simultaneous equations.

$$a + b = -11 \ldots..(1)$$

$$4a + b = 1 \ldots\ldots..(2)$$

Equation (2) – equation (1):

$3a = 12$

$a = 4$

Substitute $a = 4$ into equation (1).

$4 + b = -11$

$b = -15$

4 Write the answer.

$\therefore P(x) = 4x^3 - 15x + 2$

Factorising polynomials

When factorising a cubic or higher-degree polynomial, the first step should be to check if any of the standard methods for factorising can be used. In particular, look for a common further on, then look to see if a grouping technique can produce either a common linear factor or a difference of two squares. If the standard techniques do not work then the remainder and factor theorems can be used to factorise, since the zeros of a polynomial enable linear factors to be formed.

Cubic polynomials may have up to three zeros and therefore up to three linear factors. For example, a cubic polynomial $P(x)$ for which it is known that $P(1) = 0$, $P(2) = 0$ and $P(-4) = 0$, has 3 zeros: $x = 1$, $x = 2$ and $x = -4$. From these, its three linear factors $(x - 1)$, $(x - 2)$ and $(x + 4)$ are formed.

Integer zeros of a polynomial may be found through a trial-and-error process where factors of the polynomial's constant term are tested systematically. For the polynomial $P(x) = x^3 + x^2 - 10x + 8$, the constant term is 8 so the possibilities to test are $1, -1, 2, -2, 4, -4, 8$ and -8. This is a special case of what is known as the **rational root theorem**, where if a polynomial with integer coefficients has a rational zero $\frac{p}{q}$ then its constant term is divisible by p and its leading coefficient is divisible by q. If the polynomial is not monic, then test the factors of its constant term, or test the factors of its constant term divided by factors of its leading coefficient.

In practice, not all of the zeros need to be, nor necessarily can be, found through trial and error. For a cubic polynomial it is sufficient to find one zero by trial and error and form its corresponding linear factor using the factor theorem. Dividing this linear factor into the cubic polynomial gives a quadratic quotient and zero remainder, so the quadratic quotient is also a factor. The standard techniques for factorising quadratics can then be applied.

For the division step, long division could be used; however, it is more efficient to use a division method based on equating coefficients. With practice, this can usually be done by inspection. To illustrate, $P(x) = x^3 + x^2 - 10x + 8$ has a zero of $x = 1$ since $P(1) = 0$. Therefore $(x - 1)$ is a linear factor and $P(x) = (x - 1)(ax^2 + bx + c)$.

Note that the x^3 term of $(x - 1)(ax^2 + bx + c)$ can only be formed by the product of the x term in the first bracket with the x^2 term in the second bracket; likewise, the constant term of $(x - 1)(ax^2 + bx + c)$ can only be formed by the product of the constant terms in the first and second brackets.

The coefficients of the quadratic factor are found by equating coefficients of like terms in $x^3 + x^2 - 10x + 8 = (x - 1)(ax^2 + bx + c)$.

For x^3: $1 = a$

For constants: $8 = -c \Rightarrow c = -8$

This gives $x^3 + x^2 - 10x + 8 = (x - 1)(x^2 + bx - 8)$ which can usually be written down immediately.

For the right-hand expression $(x - 1)(x^2 + bx - 8)$, the coefficient of x^2 is formed after a little more thought. An x^2 term can be formed by the product of the x term in the first bracket with the x term in the second bracket and also by the product of the constant term in the first bracket with the x^2 term in the second bracket.

$$x^3 + x^2 - 10x + 8 = (x - 1)(x^2 + bx - 8)$$

Equating coefficients of x^2: $1 = b - 1$

$\therefore b = 2$

If preferred, the coefficients of x could be equated or used as check.

It follows that:

$$\begin{aligned} P(x) &= (x - 1)(x^2 + 2x - 8) \\ &= (x - 1)(x - 2)(x + 4) \end{aligned}$$

WORKED EXAMPLE 9

a Factorise $P(x) = x^3 - 2x^2 - 5x + 6$.

b Given that $(x + 1)$ and $(5 - 2x)$ are factors of $P(x) = -4x^3 + 4x^2 + 13x + 5$, completely factorise $P(x)$.

THINK	WRITE
a 1 The polynomial does not factorise by a grouping technique so a zero needs to be found. The factors of the constant term are potential zeros.	**a** $P(x) = x^3 - 2x^2 - 5x + 6$ The factors of 6 are $\pm1, \pm2, \pm3$ and ±6.
2 Use the remainder theorem to test systematically until a zero is obtained. Then use the factor theorem to state the corresponding linear factor.	$P(1) = 1 - 2 - 5 + 6$ $= 0$ $\therefore (x - 1)$ is a factor.

3 Express the polynomial in terms of a product of the linear factor and a general quadratic factor.

$\therefore x^3 - 2x^2 - 5x + 6 = (x - 1)(ax^2 + bx + c)$

4 State the values of a and c.

For the coefficient of x^3 to be 1, $a = 1$.
For the constant term to be 6, $c = -6$.
$\therefore x^3 - 2x^2 - 5x + 6 = (x - 1)(x^2 + bx - 6)$

5 Calculate the value of b.

Equating the coefficients of x^2 gives:

$$x^3 - 2x^2 - 5x + 6 = (x - 1)(x^2 + bx - 6)$$

$-2 = b - 1$

$b = -1$

$\therefore x^3 - 2x^2 - 5x + 6 = (x - 1)(x^2 - x - 6)$

6 Factorise the quadratic factor so the polynomial is fully factorised into its linear factors.

Hence,

$$\begin{aligned} P(x) &= x^3 - 2x^2 - 5x + 6 \\ &= (x - 1)(x^2 - x - 6) \\ &= (x - 1)(x - 3)(x + 2) \end{aligned}$$

b 1 Multiply the two given linear factors to form the quadratic factor.

b $P(x) = -4x^3 + 4x^2 + 13x + 5$
Since $(x + 1)$ and $(5 - 2x)$ are factors, then $(x + 1)(5 - 2x) = -2x^2 + 3x + 5$ is a quadratic factor.

2 Express the polynomial as a product of the quadratic factor and a general linear factor.

The remaining factor is linear.

$$\begin{aligned} \therefore P(x) &= (x + 1)(5 - 2x)(ax + b) \\ &= (-2x^2 + 3x + 5)(ax + b) \end{aligned}$$

3 Find a and b.

$-4x^3 + 4x^2 + 13x + 5 = (-2x^2 + 3x + 5)(ax + b)$
Equating coefficients of x^3 gives:

$-4 = -2a$

$\therefore a = 2$

Equating constants gives:

$5 = 5b$

$\therefore b = 1$

4 State the answer.

$$\begin{aligned} -4x^3 + 4x^2 + 13x + 5 &= (-2x^2 + 3x + 5)(2x + 1) \\ &= (x + 1)(5 - 2x)(2x + 1) \\ \therefore P(x) &= (x + 1)(5 - 2x)(2x + 1) \end{aligned}$$

Polynomial equations

If a polynomial is expressed in factorised form, then the polynomial equation can be solved using the null factor law.

$$(x - a)(x - b)(x - c) = 0$$
$$\therefore (x - a) = 0, (x - b) = 0, (x - c) = 0$$
$$\therefore x = a, x = b \text{ or } x = c$$

$x = a$, $x = b$ and $x = c$ are called the roots or the solutions to the equation $P(x) = 0$.

The factor theorem may be required to express the polynomial in factorised form.

WORKED EXAMPLE 10 **Solve the equation $3x^3 + 4x^2 = 17x + 6$.**

THINK	WRITE
1 Rearrange the equation so one side is zero.	$3x^3 + 4x^2 = 17x + 6$ $3x^3 + 4x^2 - 17x - 6 = 0$
2 Since the polynomial does not factorise by grouping techniques, use the remainder theorem to find a zero and the factor theorem to form the corresponding linear factor. *Note:* It is simpler to test for integer zeros first.	Let $P(x) = 3x^3 + 4x^2 - 17x - 6$. Test factors of the constant term: $P(1) \neq 0$ $P(-1) \neq 0$ $P(2) = 3(2)^3 + 4(2)^2 - 17(2) - 6$ $= 24 + 16 - 34 - 6$ $= 0$ Therefore $(x - 2)$ is a factor.
3 Express the polynomial as a product of the linear factor and a general quadratic factor.	$3x^3 + 4x^2 - 17x - 6 = (x - 2)(ax^2 + bx + c)$
4 Find and substitute the values of a and c.	$\therefore 3x^3 + 4x^2 - 17x - 6 = (x - 2)(3x^2 + bx + 3)$
5 Calculate b.	Equate the coefficients of x^2: $4 = b - 6$ $b = 10$
6 Completely factorise the polynomial.	$3x^3 + 4x^2 - 17x - 6 = (x - 2)(3x^2 + 10x + 3)$ $= (x - 2)(3x + 1)(x + 3)$
7 Solve the equation.	The equation $3x^3 + 4x^2 - 17x - 6 = 0$ becomes: $(x - 2)(3x + 1)(x + 3) = 0$ $x - 2 = 0,\ 3x + 1 = 0,\ x + 1 = 0$ $x = 2,\ x = -\frac{1}{3},\ x = -3$

EXERCISE 4.3 The remainder and factor theorems

PRACTISE

1 WE7 Find the remainder when $P(x) = x^3 + 4x^2 - 3x + 5$ is divided by:

a $x + 2$ b $2x - 1$

2 If $x^3 - kx^2 + 4x + 8$ leaves a remainder of 29 when it is divided by $(x - 3)$, find the value of k.

3 WE8 a Show that $(x - 2)$ is a factor of $Q(x) = 4x^4 + 4x^3 - 25x^2 - x + 6$.

b Determine the polynomial $P(x) = 3x^3 + ax^2 + bx - 2$ which leaves a remainder of -22 when divided by $(x + 1)$ and is exactly divisible by $(x - 1)$.

4 Given $(2x + a)$ is a factor of $12x^2 - 4x + a$, obtain the value(s) of a.

5 WE9 a Factorise $P(x) = x^3 + 3x^2 - 13x - 15$.

b Given that $(x + 1)$ and $(3x + 2)$ are factors of $P(x) = 12x^3 + 41x^2 + 43x + 14$, completely factorise $P(x)$.

6 Given the zeros of the polynomial $P(x) = 12x^3 + 8x^2 - 3x - 2$ are not integers, use the rational root theorem to calculate one zero and hence find the three linear factors of the polynomial.

7 WE10 Solve the equation $6x^3 + 13x^2 = 2 - x$.

8 Solve for x, $2x^4 + 3x^3 - 8x^2 - 12x = 0$.

CONSOLIDATE

9 Calculate the remainder without actual division when:

a $x^3 - 4x^2 - 5x + 3$ is divided by $(x - 1)$

b $6x^3 + 7x^2 + x + 2$ is divided by $(x + 1)$

c $-2x^3 + 2x^2 - x - 1$ is divided by $(x - 4)$

d $x^3 + x^2 + x - 10$ is divided by $(2x + 1)$

e $27x^3 - 9x^2 - 9x + 2$ is divided by $(3x - 2)$

f $4x^4 - 5x^3 + 2x^2 - 7x + 8$ is divided by $(x - 2)$.

10 a When $P(x) = x^3 - 2x^2 + ax + 7$ is divided by $(x + 2)$, the remainder is 11. Find the value of a.

b If $P(x) = 4 - x^2 + 5x^3 - bx^4$ is exactly divisible by $(x - 1)$, find the value of b.

c If $2x^3 + cx^2 + 5x + 8$ has a remainder of 6 when divided by $(2x - 1)$, find the value of c.

d Given that each of $x^3 + 3x^2 - 4x + d$ and $x^4 - 9x^2 - 7$ have the same remainder when divided by $(x + 3)$, find the value of d.

11 a Calculate the values of a and b for which $Q(x) = ax^3 + 4x^2 + bx + 1$ leaves a remainder of 39 when divided by $(x - 2)$, given $(x + 1)$ is a factor of $Q(x)$.

b Dividing $P(x) = \frac{1}{3}x^3 + mx^2 + nx + 2$ by either $(x - 3)$ or $(x + 3)$ results in the same remainder. If that remainder is three times the remainder left when $P(x)$ is divided by $(x - 1)$, determine the values of m and n.

12 a A monic polynomial of degree 3 in x has zeros of 5, 9 and –2. Express this polynomial in:

i factorised form ii expanded form.

b A polynomial of degree 3 has a leading term with coefficient -2 and zeros of $-4, -1$ and $\frac{1}{2}$. Express this polynomial in:

i factorised form ii expanded form.

13 a Given $(x - 4)$ is a factor of $P(x) = x^3 - x^2 - 10x - 8$, fully factorise $P(x)$.

b Given $(x + 12)$ is a factor of $P(x) = 3x^3 + 40x^2 + 49x + 12$, fully factorise $P(x)$.

c Given $(5x + 1)$ is a factor of $P(x) = 20x^3 + 44x^2 + 23x + 3$, fully factorise $P(x)$.

d Given $(4x - 3)$ is a factor of $P(x) = -16x^3 + 12x^2 + 100x - 75$, fully factorise $P(x)$.

e Given $(8x - 11)$ and $(x - 3)$ are factors of $P(x) = -8x^3 + 59x^2 - 138x + 99$, fully factorise $P(x)$.

f Given $(3x - 5)$ is a factor of $P(x) = 9x^3 - 75x^2 + 175x - 125$, fully factorise $P(x)$.

14 Factorise the following:

a $x^3 + 5x^2 + 2x - 8$

b $x^3 + 10x^2 + 31x + 30$

c $2x^3 - 13x^2 + 13x + 10$

d $-18x^3 + 9x^2 + 23x - 4$

e $x^3 - 7x + 6$

f $x^3 + x^2 - 49x - 49$

15 a The polynomial $24x^3 + 34x^2 + x - 5$ has three zeros, none of which are integers. Calculate the three zeros and express the polynomial as the product of its three linear factors.

b The polynomial $P(x) = 8x^3 + mx^2 + 13x + 5$ has a zero of $\frac{5}{2}$.

i State a linear factor of the polynomial.

ii Fully factorise the polynomial.

iii Calculate the value of m.

c **i** Factorise the polynomials $P(x) = x^3 - 12x^2 + 48x - 64$ and $Q(x) = x^3 - 64$.

ii Hence, show that $\dfrac{P(x)}{Q(x)} = 1 - \dfrac{12x}{x^2 + 4x + 16}$.

d A cubic polynomial $P(x) = x^3 + bx^2 + cx + d$ has integer coefficients and $P(0) = 9$. Two of its linear factors are $(x - \sqrt{3})$ and $(x + \sqrt{3})$. Calculate the third linear factor and obtain the values of b, c and d.

16 Solve the following equations for x.

a $(x + 4)(x - 3)(x + 5) = 0$

b $2(x - 7)(3x + 5)(x - 9) = 0$

c $x^3 - 13x^2 + 34x + 48 = 0$

d $2x^3 + 7x^2 = 9$

e $3x^2(3x + 1) = 4(2x + 1)$

f $8x^4 + 158x^3 - 46x^2 - 120x = 0$

17 a Show that $(x - 2)$ is a factor of $P(x) = x^3 + 6x^2 - 7x - 18$ and hence fully factorise $P(x)$ over R.

b Show that $(3x - 1)$ is the only real linear factor of $3x^3 + 5x^2 + 10x - 4$.

c Show that $(2x^2 - 11x + 5)$ is a factor of $2x^3 - 21x^2 + 60x - 25$ and hence calculate the roots of the equation $2x^3 - 21x^2 + 60x - 25 = 0$.

18 a If $(x^2 - 4)$ divides $P(x) = 5x^3 + kx^2 - 20x - 36$ exactly, fully factorise $P(x)$ and hence obtain the value of k.

b If $x = a$ is a solution to the equation $ax^2 - 5ax + 4(2a - 1) = 0$, find possible values for a.

c The polynomials $P(x) = x^3 + ax^2 + bx - 3$ and $Q(x) = x^3 + bx^2 + 3ax - 9$ have a common factor of $(x + a)$. Calculate a and b and fully factorise each polynomial.

d $(x + a)^2$ is a repeated linear factor of the polynomial $P(x) = x^3 + px^2 + 15x + a^2$. Show there are two possible polynomials satisfying this information and, for each, calculate the values of x which give the roots of the equation $x^3 + px^2 + 15x + a^2 = 0$.

MASTER

19 Specify the remainder when $(9 + 19x - 2x^2 - 7x^3)$ is divided by $(x - \sqrt{2} + 1)$.

20 Solve the equation $10x^3 - 5x^2 + 21x + 12 = 0$ expressing the values of x to 4 decimal places.

4.4 Graphs of cubic polynomials

The graph of the general cubic polynomial has an equation of the form $y = ax^3 + bx^2 + cx + d$, where a, b, c and d are real constants and $a \neq 0$. Since a cubic polynomial may have up to three linear factors, its graph may have up to three x-intercepts. The shape of its graph is affected by the number of x-intercepts.

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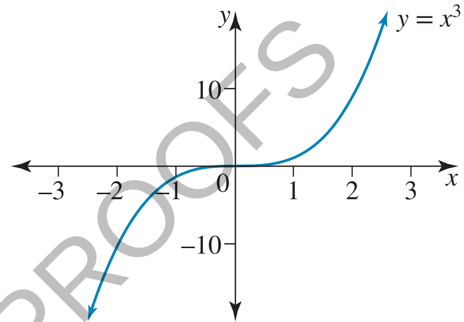
The graph of $y = x^3$ and transformations

The graph of the simplest cubic polynomial has the equation $y = x^3$.



The 'maxi–min' point at the origin is sometimes referred to as a 'saddle point'. Formally, it is called a **stationary point of inflection** (or inflexion as a variation of spelling). It is a key feature of this cubic graph.

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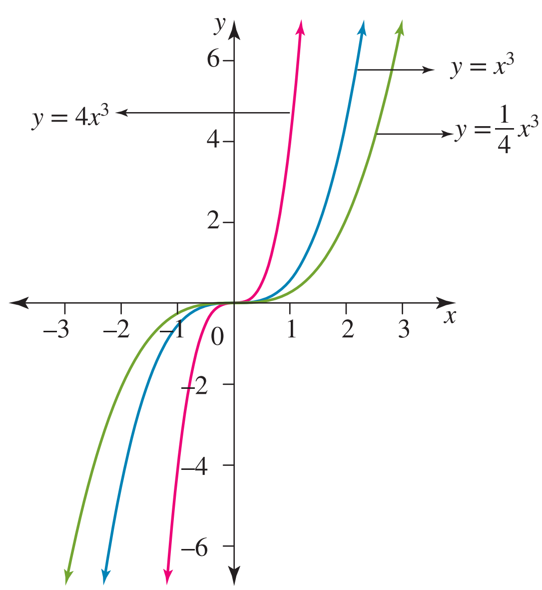
Key features of the graph of $y = x^3$:

- $(0, 0)$ is a stationary point of inflection.
- The shape of the graph changes from concave down to concave up at the stationary point of inflection.
- There is only one x-intercept.
- As the values of x become very large positive, the behaviour of the graph shows its y-values become increasingly large positive also. This means that as $x \to \infty$, $y \to \infty$. This is read as 'as x approaches infinity, y approaches infinity'.
- As the values of x become very large negative, the behaviour of the graph shows its y-values become increasingly large negative. This means that as $x \to -\infty$, $y \to -\infty$.
- The graph starts from below the x-axis and increases as x increases.

Once the basic shape is known, the graph can be dilated, reflected and translated in much the same way as the parabola $y = x^2$.

Dilation

The graph of $y = 4x^3$ will be narrower than the graph of $y = x^3$ due to the dilation factor of 4 from the x-axis. Similarly, the graph of $y = \frac{1}{4}x^3$ will be wider than the graph of $y = x^3$ due to the dilation factor of $\frac{1}{4}$ from the x-axis.

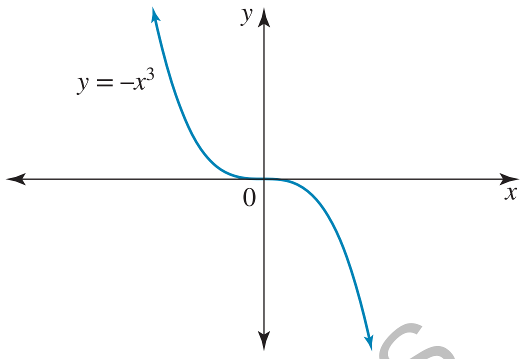


Reflection

The graph of $y = -x^3$ is the reflection of the graph of $y = x^3$ in the x-axis.

For the graph of $y = -x^3$ note that:

- as $x \to \infty$, $y \to -\infty$ and as $x \to -\infty$, $y \to \infty$
- the graph starts from above the x-axis and decreases as x increases
- at (0, 0), the stationary point of inflection, the graph changes from concave up to concave down.

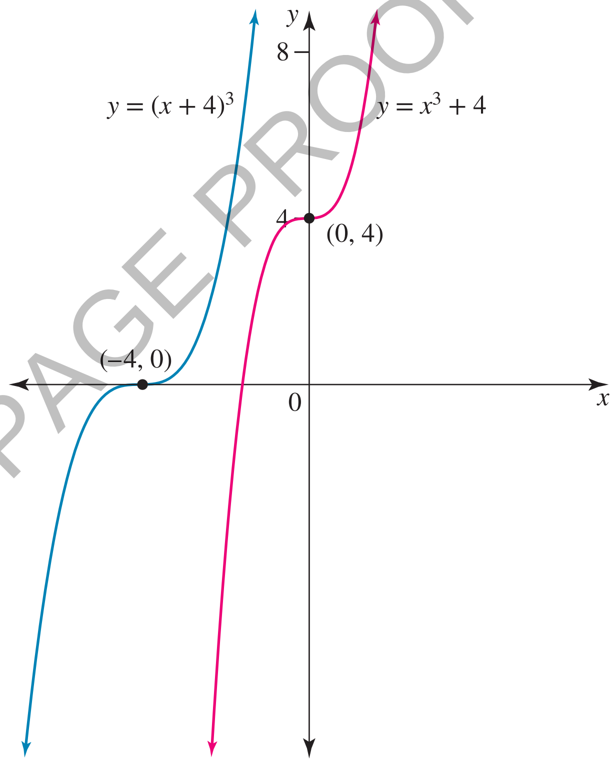


Translation

The graph of $y = x^3 + 4$ is obtained when the graph of $y = x^3$ is translated vertically upwards by 4 units. The stationary point of inflection is at the point (0, 4).

The graph of $y = (x + 4)^3$ is obtained when the graph of $y = x^3$ is translated horizontally 4 units to the left. The stationary point of inflection is at the point (−4, 0).

The transformations from the basic parabola $y = x^2$ are recognisable from the equation $y = a(x - h)^2 + k$, and the equation of the graph of $y = x^3$ can be transformed to a similar form.



The key features of the graph of $y = a(x - h)^3 + k$ are:

- **stationary point of inflection at (h, k)**
- **change of concavity at the stationary point of inflection**
- **if $a > 0$, the graph starts below the x-axis and increases, like $y = x^3$**
- **if $a < 0$, the graph starts above the x-axis and decreases, like $y = -x^3$**
- **the one x-intercept is found by solving $a(x - h)^3 + k = 0$**
- **the y-intercept is found by substituting $x = 0$.**

WORKED EXAMPLE 11

Sketch

a $y = (x + 1)^3 + 8$

b $y = 6 - \frac{1}{2}(x - 2)^3$

THINK	WRITE
a 1 State the point of inflection.	a $y = (x + 1)^3 + 8$ Point of inflection is $(-1, 8)$.

2 Calculate the y-intercept.

y-intercept: let $x = 0$

$y = (1)^3 + 8$

$= 9$

$\Rightarrow (0, 9)$

3 Calculate the x-intercept.

x-intercept: let $y = 0$

$(x + 1)^3 + 8 = 0$

$(x + 1)^3 = -8$

Take the cube root of both sides:

$x + 1 = \sqrt[3]{-8}$

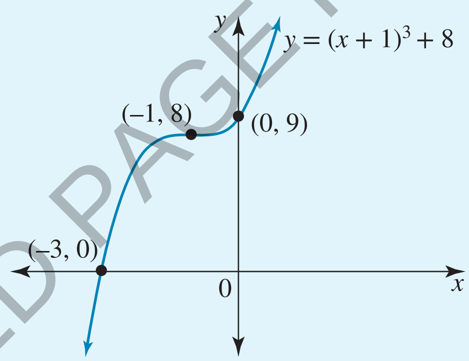
$x + 1 = -2$

$x = -3$

$\Rightarrow (-3, 0)$

4 Sketch the graph. Label the key points and ensure the graph changes concavity at the point of inflection.

The coefficient of x^3 is positive so the graph starts below the x-axis and increases.



b 1 Rearrange the equation to the $y = a(x - h)^3 + k$ form and state the point of inflection.

b $y = 6 - \frac{1}{2}(x - 2)^3$

$= -\frac{1}{2}(x - 2)^3 + 6$

Point of inflection: $(2, 6)$

2 Calculate the y-intercept.

y-intercept: let $x = 0$

$y = -\dfrac{1}{2}(-2)^3 + 6$

$= 10$

$\Rightarrow (0, 10)$

3 Calculate the x-intercept.

Note: A decimal approximation helps locate the point.

x-intercept: let $y = 0$

$-\frac{1}{2}(x - 2)^3 + 6 = 0$

$\frac{1}{2}(x - 2)^3 = 6$

$(x - 2)^3 = 12$

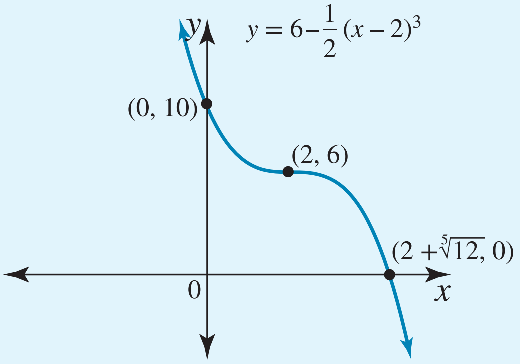
$x - 2 = \sqrt[3]{12}$

$x = 2 + \sqrt[3]{12}$

$\Rightarrow (2 + \sqrt[3]{12}, 0) \approx (4.3, 0)$

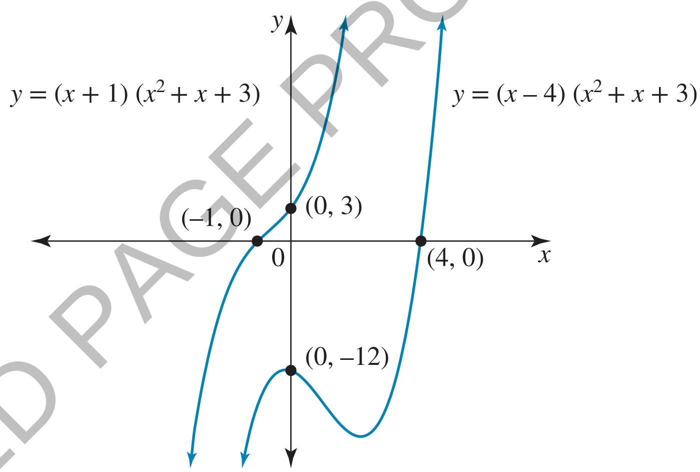
4 Sketch the graph showing all key features.

$a < 0$ so the graph starts above the x-axis and decreases.



Cubic graphs with one x-intercept but no stationary point of inflection

There are cubic graphs which have one x-intercept but no stationary point of inflection. The equations of such cubic graphs cannot be expressed in the form $y = a(x - h)^3 + k$. Their equations can be expressed as the product of a linear factor and a quadratic factor which is **irreducible**, meaning the quadratic has no real factors.



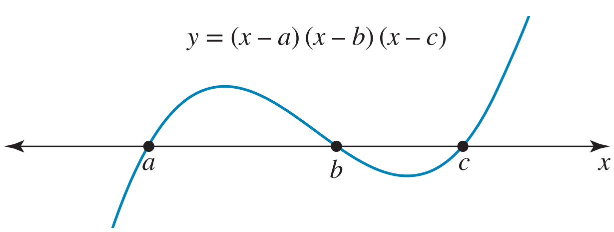
Technology is often required to sketch such graphs. Two examples, $y = (x + 1)(x^2 + x + 3)$ and $y = (x - 4)(x^2 + x + 3)$, are shown in the diagram. Each has a linear factor and the discriminant of the quadratic factor $x^2 + x + 3$ is negative; this means it cannot be further factorised over R.

Both graphs maintain the **long-term behavior** exhibited by all cubics with a positive leading-term coefficient; that is, as $x \to \infty$, $y \to \infty$ and as $x \to -\infty$, $y \to -\infty$. Every cubic polynomial must have at least one linear factor in order to maintain this long-term behaviour.

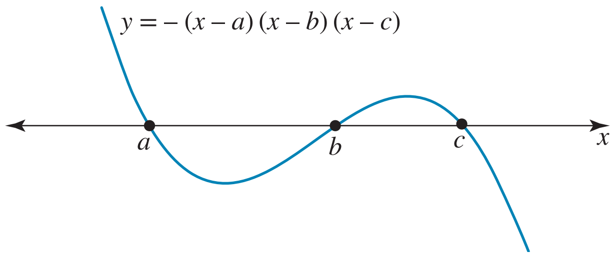
Cubic graphs with three x-intercepts

For the graph of a cubic polynomial to have three x-intercepts, the polynomial must have three distinct linear factors. This is the case when the cubic polynomial expressed as the product of a linear factor and a quadratic factor is such that the quadratic factor has two distinct linear factors.

This means that the graph of a monic cubic with an equation of the form $y = (x - a)(x - b)(x - c)$ where $a, b, c \in R$ and $a < b < c$ will have the shape of the graph shown.



If the graph is reflected in the x-axis, its equation is of the form $y = -(x - a)(x - b)(x - c)$ and the shape of its graph satisfies the long-term behaviour that as $x \to \pm\infty$, $y \to \mp\infty$.



It is important to note the graph is not a quadratic so the maximum and minimum turning points do not lie halfway between the x-intercepts. In a later chapter we will learn how to locate these points without using technology.

To sketch the graph, it is usually sufficient to identify the x- and y-intercepts and to ensure the shape of the graph satisfies the long-term behaviour requirement determined by the sign of the leading term.

WORKED EXAMPLE 12

Sketch the following without attempting to locate turning points.

a $y = (x - 1)(x - 3)(x + 5)$ **b** $y = (x + 1)(2x - 5)(6 - x)$

THINK

a 1 Calculate the x-intercepts.

WRITE

a $y = (x - 1)(x - 3)(x + 5)$

x-intercepts: let $y = 0$

$(x - 1)(x - 3)(x + 5) = 0$

$x = 1,\ x = 3,\ x = -5$

$\Rightarrow (-5, 0), (1, 0), (3, 0)$ are the x-intercepts.

2 Calculate the y-intercept.

y-intercept: let $x = 0$

$y = (-1)(-3)(5)$

$= 15$

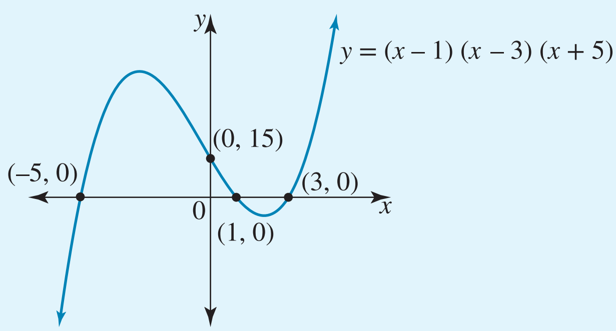
$\Rightarrow (0, 15)$ is the y-intercept.

3 Determine the shape of the graph.

Multiplying together the terms in x from each bracket gives x^3, so its coefficient is positive.

The shape is of a positive cubic.

4 Sketch the graph.



b 1 Calculate the x-intercepts.

b $y = (x + 1)(2x - 5)(6 - x)$

x-intercepts: let $y = 0$

$(x + 1)(2x - 5)(6 - x) = 0$

$x + 1 = 0,\ 2x - 5 = 0,\ 6 - x = 0$

$x = -1,\ x = 2.5,\ x = 6$

$\Rightarrow (-1, 0), (2.5, 0), (6, 0)$ are the x-intercepts.

2 Calculate the y-intercept.

y-intercept: let $x = 0$

$y = (1)(-5)(6)$

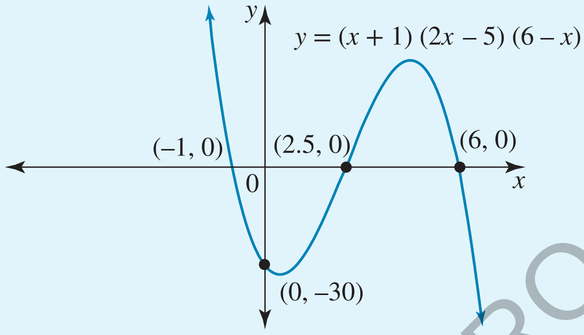
$= -30$

$\Rightarrow (0, -30)$ is the y-intercept.

3 Determine the shape of the graph.

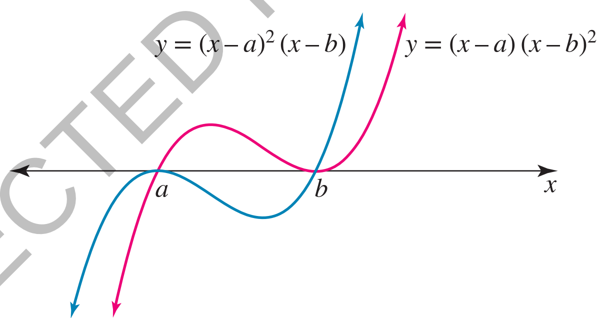
Multiplying the terms in x from each bracket gives $(x) \times (2x) \times (-x) = -2x^3$ so the shape is of a negative cubic.

4 Sketch the graph.



Cubic graphs with two x-intercepts

If a cubic has two x-intercepts one at $x = a$ and one at $x = b$ then in order to satisfy the long-term behaviour required of any cubic, the graph either touches the x-axis at $x = a$ and turns, or it touches the x-axis at $x = b$ and turns. One of the x-intercepts must be a turning point.



Thinking of the cubic polynomial as the product of a linear and a quadratic factor, for its graph to have two instead of three x-intercepts, the quadratic factor must have two identical factors. Either the factors of the cubic are $(x - a)(x - a)(x - b) = (x - a)^2(x - b)$ or the factors are $(x - a)(x - b)(x - b) = (x - a)(x - b)^2$. The repeated factor identifies the x-intercept which is the turning point. The repeated factor is said to be of multiplicity 2 and the single factor of multiplicity 1.

The graph of a cubic polynomial with equation of the form $y = (x - a)^2(x - b)$ has a turning point on the x-axis at $(a, 0)$ and a second x-intercept at $(b, 0)$. The graph is said to *touch* the x-axis at $x = a$ and *cut* it at $x = b$.

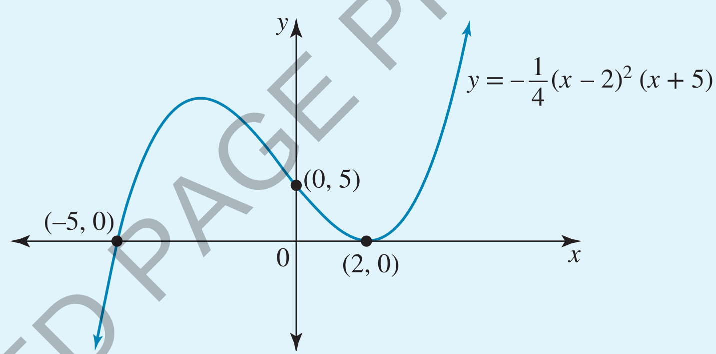
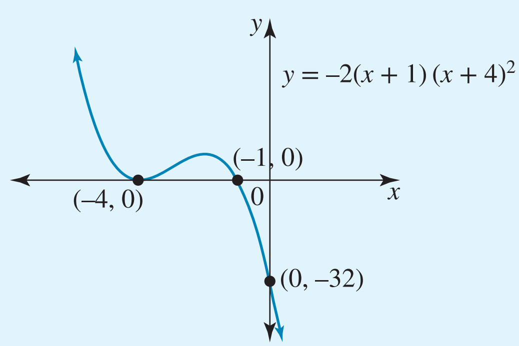
Although the turning point on the x-axis must be identified when sketching the graph, there will be a second turning point that cannot yet be located without technology.

Note that a cubic graph whose equation has a repeated factor of multiplicity 3, such as $y = (x - h)^3$, would have only one x-intercept as this is a special case of $y = a(x - h)^3 + k$ with $k = 0$. The graph would cut the x-axis at its stationary point of inflection $(h, 0)$.

WORKED EXAMPLE 13

Sketch the graphs of:

a $y = \frac{1}{4}(x - 2)^2(x + 5)$ **b** $y = -2(x + 1)(x + 4)^2$

THINK	WRITE
a 1 Calculate the x-intercepts and interpret the multiplicity of each factor.	**a** $y = \frac{1}{4}(x - 2)^2(x + 5)$ x-intercepts: let $y = 0$ $\frac{1}{4}(x - 2)^2(x + 5) = 0$ $\therefore x = 2$ (touch), $x = -5$ (cut) x-intercept at $(-5, 0)$ and turning-point x-intercept at $(2, 0)$
2 Calculate the y-intercept.	y-intercept: let $x = 0$ $y = \frac{1}{4}(-2)^2(5)$ $= 5$ $\Rightarrow (0, 5)$
3 Sketch the graph.	
b 1 Calculate the x-intercepts and interpret the multiplicity of each factor.	**b** $y = -2(x + 1)(x + 4)^2$ x-intercepts: let $y = 0$ $-2(x + 1)(x + 4)^2 = 0$ $(x + 1)(x + 4)^2 = 0$ $\therefore x = -1$ (cut), $x = -4$ (touch) x-intercept at $(-1, 0)$ and turning-point x-intercept at $(-4, 0)$
2 Calculate the y-intercept.	y-intercept: let $x = 0$ $y = -2(1)(4)^2$ $= -32$ y-intercept at $(0, -32)$
3 Sketch the graph.	

Cubic graphs in the general form $y = ax^3 + bx^2 + cx + d$

If the cubic polynomial with equation $y = ax^3 + bx^2 + cx + d$ can be factorised, then the shape of its graph and its key features can be determined. Standard factorisation techniques such as grouping terms together may be sufficient, or the factor theorem may be required in order to obtain the factors.

The sign of a, the coefficient of x^3, determines the long-term behaviour the graph exhibits. For $a > 0$ as $x \to \pm\infty$, $y \to \pm\infty$; for $a < 0$ as $x \to \pm\infty$, $y \to \mp\infty$.

The value of d determines the y-intercept.

The factors determine the x-intercepts and the multiplicity of each factor will determine how the graph intersects the x-axis.

Every cubic graph must have at least one x-intercept and hence the polynomial must have at least one linear factor. Considering a cubic as that's the product of a linear and a quadratic factor, it is the quadratic factor which determines whether there is more than one x-intercept.

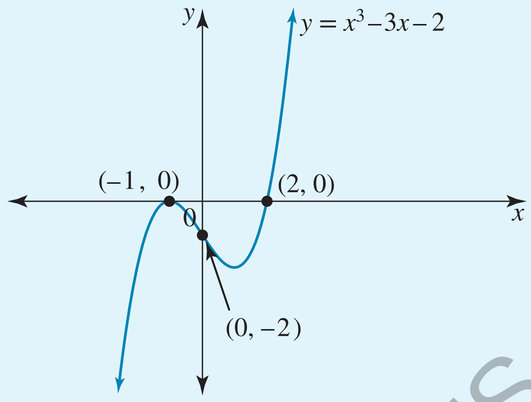
Graphs which have only one x-intercept may be of the form $y = a(x - h)^3 + k$ where the stationary point of inflection is a major feature. Recognition of this equation from its expanded form would require the expansion of a perfect cube to be recognised, since $a(x^3 - 3x^2h + 3xh^2 - h^3) + k = a(x - h)^3 + k$. However, as previously noted, not all graphs with only one x-intercept have a stationary point of inflection.

WORKED EXAMPLE 14

Sketch the graph of $y = x^3 - 3x - 2$, without attempting to obtain any turning points that do not lie on the coordinate axes.

THINK	WRITE
1 Obtain the y-intercept first since it is simpler to obtain from the expanded form.	$y = x^3 - 3x - 2$ y-intercept: $(0, -2)$
2 Factorisation will be needed in order to obtain the x-intercepts.	x-intercepts: let $y = 0$ $x^3 - 3x - 2 = 0$
3 The polynomial does not factorise by grouping so the factor theorem needs to be used.	Let $P(x) = x^3 - 3x - 2$ $P(1) \neq 0$ $P(-1) = -1 + 3 - 2 = 0$ $\therefore (x + 1)$ is a factor $x^3 - 3x - 2 = (x + 1)(x^2 + bx - 2)$ $= (x + 1)(x^2 - x - 2)$ $= (x + 1)(x - 2)(x + 1)$ $= (x + 1)^2(x - 2)$ $\therefore x^3 - 3x - 2 = 0$ $\Rightarrow (x + 1)^2(x - 2) = 0$ $\therefore x = -1, 2$
4 What is the nature of these x-intercepts?	$y = P(x) = (x + 1)^2(x - 2)$ $x = -1$ (touch) and $x = 2$ (cut) Turning point at $(-1, 0)$

5 Sketch the graph.



EXERCISE 4.4 Graphs of cubic polynomials

PRACTISE

1 WE11 Sketch the graphs of these polynomials.

a $y = (x - 1)^3 - 8$ b $y = 1 - \frac{1}{36}(x + 6)^3$

2 State the coordinates of the point of inflection and sketch the graph of the following.

a $y = \left(\frac{x}{2} - 3\right)^3$ b $y = 2x^3 - 2$

3 WE12 Sketch the following, without attempting to locate turning points.

a $y = (x + 1)(x + 6)(x - 4)$ b $y = (x - 4)(2x + 1)(6 - x)$

4 Sketch $y = 3x(x^2 - 4)$.

5 WE13 Sketch the graphs of these polynomials.

a $y = \frac{1}{9}(x - 3)^2(x + 6)$ b $y = -2(x - 1)(x + 2)^2$

6 Sketch $y = 0.1x(10 - x)^2$ and hence shade the region for which $y \le 0.1x(10 - x)^2$.

7 WE14 Sketch the graph of $y = x^3 - 3x^2 - 10x + 24$ without attempting to obtain any turning points that do not lie on the coordinate axes.

8 a Sketch the graph of $y = -x^3 + 3x^2 + 10x - 30$ without attempting to obtain any turning points that do not lie on the coordinate axes.

b Determine the coordinates of the stationary point of inflection of the graph with equation $y = x^3 + 3x^2 + 3x + 2$ and sketch the graph.

CONSOLIDATE

9 a Sketch the graphs of $y = x^3$, $y = 3x^3$, $y = x^3 + 3$ and $y = (x + 3)^3$ on the one set of axes.

b Sketch the graphs of $y = -x^3$, $y = -3x^3$, $y = -x^3 + 3$ and $y = -(x + 3)^3$ on the one set of axes.

10 Sketch the graphs of the following, identifying all key points.

a $y = (x + 4)^3 - 27$ b $y = 2(x - 1)^3 + 10$

c $y = 27 + 2(x - 3)^3$ d $y = 16 - 2(x + 2)^3$

e $y = -\frac{3}{4}(3x + 4)^3$ f $y = 9 + \frac{x^3}{3}$

11 Sketch the graphs of the following, without attempting to locate any turning points that do not lie on the coordinate axes.

a $y = (x - 2)(x + 1)(x + 4)$

b $y = -0.5x(x + 8)(x - 5)$

c $y = (x + 3)(x - 1)(4 - x)$

d $y = \frac{1}{4}(2 - x)(6 - x)(4 + x)$

e $y = 0.1(2x - 7)(x - 10)(4x + 1)$

f $y = 2\left(\frac{x}{2} - 1\right)\left(\frac{3x}{4} + 2\right)\left(x - \frac{5}{8}\right)$

12 Sketch the graphs of the following, without attempting to locate any turning points that do not lie on the coordinate axes.

a $y = -(x + 4)^2(x - 2)$

b $y = 2(x + 3)(x - 3)^2$

c $y = (x + 3)^2(4 - x)$

d $y = \frac{1}{4}(2 - x)^2(x - 12)$

e $y = 3x(2x + 3)^2$

f $y = -0.25x^2(2 - 5x)$

13 Sketch the graphs of the following, showing any intercepts with the coordinate axes and any stationary point of inflection.

a $y = (x + 3)^3$

b $y = (x + 3)^2(2x - 1)$

c $y = (x + 3)(2x - 1)(5 - x)$

d $2(y - 1) = (1 - 2x)^3$

e $4y = x(4x - 1)^2$

f $y = -\frac{1}{2}(2 - 3x)(3x + 2)(3x - 2)$

14 Factorise, if possible, and then sketch the graphs of the cubic polynomials with equations given by:

a $y = 9x^2 - 2x^3$

b $y = 9x^3 - 4x$

c $y = 9x^2 - 3x^3 + x - 3$

d $y = 9x(x^2 + 4x + 3)$

e $y = 9x^3 + 27x^2 + 27x + 9$

f $y = -9x^3 - 9x^2 + 9x + 9$

15 Sketch, without attempting to locate any turning points that do not lie on the coordinate axes.

a $y = 2x^3 - 3x^2 - 17x - 12$

b $y = 6 - 55x + 57x^2 - 8x^3$

c $y = x^3 - 17x + 4$

d $y = 6x^3 - 13x^2 - 59x - 18$

e $y = -5x^3 - 7x^2 + 10x + 14$

f $y = -\frac{1}{2}x^3 + 14x - 24$

16 Consider $P(x) = 30x^3 + kx^2 + 1$.

a Given $(3x - 1)$ is a factor, find the value of k.

b Hence express $P(x)$ as the product of its linear factors.

c State the values of x for which $P(x) = 0$.

d Sketch the graph of $y = P(x)$.

e Does the point $(-1, -40)$ lie on the graph of $y = P(x)$? Justify your answer.

f On your graph shade the region for which $y \geq P(x)$.

17 **a** Express $-\frac{1}{2}x^3 + 6x^2 - 24x + 38$ in the form $a(x - b)^3 + c$.

b Hence sketch the graph of $y = -\frac{1}{2}x^3 + 6x^2 - 24x + 38$.

18 Consider $y = x^3 - 5x^2 + 11x - 7$.

a Show that the graph of $y = x^3 - 5x^2 + 11x - 7$ has only one x-intercept.

b Show that $y = x^3 - 5x^2 + 11x - 7$ cannot be expressed in the form $y = a(x - b)^3 + c$.

c Describe the behaviour of the graph as $x \to \infty$.

d Given the graph of $y = x^3 - 5x^2 + 11x - 7$ has no turning points, draw a sketch of the graph.

MASTER

19 **a** Sketch, locating turning points, the graph of $y = x^3 + 4x^2 - 44x - 96$.

b Show that the turning points are not placed symmetrically in the interval between the adjoining x-intercepts.

20 Sketch, locating intercepts with the coordinate axes and any turning points. Express values to 1 decimal place where appropriate.

a $y = 10x^3 - 20x^2 - 10x - 19$ **b** $y = -x^3 + 5x^2 - 11x + 7$

c $y = 9x^3 - 70x^2 + 25x + 500$

4.5 Equations of cubic polynomials

studyon

Units 1 & 2

AOS 1

Topic 3

Concept 4

Equations of cubic polynomials
Concept summary

eBook*plus*

Interactivity
Graph plotter: 1, 2, 3 intercepts
int-2567

The equation $y = ax^3 + bx^2 + cx + d$ contains four unknown coefficients that need to be specified, so four pieces of information are required to determine the equation of a cubic graph, unless the equation is written in turning point form when 3 pieces of information are required.

As a guide:

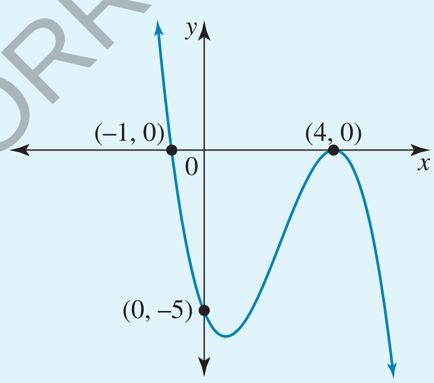
- **If there is a stationary point of inflection given, use the $y = a(x - h)^3 + k$ form.**
- **If the x-intercepts are given, use the $y = a(x - x_1)(x - x_2)(x - x_3)$ form, or the repeated factor form $y = a(x - x_1)^2(x - x_2)$ if there is a turning point at one of the x-intercepts.**
- **If four points on the graph are given, use the $y = ax^3 + bx^2 + cx + d$ form.**

WORKED EXAMPLE 15

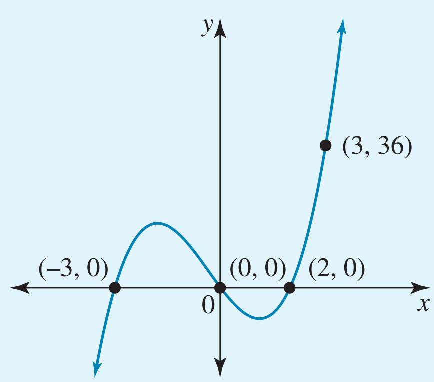
Determine the equation for each of the following graphs.

a **The graph of a cubic polynomial which has a stationary point of inflection at the point $(-7, 4)$ and an x-intercept at $(1, 0)$.**

b



c



THINK	WRITE
a **1** Consider the given information and choose the form of the equation to be used.	**a** Stationary point of inflection is given. Let $y = a(x - h)^3 + k$ Point of inflection is $(-7, 4)$. $\therefore y = a(x + 7)^3 + 4$

2 Calculate the value of a.

Note: The coordinates of the given points show the y-values decrease as the x-values increase, so a negative value for a is expected.

Substitute the given x-intercept point $(1, 0)$.

$$0 = a(8)^3 + 4$$
$$(8)^3 a = -4$$
$$a = \frac{-4}{8 \times 64}$$
$$a = -\frac{1}{128}$$

3 Write the equation of the graph.

The equation is $y = -\frac{1}{128}(x + 7)^3 + 4$.

b 1 Consider the given information and choose the form of the equation to be used.

b Two x-intercepts are given.
One shows a turning point at $x = 4$ and the other a cut at $x = -1$.
Let the equation be $y = a(x + 1)(x - 4)^2$.

2 Calculate the value of a.

Substitute the given y-intercept point $(0, -5)$.

$$-5 = a(1)(-4)^2$$
$$-5 = a(16)$$
$$a = -\frac{5}{16}$$

3 Write the equation of the graph.

The equation is $y = -\frac{5}{16}(x + 1)(x - 4)^2$.

c 1 Consider the given information and choose the form of the equation to be used.

c Three x-intercepts are given.
Let the equation be

$$y = a(x + 3)(x - 0)(x - 2)$$
$$\therefore y = ax(x + 3)(x - 2)$$

2 Calculate the value of a.

Substitute the given point $(3, 36)$.

$$36 = a(3)(6)(1)$$
$$36 = 18a$$
$$a = 2$$

3 Write the equation of the graph.

The equation is $y = 2x(x + 3)(x - 2)$.

Cubic inequations

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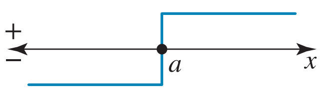
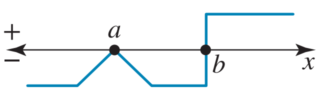
Interactivity
Cubic inequations
int-2568

The sign diagram for a cubic polynomial can be deduced from the shape of its graph and its x-intercepts. The values of the zeros of the polynomial are those of the x-intercepts. For a cubic polynomial with a positive coefficient of x^3, the sign diagram starts from below the x-axis. For examples below, assume $a < b < c$.

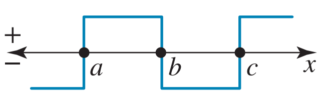
One zero $(x - a)^3$ or $(x - a) \times$ irreducible quadratic factor



Two zeros $(x - a)^2(x - b)$



Three zeros $(x - a)(x - b)(x - c)$



At a zero of multiplicity 2, the sign diagram touches the x-axis. For a zero of odd multiplicity, either multiplicity 1 or multiplicity 3, the sign diagram cuts the x-axis. This 'cut and touch' nature applies if the coefficient of x^3 is negative; however, the sign diagram would start from above the x-axis in that case.

To solve a cubic inequation:

- **Rearrange the terms in the inequation, if necessary, so that one side of the inequation is 0.**
- **Factorise the cubic expression and calculate its zeros.**
- **Draw the sign diagram, or the graph, of the cubic.**
- **Read from the sign diagram the set of values of x which satisfy the inequation.**

An exception applies to inequations of forms such as $a(x - h)^3 + k > 0$. These inequations are solved in a similar way to solving linear inequations without the need for a sign diagram or a graph. Note the similarity between the sign diagram of the cubic polynomial with one zero and the sign diagram of a linear polynomial.

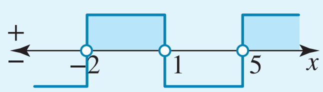
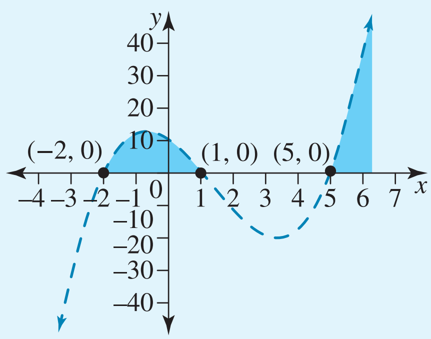
WORKED EXAMPLE 16

Solve the inequations.

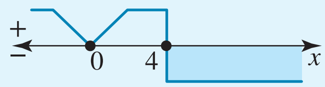
a $(x + 2)(x - 1)(x - 5) > 0$

b $\{x : 4x^2 \le x^3\}$

c $(x - 2)^3 - 1 > 0$

THINK	WRITE
a 1 Read the zeros from each factor.	**a** $(x + 2)(x - 1)(x - 5) > 0$ Zeros: $x = -2,\ x = 1,\ x = 5$
2 Consider the leading-term coefficient to draw the sign diagram. This is a positive cubic.	
3 State the intervals in which the required solutions lie.	$-2 < x < 1$ or $x > 5$
4 Show the solution on a graph.	
b 1 Rearrange so one side is 0.	**b** $4x^2 \le x^3$ $4x^2 - x^3 \le 0$ $\therefore x^2(4 - x) \le 0$
2 Factorise to calculate the zeros.	Let $x^2(4 - x) = 0$ $x^2 = 0$ or $4 - x = 0$ $x = 0$ or $x = 4$ Zeros: $x = 0$ (multiplicity 2), $x = 4$

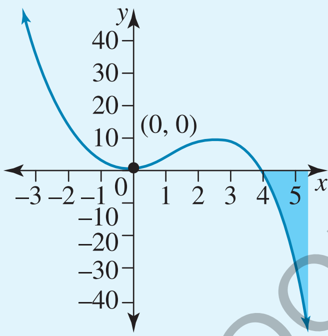
3 Consider the leading-term coefficient to draw the sign diagram. $4x^2 - x^3$ is a negative cubic.



4 State the answer from the sign diagram, using set notation.

$\{x : x \geq 4\} \cup \{0\}$

5 Show the solution on a graph.



c 1 Solve for x.

c $(x - 2)^3 - 1 > 0$

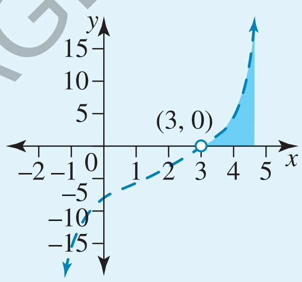
$(x - 2)^3 > 1$

$(x - 2) > \sqrt[3]{1}$

$x - 2 > 1$

$x > 3$

2 Show the solution on a graph.



Intersections of cubic graphs with linear and quadratic graphs

If $P(x)$ is a cubic polynomial and $Q(x)$ is either a linear or a quadratic polynomial, then at the intersections of the graphs of $y = P(x)$ and $y = Q(x)$, $P(x) = Q(x)$. Hence the x-coordinates of the points of intersection are the roots of the equation $P(x) - Q(x) = 0$. This is a cubic equation since $P(x) - Q(x)$ is a polynomial of degree 3.

WORKED EXAMPLE 17

Sketch the graphs of $y = x(x - 1)(x + 1)$ and $y = x$ and calculate the coordinates of the points of intersection. Hence state the values of x for which $x > x(x - 1)(x + 1)$.

THINK

1 Sketch the graphs.

WRITE

$y = x(x - 1)(x + 1)$

This is a positive cubic.

x-intercepts: let $y = 0$

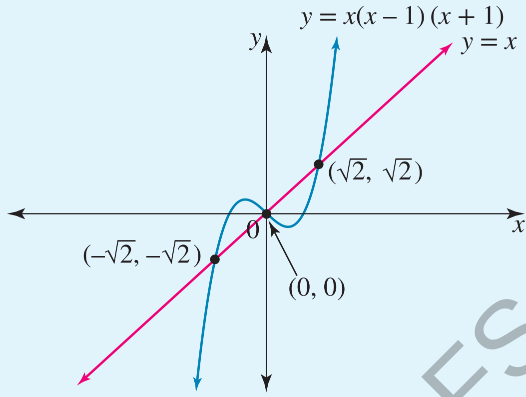
$x(x - 1)(x + 1) = 0$

$x = 0, \ x = \pm 1$

$(-1, 0), (0, 0), (1, 0)$ are the three x-intercepts.

y-intercept is $(0, 0)$.

Line: $y = x$ passes through $(0, 0)$ and $(1, 1)$.



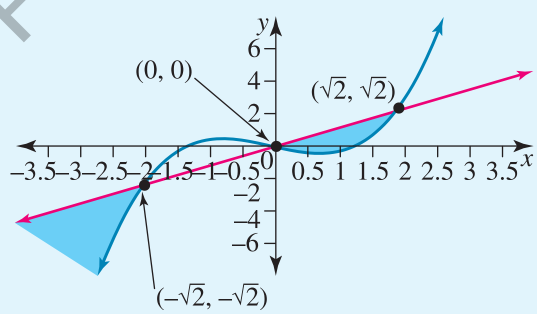
2 Calculate the coordinates of the points of intersections.

At intersection:

$$x(x-1)(x+1) = x$$
$$x(x^2-1) - x = 0$$
$$x^3 - 2x = 0$$
$$x(x^2-2) = 0$$
$$x = 0,\ x^2 = 2$$
$$x = 0,\ x = \pm\sqrt{2}$$

Substituting these x-values in the equation of the line $y = x$, the points of intersection are $(0, 0), (\sqrt{2}, \sqrt{2}), (-\sqrt{2}, -\sqrt{2})$.

3 Show the regions of the graph where $x > x(x-1)(x+1)$



4 Use the diagram to state the intervals for which the given inequation holds.

$x > x(x-1)(x+1)$ for
$\{x: x < \sqrt{2}\} \cup \{x: 0 < x -\sqrt{2}\}$

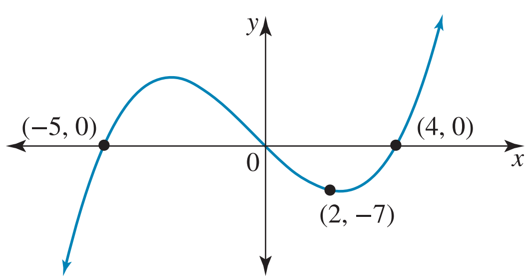
EXERCISE 4.5 Equations of cubic polynomials

PRACTISE

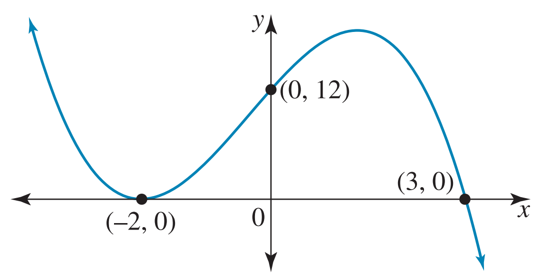
1 WE15 Determine the equation of each of the following graphs.

a The graph of a cubic polynomial which has a stationary point of inflection at the point $(3, -7)$ and an x-intercept at $(10, 0)$.

b



c



2 Use simultaneous equations to determine the equation of the cubic graph containing the points $(0, 3)$, $(1, 4)$, $(-1, 8)$, $(-2, 7)$.

3 **WE16** Solve the inequations.

a $(x - 2)^2(6 - x) > 0$ b $\{x : 4x \leq x^3\}$ c $2(x + 4)^3 - 16 < 0$

4 Calculate $\{x : 3x^3 + 7 > 7x^2 + 3x\}$.

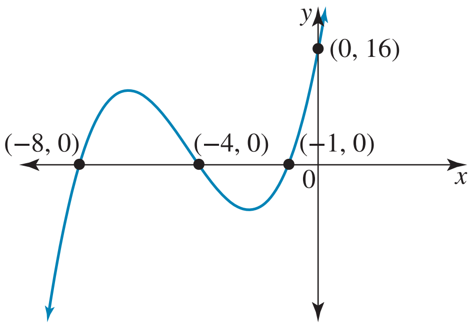
5 **WE17** Sketch the graphs of $y = (x + 2)(x - 1)^2$ and $y = -3x$ and calculate the coordinates of the points of intersection. Hence state the values of x for which $-3x < (x + 2)(x - 1)^2$.

6 Calculate the coordinates of the points of intersection of $y = 4 - x^2$ and $y = 4x - x^3$ and then sketch the graphs on the same axes.

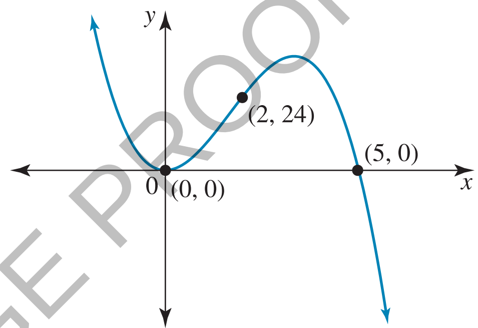
CONSOLIDATE

7 Determine the equation for each of the following graphs of cubic polynomials.

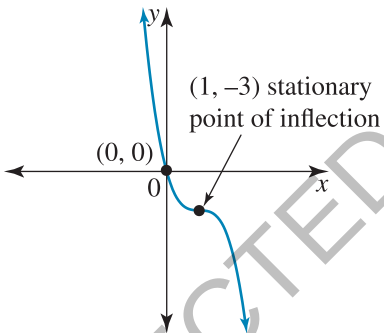
a



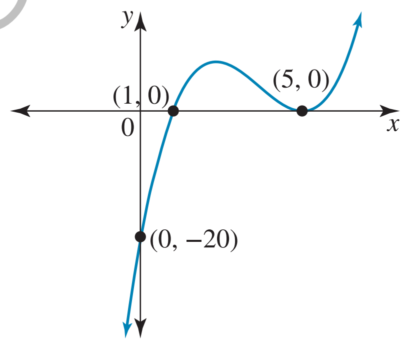
b



c



d



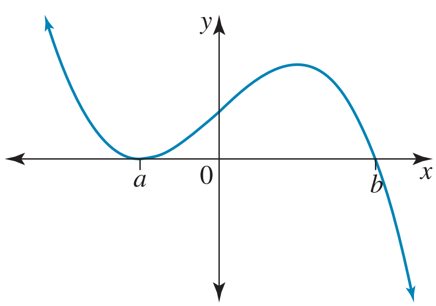
8 a Give the equation of the graph which has the same shape as $y = -2x^3$ and a point of inflection at $(-6, -7)$.

b Calculate the y-intercept of the graph which is created by translating the graph of $y = x^3$ two units to the right and four units down.

c A cubic graph has a stationary point of inflection at $(-5, 2)$ and a y-intercept of $(0, -23)$. Calculate its exact x-intercept.

d A curve has the equation $y = ax^3 + b$ and contains the points $(1, 3)$ and $(-2, 39)$. Calculate the coordinates of its stationary point of inflection.

9 The graph of $y = P(x)$ shown is the reflection of a monic cubic polynomial in the x-axis. The graph touches the x-axis at $x = a$, $a < 0$ and cuts it at $x = b$, $b > 0$.



a Form an expression for the equation of the graph.

b Use the graph to find $\{x : P(x) \geq 0\}$.

c How far horizontally to the left would the graph need to be moved so that both of its x-intercepts are negative?

d How far horizontally to the right would the graph need to be moved so that both of its x-intercepts are positive?

10 A graph of a cubic polynomial with equation $y = x^3 + ax^2 + bx + 9$ has a turning point at $(3, 0)$.

a State the factor of the equation with greatest multiplicity.
b Determine the other x-intercept.
c Calculate the values of a and b.

11 Solve the cubic inequations.

a $(x - 2)(x + 1)(x + 9) \geq 0$
b $x^2 - 5x^3 < 0$
c $8(x - 2)^3 - 1 > 0$
d $x^3 + x \leq 2x^2$
e $5x^3 + 6x^2 - 20x - 24 < 0$
f $2(x + 1) - 8(x + 1)^3 < 0$

12 Find the coordinates of the points of intersection of the following.

a $y = 2x^3$ and $y = x^2$
b $y = 2x^3$ and $y = x - 1$
c Illustrate the answers to parts **a** and **b** with a graph.
d Solve the inequation $2x^3 - x^2 \leq 0$ algebraically and explain how you could use your graph from part **c** to solve this inequation.

13 **a** The number of solutions to the equation $x^3 + 2x - 5 = 0$ can be found by determining the number of intersections of the graphs of $y = x^3$ and a straight line. What is the equation of this line and how many solutions does $x^3 + 2x - 5 = 0$ have?
b Use a graph of a cubic and a linear polynomial to determine the number of solutions to the equation $x^3 + 3x^2 - 4x = 0$.
c Use a graph of a cubic and a quadratic polynomial to determine the number of solutions to the equation $x^3 + 3x^2 - 4x = 0$.
d Solve the equation $x^3 + 3x^2 - 4x = 0$.

14 The graph of a polynomial of degree 3 cuts the x-axis at $x = 1$ and at $x = 2$. It cuts the y-axis at $y = 12$.

a Explain why this is insufficient information to completely determine the equation of the polynomial.
b Show that this information identifies a family of cubic polynomials with equation $y = ax^3 + (6 - 3a)x^2 + (2a - 18)x + 12$.
c On the same graph, sketch the two curves in the family for which $a = 1$ and $a = -1$.
d Determine the equation of the curve for which the coefficient of x^2 is 15. Specify the x-intercepts and sketch this curve.

15 The graph with equation $y = (x + a)^3 + b$ passes through the three points $(0, 0), (1, 7), (2, 26)$.

a Use this information to determine the values of a and b.
b Find the points of intersection of the graph with the line $y = x$.
c Sketch both graphs in part **b** on the same axes.
d Hence, with the aid of the graphs, find $\{x : x^3 + 3x^2 + 2x > 0\}$.

16 **a** Show the line $y = 3x + 2$ is a tangent to the curve $y = x^3$ at the point $(-1, -1)$.
b What are the coordinates of the point where the line cuts the curve?
c Sketch the curve and its tangent on the same axes.
d Investigate for what values of m will the line $y = mx + 2$ have one, two or three intersections with the curve $y = x^3$.

MASTER

17 Give the equation of the cubic graph containing the points $(-2, 53)$, $(-1, -6)$, $(2, 33)$, $(4, -121)$.

18 a Write down, to 2 decimal places, the coordinates of the points of intersection of $y = (x + 1)^3$ and $y = 4x + 3$.

b Form the cubic equation $ax^3 + bx^2 + cx + d = 0$ for which the x-coordinates of the points of intersection obtained in part a are the solution.

c What feature of the graph of $y = ax^3 + bx^2 + cx + d$ would the x-coordinates of these points of intersection be?

Omar Khayyăm (1050–1123) is not only a brilliant poet, but also an extraordinary mathematician, and is noted for linking algebra with geometry by solving cubic equations as the intersection of two curves.

4.6 Cubic models and applications

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Units 1 & 2

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Topic 3

Concept 5

Cubic models and applications
Concept summary

Practical situations which use cubic polynomials as models are likely to require a restriction of the possible values the variable may take. This is called a **domain** restriction. The domain is the set of possible values of the variable that the polynomial may take. We shall look more closely at domains in later chapters.

The polynomial model should be expressed in terms of one variable.

Applications of cubic models where a maximum or minimum value of the model is sought will require identification of turning point coordinates. In a later chapter we will see how this is done. For now, obtaining turning points may require the use of graphing or CAS technology.

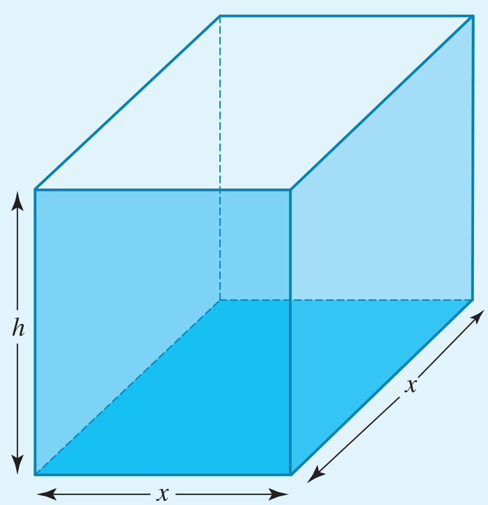
WORKED EXAMPLE 18

A rectangular storage container is designed to have an open top and a square base.

The base has side length x cm and the height of the container is h cm. The sum of its dimensions (the sum of the length, width and height) is 48 cm.

a Express h in terms of x.

b Show that the volume V cm^3 of the container is given by $V = 48x^2 - 2x^3$.

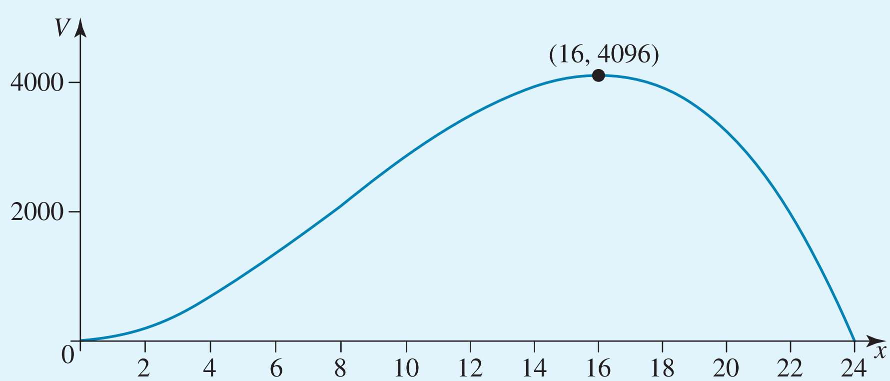


c State any restrictions on the values x can take.

d Sketch the graph of V against x for appropriate values of x, given its maximum turning point has coordinates $(16, 4096)$.

e Calculate the dimensions of the container with the greatest possible volume.

THINK	WRITE
a Write the given information as an equation connecting the two variables.	**a** Sum of dimensions is 48 cm. $x + x + h = 48$ $h = 48 - 2x$
b Use the result from part **a** to express the volume in terms of one variable and prove the required statement.	**b** The formula for volume of a cuboid is $V = lwh$ $\therefore V = x^2 h$ Substitute $h = 48 - 2x$. $V = x^2(48 - 2x)$ $\therefore V = 48x^2 - 2x^3$, as required
c State the restrictions. *Note:* It could be argued that the restriction is $0 < x < 24$ because when $x = 0$ or $x = 48$ there is no storage container, but we are adopting the closed convention.	**c** Length cannot be negative, so $x \geq 0$. Height cannot be negative, so $h \geq 0$. $48 - 2x \geq 0$ $-2x \geq -48$ $\therefore x \leq 24$ Hence the restriction is $0 \leq x \leq 24$.
d Draw the cubic graph but only show the section of the graph for which the restriction applies. Label the axes with the appropriate symbols and label the given turning point.	**d** $V = 48x^2 - 2x^3$ $= 2x^2(24 - x)$ x-intercepts: let $V = 0$ $x^2 = 0$ or $24 - x = 0$ $\therefore x = 0$ (touch), $x = 24$ (cut) $(0, 0), (24, 0)$ are the x-intercepts. This is a negative cubic. Maximum turning point $(16, 4096)$ Draw the section for which $0 \leq x \leq 24$.



e 1 Calculate the required dimensions.

Note: The maximum turning point (x, V) gives the maximum value of V and the value of x when this maximum occurs.

e The maximum turning point is $(16, 4096)$. This means the greatest volume is 4096 cm^3. It occurs when $x = 16$.
$\therefore h = 48 - 2(16) \Rightarrow h = 16$
Dimensions: length = 16 cm, width = 16 cm, height = 16 cm

2 State the answer.

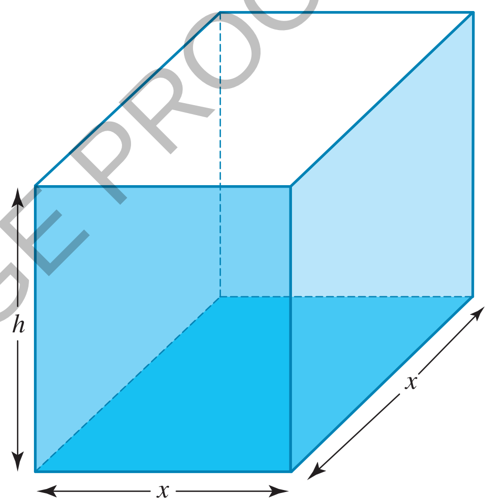
The container has the greatest volume when it is a cube of edge 16 cm.

EXERCISE 4.6 Cubic models and applications

PRACTISE

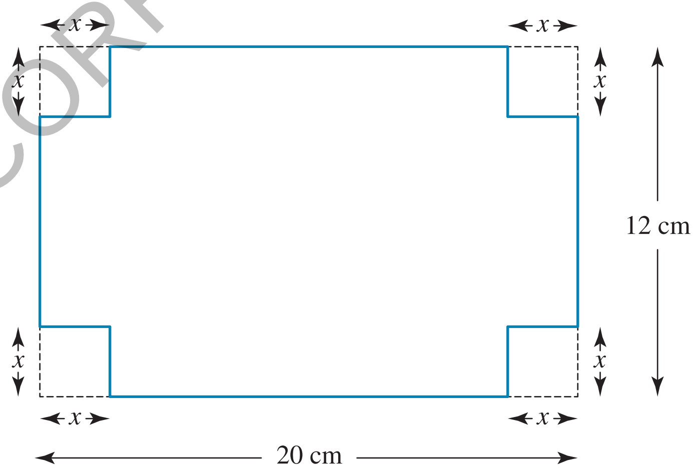
1 **WE18** A rectangular storage container is designed to have an open top and a square base.

The base has side length x metres and the height of the container is h metres. The total length of its 12 edges is 6 metres.



a Express h in terms of x.
b Show that the volume $V \text{ m}^3$ of the container is given by $V = 1.5x^2 - 2x^3$.
c State any restrictions on the values x can take.
d Sketch the graph of V against x for appropriate values of x, given its maximum turning point has coordinates $(0.5, 0.125)$.
e Calculate the dimensions of the container with the greatest possible volume.

2 A rectangular box with an open top is to be constructed from a rectangular sheet of cardboard measuring 20 cm by 12 cm by cutting equal squares of side length x cm out of the four corners and folding the flaps up.



The box has length l cm, width w cm and volume $V \text{ cm}^3$.

a Express l and w in terms of x and hence express V in terms of x.
b State any restrictions on the values of x.

c Sketch the graph of V against x for appropriate values of x, given the unrestricted graph would have turning points at $x = 2.43$ and $x = 8.24$.

d Calculate the length and width of the box with maximum volume and give this maximum volume to the nearest whole number.

CONSOLIDATE

3 The cost C dollars for an artist to produce x sculptures by contract is given by $C = x^3 + 100x + 2000$. Each sculpture is sold for \$500 and as the artist only makes the sculptures by order, every sculpture produced will be paid for. However, too few sales will result in a loss to the artist.

a Show the artist makes a loss if only 5 sculptures are produced and a profit if 6 sculptures are produced.

b Show that the profit, P dollars, from the sale of x sculptures is given by $P = -x^3 + 400x - 2000$.

c What will happen to the profit if a large number of sculptures are produced? Why does this effect occur?

d Calculate the profit (or loss) from the sale of:

i 16 sculptures **ii** 17 sculptures.

e Use the above information to sketch the graph of the profit P for $0 \leq x \leq 20$. Place its intersection with the x-axis between two consecutive integers but don't attempt to obtain its actual x-intercepts.

f In order to guarantee a profit is made, how many sculptures should the artist produce?

4 The number of bacteria in a slow-growing culture at time t hours after 9 am is given by $N = 54 + 23t + t^3$

a What is the initial number of bacteria at 9 am?

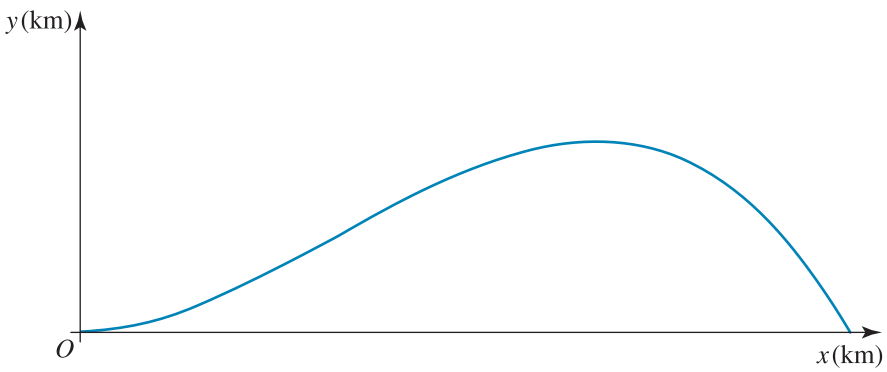
b How long does it take for the initial number of bacteria to double?

c How many bacteria are there by 1 pm?

d Once the number of bacteria reaches 750, the experiment is stopped. At what time of the day does this happen?

5 Engineers are planning to build an underground tunnel through a city to ease traffic congestion.

The cross-section of their plan is bounded by the curve shown.

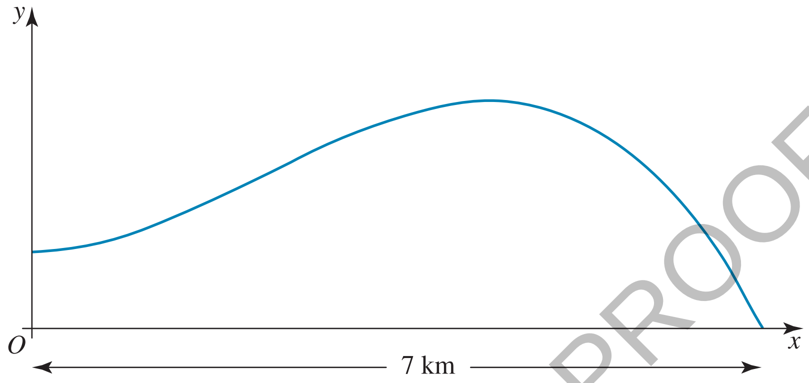


The equation of the bounding curve is $y = ax^2(x - b)$ and all measurements are in kilometres.

It is planned that the greatest breadth of the bounding curve will be 6 km and the greatest height will be 1 km above this level at a point 4 km from the origin.

a Determine the equation of the bounding curve.

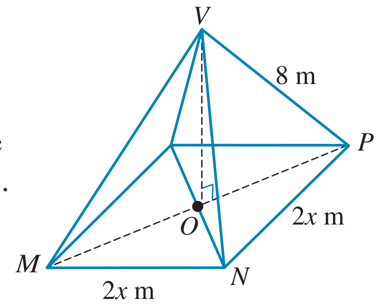
b If the greatest breadth of the curve was extended to 7 km, what would be the greatest height of the curve above this new lowest level?



6 Find the smallest positive integer and the largest negative integer for which the difference between the square of 5 more than this number and the cube of 1 more than the number exceeds 22.

7 A tent used by a group of bushwalkers is in the shape of a square-based right pyramid with a slant height of 8 metres.

For the figure shown, let OV, the height of the tent, be h metres and the edge of the square base be $2x$ metres.



a Use Pythagoras' theorem to express the length of the diagonal of the square base of the tent in terms of x.

b Use Pythagoras' theorem to show $2x^2 = 64 - h^2$.

c The volume V of a pyramid is found using the formula $V = \frac{1}{3}Ah$ where A is the area of the base of the pyramid. Use this formula to show that the volume of space contained within the bushwalkers' tent is given by $V = \frac{1}{3}(128h - 2h^3)$.

d **i** If the height of the tent is 3 metres, what is the volume?

ii What values for the height does this mathematical model allow?

e Sketch the graph of $V = \dfrac{1}{3}(128h - 2h^3)$ for appropriate values of h and estimate the height for which the volume is greatest.

f The greatest volume is found to occur when the height is half the length of the base. Use this information to calculate the height which gives the greatest volume and compare this value with your estimate from your graph in part **e**.

8 A cylindrical storage container is designed so that it is open at the top and has a surface area of 400π cm^2. Its height is h cm and its radius is r cm.

a Show that $h = \dfrac{400 - r^2}{2r}$.

b Show that the volume V cm^3 the container can hold is given by $V = 200\pi r - \frac{1}{2}\pi r^3$.

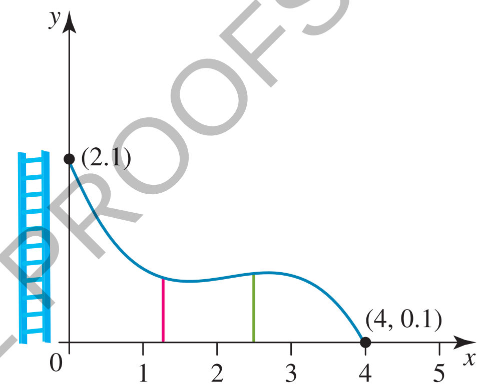
c State any restrictions on the values r can take.

d Sketch the graph of V against r for appropriate values of r.

e Find the radius and height of the container if the volume is 396π cm^3.

f State the maximum possible volume to the nearest cm^2 if the maximum turning point on the graph of $y = 200\pi x - \frac{1}{2}\pi x^3$ has an x-coordinate of $\frac{20}{\sqrt{3}}$.

9 A new playground slide for children is to be constructed at a local park. At the foot of the slide the children climb a vertical ladder to reach the start of the slide. The slide must start at a height of 2.1 metres above the ground and end at a point 0.1 metres above the ground and 4 metres horizontally from its foot. A model for the slide is $h = ax^3 + bx^2 + cx + d$ where h metres is the height of the slide above ground level at a horizontal distance of x metres from its foot. The foot is at the origin.



The ladder supports the slide at one end and the slide also requires two vertical struts as support. One strut of length 1 metre is placed at a point 1.25 metres horizontally from the foot of the slide and the other is placed at a point 1.5 metres horizontally from the end of the slide and is of length 1.1 metres.

a Give the coordinates of 4 points which lie on the cubic graph of the slide.

b State the value of d in the equation of the slide.

c Form a system of 3 simultaneous equations, the solutions to which give the coefficients a, b, c in the equation of the slide.

d The equation of the slide can be shown to be $y = -0.164x^3 + x^2 - 1.872x + 2.1$. Use this equation to calculate the length of a third strut thought necessary at $x = 3.5$. Give your answer to 2 decimal places.

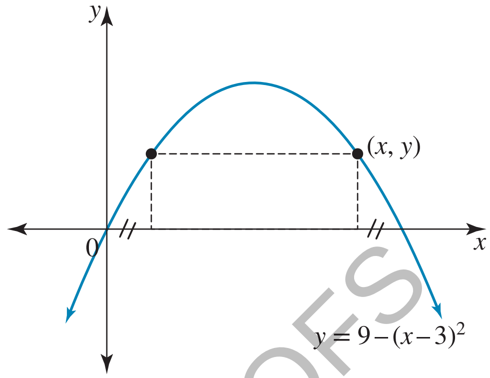
10 Since 1988, the world record times for the men's 100-m sprint can be roughly approximated by the cubic model $T(t) = -0.00005(t - 6)^3 + 9.85$ where T is the time in seconds and t is the number of years since 1988.

a In 1991 the world record was 9.86 seconds and in 2008 the record was 9.72 seconds. Compare these times with those predicted by the cubic model.

b Sketch the graph of T versus t from 1988 to 2008.

c What does the model predict for 2016? Is the model likely to be a good predictor beyond 2016?

11 A rectangle is inscribed under the parabola $y = 9 - (x - 3)^2$ so that two of its corners lie on the parabola and the other two lie on the x-axis at equal distances from the intercepts the parabola makes with the x-axis.



a Calculate the x-intercepts of the parabola.

b Express the length and width of the rectangle in terms of x.

c Hence show that the area of the rectangle is given by $A = -2x^3 + 18x^2 - 36x$.

d For what values of x is this a valid model of the area?

e Calculate the value(s) of x for which $A = 16$.

12 A pathway through the countryside passes through 5 scenic points. Relative to a fixed origin, these points have coordinates $A(-3, 0)$, $B(-\sqrt{3}, -12\sqrt{3})$, $C(\sqrt{3}, 12\sqrt{3})$, $D(3, 0)$; the fifth scenic point is the origin, $O(0, 0)$. The two-dimensional shape of the path is a cubic polynomial.

a State the maximum number of turning points and x-intercepts that a cubic graph can have.

b Determine the equation of the pathway through the 5 scenic points.

c Sketch the path, given that points B and C are turning points of the cubic polynomial graph.

d It is proposed that another pathway be created to link B and C by a direct route. Show that if a straight-line path connecting B and C is created, it will pass through O and give the equation of this line.

e An alternative plan is to link B and C by a cubic path which has a stationary point of inflection at O. Determine the equation of this path.

MASTER

Use CAS technology to answer the following questions.

13 a Consider question 7e.

Use the calculator to sketch the graph of $V = \frac{1}{3}(128h - 2h^3)$ and state the height for which the volume is greatest, correct to 2 decimal places.

b Consider question 7f.

Show that the greatest volume occurs when the height is half the length of the base.

c Consider question 8f.

Use technology to obtain the maximum volume and calculate the corresponding dimensions of the cylindrical storage container, expressed to 1 decimal place.

14 a Consider question 9c.

Use technology to obtain the equation of the slide.

b Consider question 11.

Calculate, to 3 decimal places, the length and width of the rectangle which has the greatest area.

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4.7 Review

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The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions without the use of CAS technology
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

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Activities

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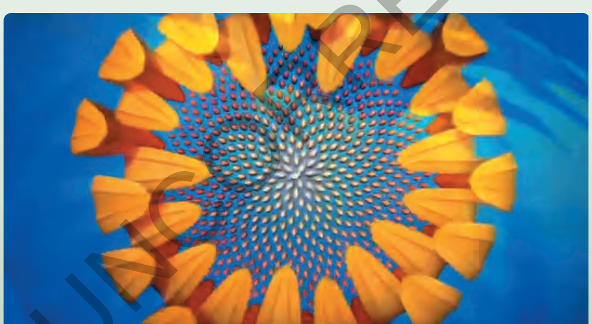
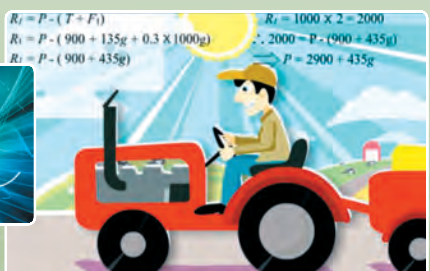
Interactivities

A comprehensive set of relevant interactivities to bring difficult mathematical concepts to life can be found in the Resources section of your eBookPLUS.





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Units 1 & 2 **Cubic polynomials**

Sit topic test

4 Answers

EXERCISE 4.2

1 A: Degree 5; leading coefficient 4; constant term 12; coefficients $\in Z$

C: Degree 2; leading coefficient -0.2; constant term 5.6; coefficients $\in Q$

2 $y^7 + 2y^5 - \sqrt{2}y^2 + 4$ (answers may vary)

3 a 15 b 10

4 21

5 $a = 2$; $b = -13$; $c = 6$

6 $(x+2)^3 = x^2(x+1) + 5x(x+2) + 2(x+3) + 2$

7 $p = 5$; $q = -4$

8 $x^7 + 4x^5 - 6x^4 + 5x^3 - 4x^2 - 5x + 2$; degree 7

9 a $\dfrac{x-12}{x+3} = 1 - \dfrac{15}{x+3}$; quotient is 1; remainder is -15.

b $\dfrac{4x+7}{2x+1} = 2 + \dfrac{5}{2x+1}$

10 $a = 3,\ b = -12,\ c = 29$;

$\dfrac{3x^2 - 6x + 5}{x+2} = 3x - 12 + \dfrac{29}{x+2}$;

$P(x) = 3x - 12,\ d = 29$

11 a $\dfrac{2x^3 - 5x^2 + 8x + 6}{x-2} = 2x^2 - x + 6 + \dfrac{18}{x-2}$; quotient is $2x^2 - x + 6$; remainder is 18.

b $\dfrac{x^3+10}{1-2x} = -\dfrac{1}{2}x^2 - \dfrac{1}{4}x - \dfrac{1}{8} + \dfrac{81}{8(1-2x)}$;

remainder is $\dfrac{81}{8}$.

12 Quotient is $x^2 - x + 3$; remainder is 0.

13 a A, B, D, F are polynomials.

	Degree	Type of coefficient	Leading term	Constant term
A	5	Q	$3x^5$	12
B	4	R	$-5x^4$	9
D	4	Z	$-18x^4$	0
F	6	N	$49x^6$	9

b C is not a polynomial due to $\sqrt{4x^5} = 2x^{\frac{1}{2}}$ term.

E is not a polynomial due to $\dfrac{5}{3x^2} = \dfrac{5}{3}x^{-2}$ term.

14 a 78 b -12

c 0 d -6

e -6 f -5.868

15 a $-14a$

b $h^2 - 5h - 4$

c $2xh + h^2 - 7h$

16 a $k = 25$ b $a = -9$

c $n = -\frac{1}{5}$ d $b = 4$; $c = 5$

17 a $a = 3$; $b = -2$; $c = -8$

b $m = -3$; $n = -10$; $p = 24$

c $a = 1$; $b = 7$; $c = -41$; $x^2 - 14x + 8 = (x-7)^2 - 41$

d $4x^3 + 2x^2 - 7x + 1$
$= 4x^2(x+1) - 2x(x+1) - 5(x+1) + 6$

18 a i $3x^4 + 7x + 12$

ii $6x^4 + 10x^2 - 21x - 31$

iii $6x^6 - 21x^5 - 29x^4 - 14x^3 - 20x^2 - 7x - 11$

b i m ii m

iii $m + n$

19 a $a = -3$; $b = -1$; $p = 4$; $q = -2$

b i $4x^4 + 12x^3 + 13x^2 + 6x + 1 = (2x^2 + 3x + 1)^2$

ii $2x^2 + 3x + 1$

20 a $k = -13$; $m = -5$; $n = 6$

b $(x-3)^2(x-2)^2(x+3)(x+2)$

21

	Quotient	Remainder
a	1	9
b	4	1
c	$x + 7$	-10
d	$2x - 12$	27
e	$x^2 + 5x + 12$	41
f	$x^2 - 7x + 2$	0

22 a $4x^2 + x - 3 + \dfrac{18}{1+2x}$

b $2x^2 + 3x + 5 + \dfrac{20}{2x-3}$

c $x^2 + 6 - \dfrac{48}{x+6}$

d $x^2 - x + 1 + \dfrac{1}{x+1}$

e $x^2 + x + \dfrac{3x+5}{x^2-1}$

f $-2x - 2 - \dfrac{7x+12}{(x+2)(x-3)}$

23 a $2x^2 - \dfrac{13}{2}x - \dfrac{139}{4(2x+3)} + \dfrac{49}{4}$

b Remainder is $-\dfrac{139}{4}$; quotient is $2x^2 - \dfrac{13}{2}x + \dfrac{49}{4}$.

c Dividend equals $-\dfrac{139}{4}$.

d Divisor equals 0.

24 a Define using CAS technology.

b $\frac{349}{9}$

c $24n^2 + 24an - 16n - 154$

d 9

EXERCISE 4.3

1 a The remainder is 19.

b The remainder is $\frac{37}{8}$.

2 $k = 2$

3 a Proof required to show $Q(2) = 0$

b $a = -9$; $b = 8$; $P(x) = 3x^3 - 9x^2 + 8x - 2$

4 $a = 0$; $a = -1$

5 a $P(-1) = 0 \Rightarrow (x + 1)$ is a factor; $P(x) = (x + 1)(x + 5)(x - 3)$

b $P(x) = (x + 1)(3x + 2)(4x + 7)$

6 Linear factors are $(2x - 1)$, $(2x + 1)$ and $(3x + 2)$.

7 $x = -2, \frac{1}{3}, -\frac{1}{2}$

8 $x = 0, x = -\frac{3}{2}, x = 2, x = -2$

9 a –5 **b** 2

c –101 **d** $-10\frac{3}{8}$

e 0 **f** 26

10 a –10 **b** 8

c –19 **d** –19

11 a $a = 2$; $b = 3$ **b** $m = -\frac{2}{3}$; $n = -3$

12 a i $(x - 5)(x - 9)(x + 2)$

ii $x^3 - 12x^2 + 17x + 90$

b i $(x + 4)(x + 1)(1 - 2x)$

ii $-2x^3 - 9x^2 - 3x + 4$

13 a $(x - 4)(x + 1)(x + 2)$

b $(x + 12)(3x + 1)(x + 1)$

c $(5x + 1)(2x + 3)(2x + 1)$

d $(4x - 3)(5 - 2x)(5 + 2x)$

e $-(x - 3)^2(8x - 11)$

f $(3x - 5)^2(x - 5)$

14 a $(x - 1)(x + 2)(x + 4)$

b $(x + 2)(x + 3)(x + 5)$

c $(x - 2)(2x + 1)(x - 5)$

d $(x + 1)(3x - 4)(1 - 6x)$

e $(x - 1)(x + 3)(x - 2)$

f $(x + 1)(x - 7)(x + 7)$

15 a $(2x + 1)(3x - 1)(4x + 5)$

b i $(2x - 5)$

ii $(2x - 5)(4x + 1)(x - 1)$

iii $m = -26$

c i $P(x) = (x - 4)^3$; $Q(x) = (x - 4)(x^2 + 4x + 16)$

ii Proof required

d Third factor is $(x - 3)$; $b = c = -3$; $d = 9$

16 a $-4, 3, -5$ **b** $7, -\frac{5}{3}, 9$

c $-1, 6, 8$ **d** $1, -\frac{3}{2}, -3$

e $1, -\frac{2}{3}$ **f** $0, 1, -\frac{3}{4}, -20$

17 a Proof required; $(x - 2)(x + 4 - \sqrt{7})(x + 4 + \sqrt{7})$

b Proof required

c $(2x - 1)(x - 5)^2$; equation has roots $x = \frac{1}{2}$, $x = 5$

18 a $(x - 2)(x + 2)(5x + 9)$; $k = 9$

b $a = 1$; $a = 2$

c $a = -3$; $b = 1$; $P(x) = (x - 3)(x^2 + 1)$, $Q(x) = (x - 3)(x + 3)(x + 1)$

d $x^3 + 7x^2 + 15x + 9$; $x = -3$, $x = -1$; $x^3 - 9x^2 + 15x + 25$: $x = 5$, $x = -1$

19 $-12\sqrt{2} + 33$

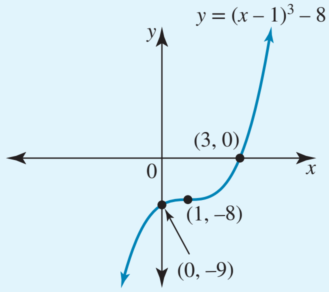
20 -0.4696

EXERCISE 4.4

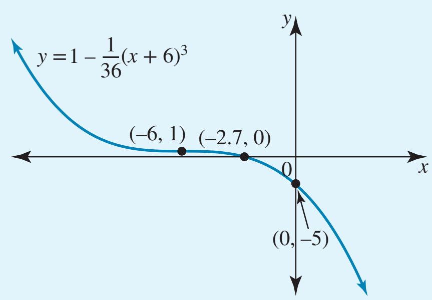
1

	Inflection point	y-intercept	x-intercept
a	(1, –8)	(0, –9)	(3, 0)
b	(–6, 1)	(0, –5)	(–2.7, 0) approx.

a



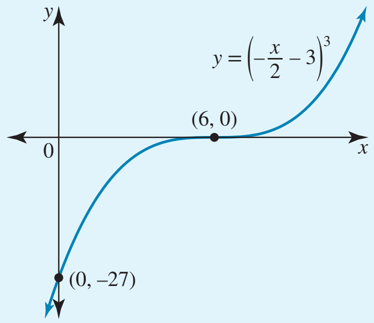
b



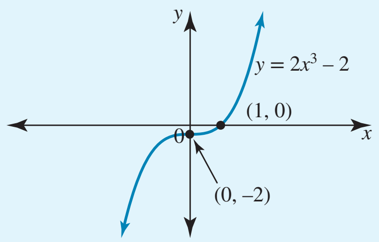
2

	Inflection point	y-intercept	x-intercept
a	(6, 0)	(0, −27)	(6, 0)
b	(0, −2)	(0, −2)	(1, 0)

a



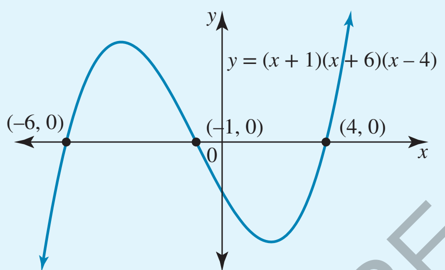
b



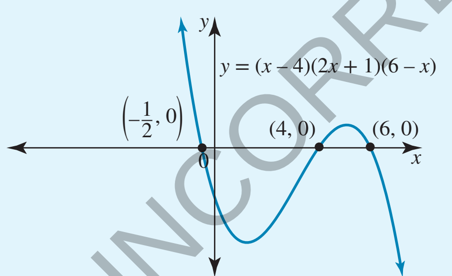
3

	y-intercept	x-intercepts
a	(0, −24)	(−6, 0), (−1, 0), (4, 0)
b	(0, −24)	$\left(-\frac{1}{2}, 0\right)$, (2, 0), (4, 0)

a

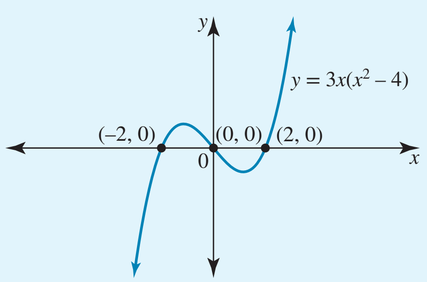


b



4

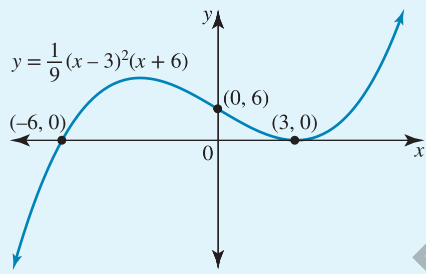
y-intercept	x-intercepts
(0, 0)	(−2, 0), (0, 0), (2, 0)



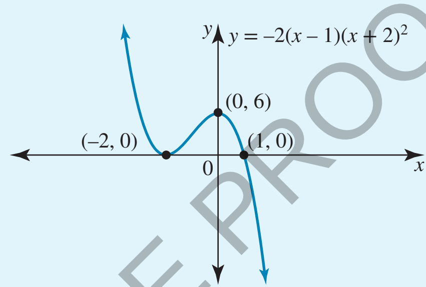
5

	y-intercept	x-intercepts
a	(0, 6)	(−6, 0) and (3, 0) which is a turning point
b	(0, 8)	(−2, 0) is a turning point and (1, 0)

a

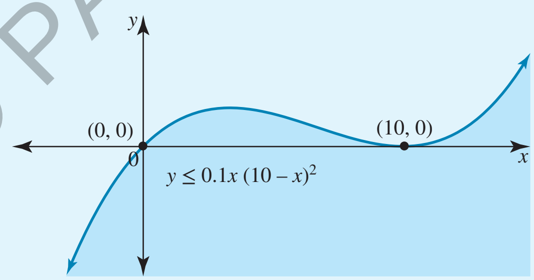


b



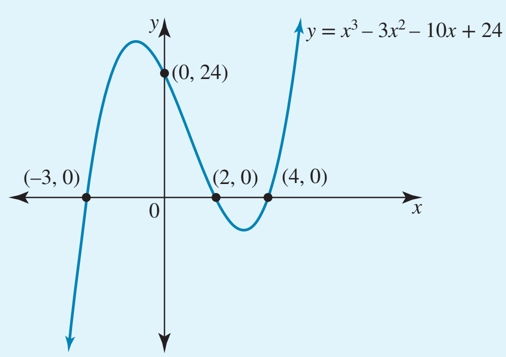
6

y-intercept	x-intercept
(0, 0)	(0, 0) and (10, 0) is a turning point



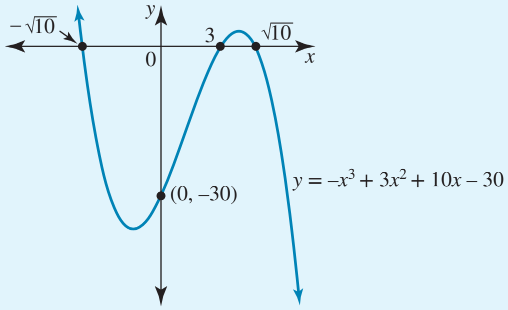
7 $x^3 - 3x^2 - 10x + 24 = (x - 2)(x - 4)(x + 3)$

y-intercept	x-intercepts
(0, 24)	(−3, 0), (2, 0), (4, 0)

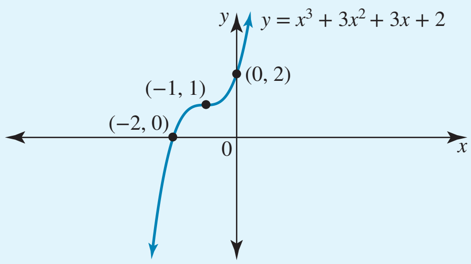


8 a $-x^3 + 3x^2 + 10x - 30 = -(x - 3)\left(x - \sqrt{10}\right)\left(x + \sqrt{10}\right)$

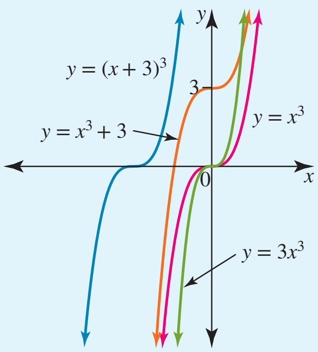
y-intercept	x-intercepts
(0, −30)	$\left(-\sqrt{10}, 0\right)$, (3, 0), $\left(\sqrt{10}, 0\right)$



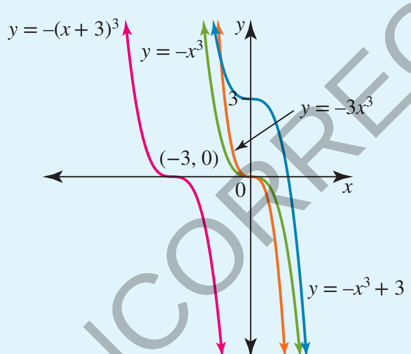
b Stationary point of inflection $(-1, 1)$; y-intercept $(0, 2)$; x-intercept $(-2, 0)$



9 a



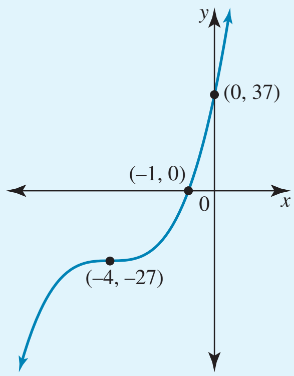
b



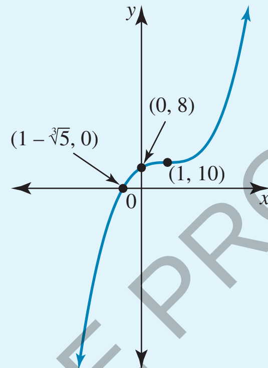
10

	Inflection point	y-intercept	x-intercept
a	$(-4, -27)$	$(0, 37)$	$(-1, 0)$
b	$(1, 10)$	$(0, 8)$	$(-0.7, 0)$ approx.
c	$(3, 27)$	$(0, -27)$	$(0.6, 0)$ approx.
d	$(-2, 16)$	$(0, 0)$	$(0, 0)$
e	$\left(-\frac{4}{3}, 0\right)$	$(0, -48)$	$\left(-\frac{4}{3}, 0\right)$
f	$(0, 9)$	$(0, 9)$	$(-3, 0)$

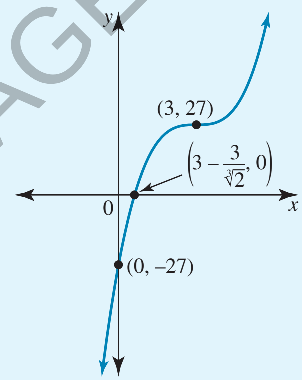
a



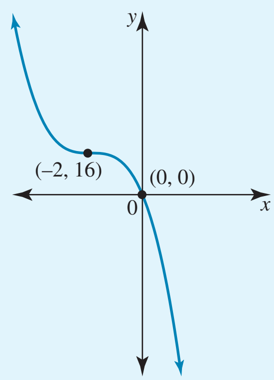
b



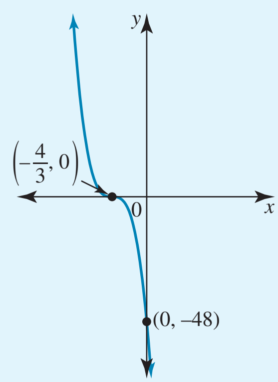
c



d



e



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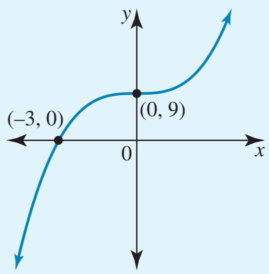

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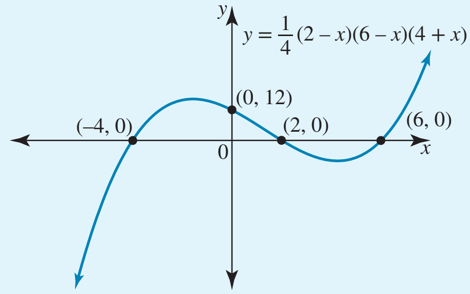
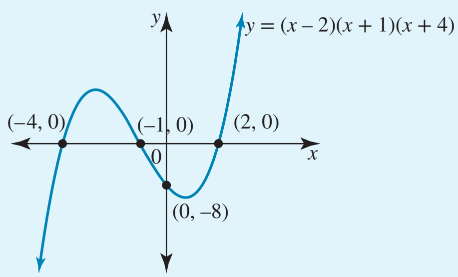


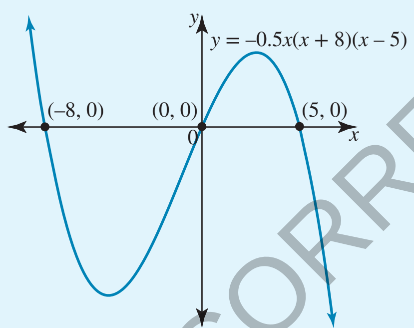
11

	y-intercept	x-intercepts
a	$(0, -8)$	$(-4, 0), (-1, 0), (2, 0)$
b	$(0, 0)$	$(-8, 0), (0, 0), (5, 0)$
c	$(0, -12)$	$(-3, 0), (1, 0), (4, 0)$
d	$(0, 12)$	$(-4, 0), (2, 0), (6, 0)$
e	$(0, 7)$	$\left(-\frac{1}{4}, 0\right), \left(\frac{7}{2}, 0\right), (10, 0)$
f	$\left(0, \frac{5}{2}\right)$	$\left(-\frac{8}{3}, 0\right), \left(\frac{5}{8}, 0\right), (2, 0)$

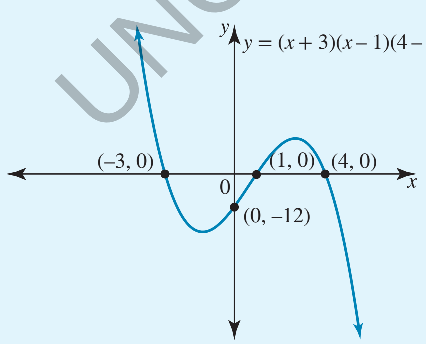
a



b



c



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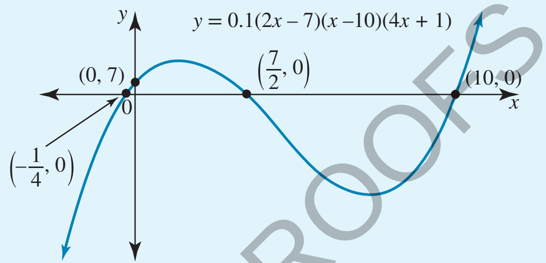
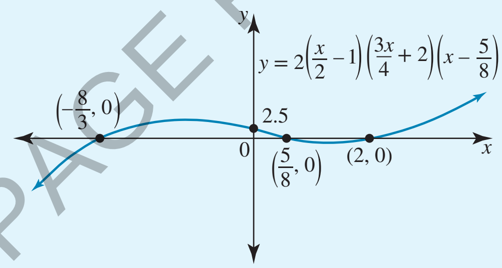


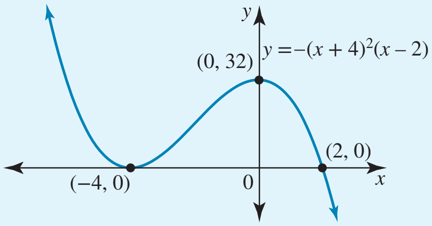
f



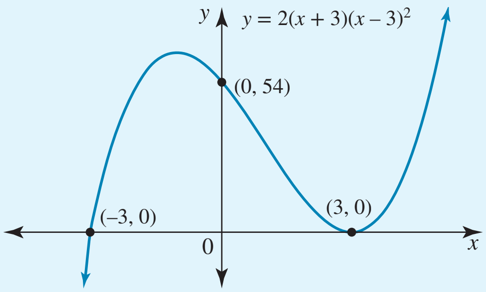
12

	y-intercept	x-intercepts
a	$(0, 32)$	$(-4, 0)$ is a turning point; $(2, 0)$ is a cut
b	$(0, 54)$	$(3, 0)$ is a turning point; $(-3, 0)$ is a cut
c	$(0, 36)$	$(-3, 0)$ is a turning point; $(4, 0)$ is a cut
d	$(0, -12)$	$(2, 0)$ is a turning point; $(12, 0)$ is a cut
e	$(0, 0)$	$\left(-\frac{3}{2}, 0\right)$ is a turning point; $(0, 0)$ is a cut
f	$(0, 0)$	$(0, 0)$ is a turning point; $(0.4, 0)$ is a cut

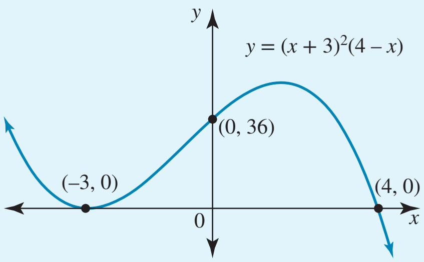
a



b



c



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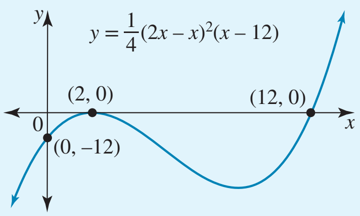

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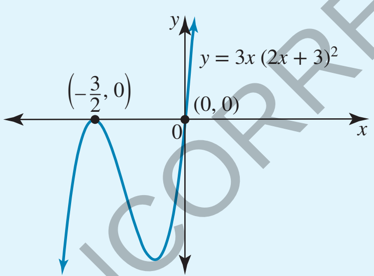
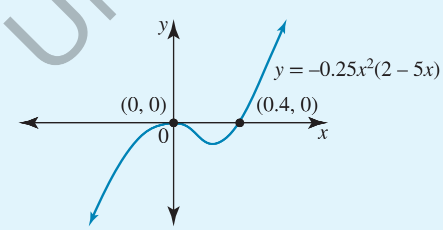


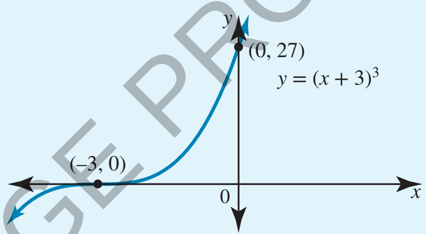
f



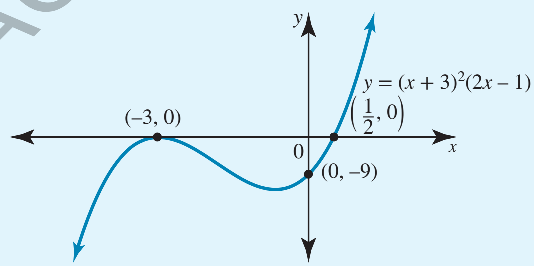
13

	Stationary point of inflection	y-intercept	x-intercepts
a	$(-3, 0)$	$(0, 27)$	$(-3, 0)$
b	none	$(0, -9)$	$(-3, 0)$ is a turning point; $\left(\frac{1}{2}, 0\right)$ is a cut
c	none	$(0, -15)$	$(-3, 0), \left(\frac{1}{2}, 0\right), (5, 0)$
d	$\left(\frac{1}{2}, 1\right)$	$\left(0, \frac{3}{2}\right)$	$(1.1, 0)$ approx.
e	none	$(0, 0)$	$\left(\frac{1}{4}, 0\right)$ is a turning point; $(0, 0)$ is a cut
f	none	$(0, 4)$	$\left(\frac{2}{3}, 0\right)$ is a turning point; $\left(-\frac{2}{3}, 0\right)$ is a cut

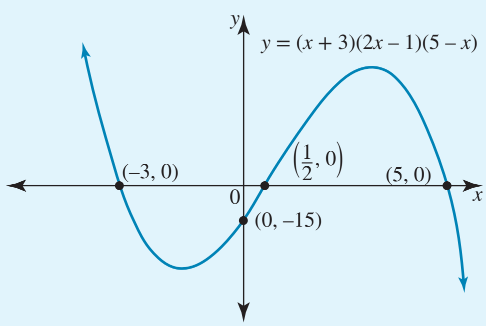
a



b



c



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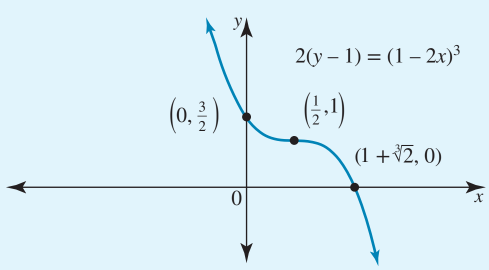

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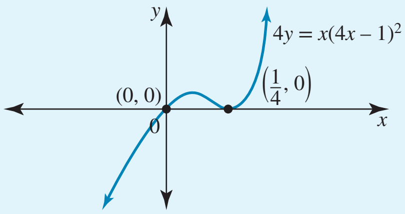
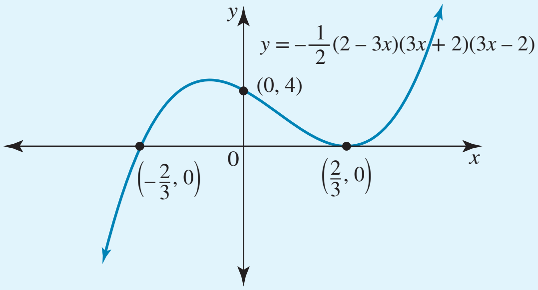
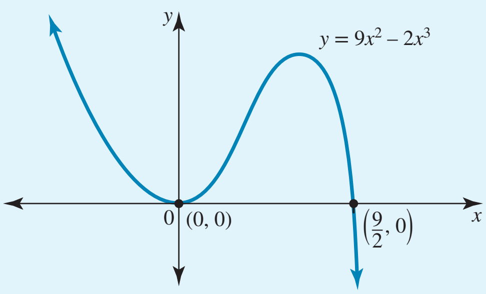


f



14 See table at foot of page.*

a



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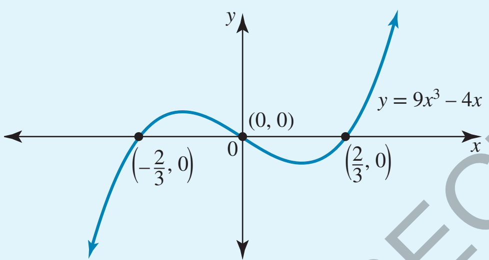

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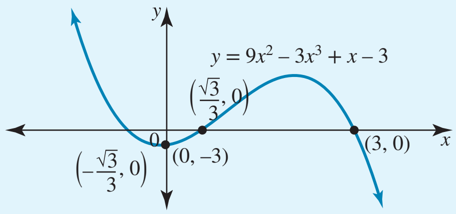

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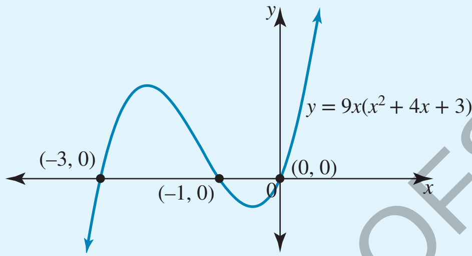

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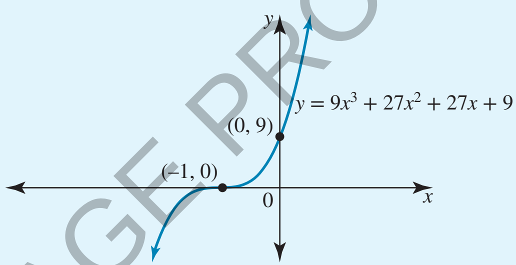
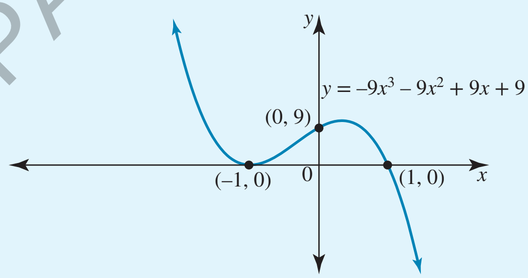


f

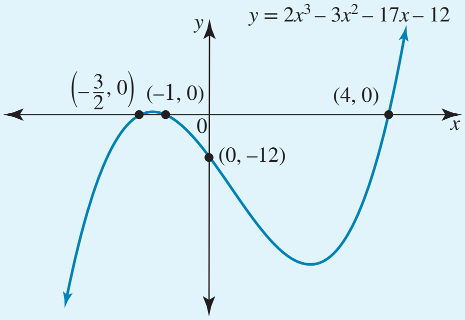


*14

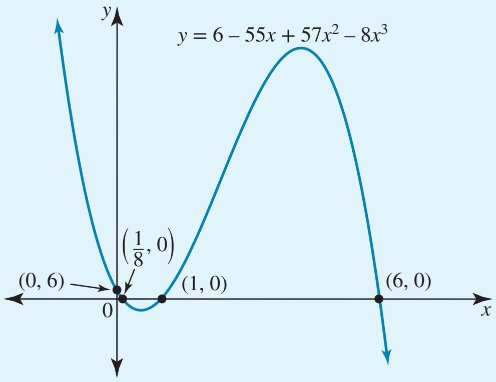
	Factorised form	Stationary point of inflection	y-intercept	x-intercepts
a	$y = x^2(9 - 2x)$	none	(0, 0)	(0, 0) is a turning point; $\left(\frac{9}{2}, 0\right)$ is a cut
b	$y = x(3x - 2)(3x + 2)$	none	(0, 0)	$\left(-\frac{2}{3}, 0\right)$, (0, 0), $\left(\frac{2}{3}, 0\right)$
c	$y = -3(x - 3)\left(x - \frac{\sqrt{3}}{3}\right)\left(x + \frac{\sqrt{3}}{3}\right)$	none	(0, −3)	$\left(-\frac{\sqrt{3}}{3}, 0\right)$, $\left(\frac{\sqrt{3}}{3}, 0\right)$, (3, 0)
d	$y = 9x(x + 1)(x + 3)$	none	(0, 0)	(−3, 0), (−1, 0), (0, 0)
e	$y = 9(x + 1)^3$	(−1, 0)	(0, 9)	(−1, 0)
f	$y = -9(x + 1)^2(x - 1)$	none	(0, 9)	(−1, 0) is a turning point; (1, 0) is a cut

15 See table at foot of page.*

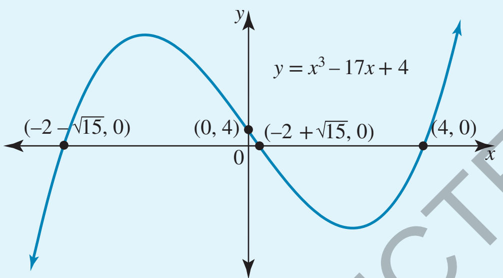
a



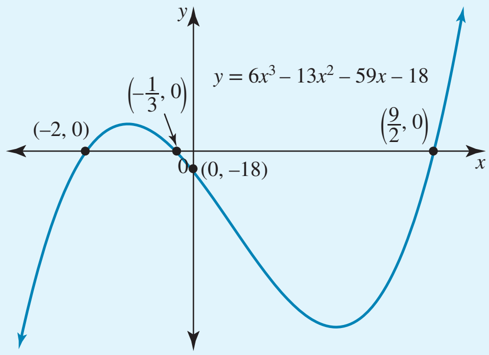
b



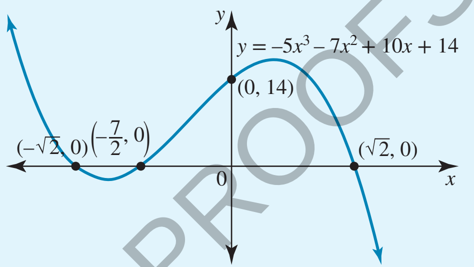
c



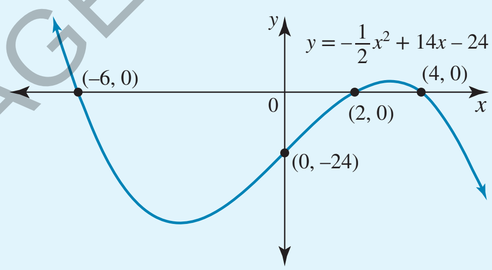
d



e



f



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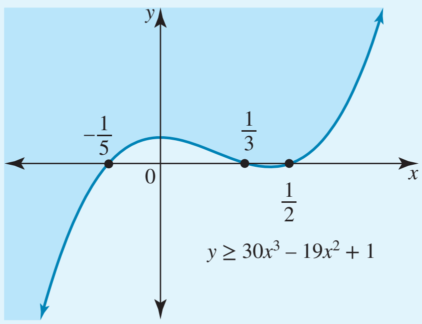
*15	Factorised form	y-intercept	x-intercepts
a	$y = (x + 1)(2x + 3)(x - 4)$	(0, −12)	$\left(-\frac{3}{2}, 0\right)$, (−1, 0), (4, 0)
b	$y = -(x - 1)(8x - 1)(x - 6)$	(0, 6)	$\left(\frac{1}{8}, 0\right)$, (1, 0), (6, 0)
c	$y = (x - 4)(x + 2 - \sqrt{5})(x + 2 + \sqrt{5})$	(0, 4)	$(-2 - \sqrt{5}, 0)$, $(-2 + \sqrt{5}, 0)$, (4, 0)
d	$y = (x + 2)(3x + 1)(2x - 9)$	(0, −18)	(−2, 0), $\left(-\frac{1}{3}, 0\right)$, $\left(\frac{9}{2}, 0\right)$
e	$y = (5x + 7)(\sqrt{2} - x)(\sqrt{2} + x)$	(0, 14)	$(-\sqrt{2}, 0)$, $\left(-\frac{7}{2}, 0\right)$, $(\sqrt{2}, 0)$
f	$y = -\frac{1}{2}(x - 2)(x + 6)(x - 4)$	(0, −24)	(−6, 0), (2, 0), (4, 0)

16 a $k = -19$

b $P(x) = (3x - 1)(5x + 1)(2x - 1)$

c $x = -\frac{1}{5}, \frac{1}{3}, \frac{1}{2}$

d y-intercept $(0, 1)$; x-intercepts $\left(-\frac{1}{5}, 0\right), \left(\frac{1}{3}, 0\right), \left(\frac{1}{2}, 0\right)$

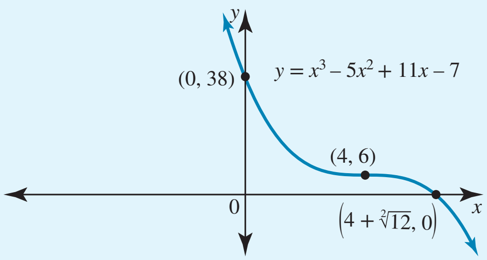


e No

f Shade the closed region above the graph of $y = P(x)$.

17 a $-\frac{1}{2}(x - 4)^3 + 6$

b



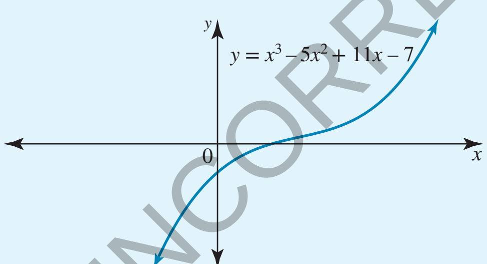
Stationary point of inflection $(4, 6)$; y-intercept $(0, 38)$; x-intercept approximately $(6.3, 0)$

18 a Proof required — check with your teacher. $y = (x - 1)(x^2 - 4x + 7)$

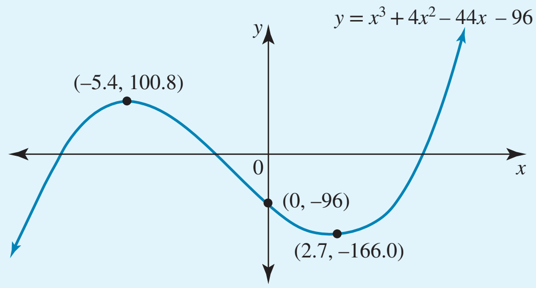
b Proof required — check with your teacher.

c $y \to \infty$

d



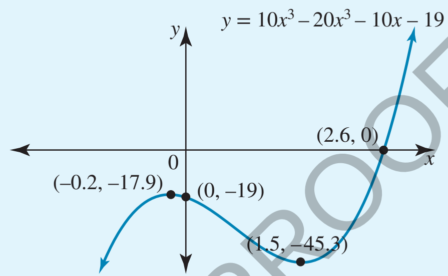
19 a Maximum turning point $(-5.4, 100.8)$; minimum turning point $(2.7, -166.0)$; y-intercept $(0, -96)$; x-intercepts $(-8, 0), (-2, 0), (6, 0)$



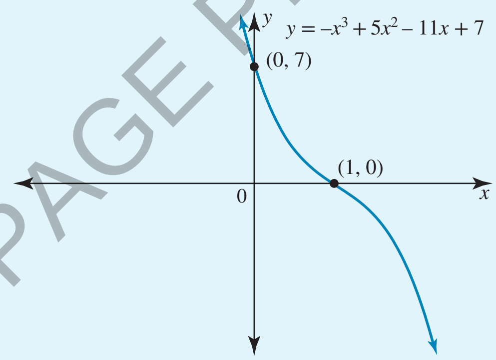
b Proof required

20 See table at foot of page.*

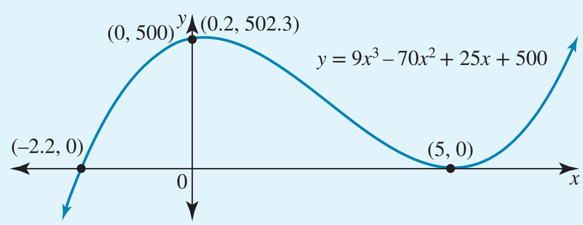
a



b



c



EXERCISE 4.5

1 a $y = \frac{1}{49}(x - 3)^3 - 7$

b $y = 0.25x(x + 5)(x - 4)$

c $y = -(x + 2)^2(x - 3)$

2 $y = 2x^3 + 3x^2 - 4x + 3$

3 a $x < 6, x \neq 2$

b $\{x : -2 \leq x \leq 0\} \cup \{x : x \geq 2\}$

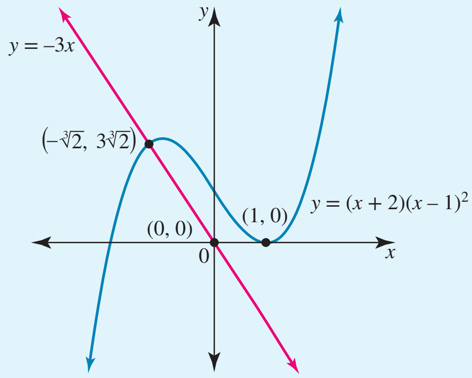
c $x < -2$

*20

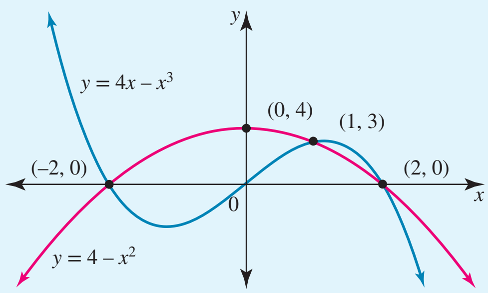
	Maximum turning point	Minimum turning point	y-intercept	x-intercept
a	$(-0.2, -17.9)$	$(1.5, -45.3)$	$(0, -19)$	$(2.6, 0)$
b	none	none	$(0, 7)$	$(1, 0)$
c	$(0.2, 502.3)$	$(5, 0)$	$(0, 500)$	$(-2.2, 0), (5, 0)$

4 $\{x : -1 < x < 1\} \cup \left\{x : x > \frac{7}{3}\right\}$

5 Point of intersection is $\left(-\sqrt[3]{2}, 3\sqrt[3]{2}\right)$.
When $x > -\sqrt[3]{2}$, $-3x < (x + 2)(x - 1)^2$.



6 Points of intersection are $(1, 3)$, $(2, 0)$, $(-2, 0)$.



7 a $y = \frac{1}{2}(x + 8)(x + 4)(x + 1)$
b $y = -2x^2(x - 5)$
c $y = -3(x - 1)^3 - 3$
d $y = \frac{4}{5}(x - 1)(x - 5)^2$

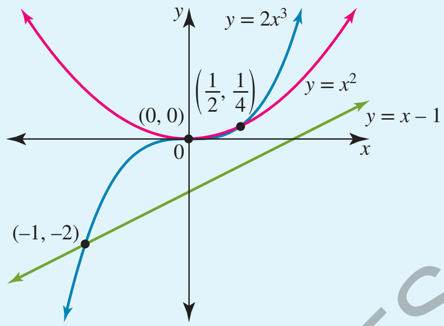
8 a $y = -2(x + 6)^3 - 7$
b $(0, -12)$
c $\left(\sqrt[3]{10} - 5, 0\right)$
d $(0, 7)$

9 a $y = -(x - a)^2(x - b)$
b $\{x : x \le b\}$
c More than b units to the left
d More than $-a$ units to the right

10 a $(x - 3)^2$
b $(-1, 0)$
c $a = -5, b = 3$

11 a $-9 \le x \le -1$ or $x \ge 2$
b $x > \frac{1}{5}$
c $x > 2.5$
d $x \le 0$ or $x = 1$
e $x < -2$ or $-\frac{6}{5} < x < 2$
f $-\frac{3}{2} < x < -1$ or $x > -\frac{1}{2}$

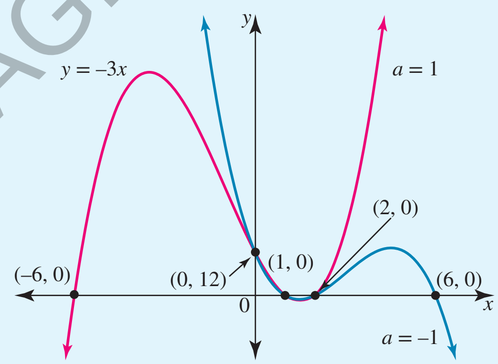
12 a $(0, 0), \left(\frac{1}{2}, \frac{1}{4}\right)$
b $(-1, -2)$
c



d $x \le 0.5$

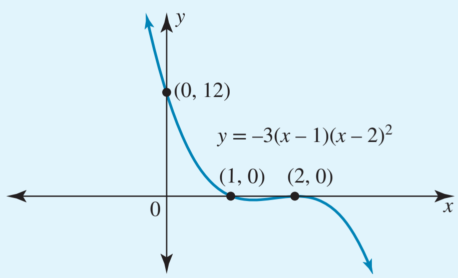
13 a $y = -2x + 5$; 1 solution
b $y = x^3 + 3x^2$, $y = 4x$; 3 solutions
c There are 3 solutions. One method is to use $y = x^3$, $y = -3x^2 + 4x$.
d $x = -4, 0, 1$

14 a Fewer than 4 pieces of information are given.
b Proof required — check with your teacher.
c



If $a = 1$, graph contains the points $(1, 0)$, $(2, 0)$, $(0, 12)$, $(-6, 0)$; if $a = -1$, the points are $(1, 0)$, $(2, 0)$, $(0, 12)$, $(6, 0)$.

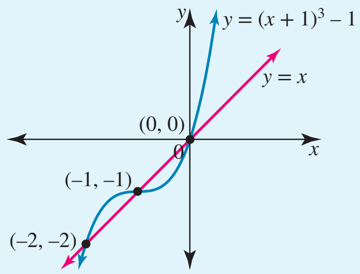
d Equation is $y = -3x^3 + 15x^2 - 24x + 12$ or $y = -3(x - 1)(x - 2)^2$; x-intercepts are $(1, 0)$, $(2, 0)$; $(2, 0)$ is a maximum turning point.



15 a $a = 1; b = -1$

b $(-2, -2), (-1, -1), (0, 0)$

c

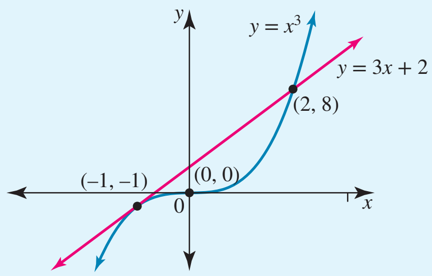


d $\{x : -2 < x < -1\} \cup \{x : x > 0\}$

16 a Proof required — check with your teacher.

b $(2, 8)$

c



d One intersection if $m < 3$; two intersections if $m = 3$; three intersections if $m > 3$

17 $y = -6x^3 + 12x^2 + 19x - 5$

18 a $(-3.11, -9.46), (-0.75, 0.02), (0.86, 6.44)$

b $x^3 + 3x^2 - x - 2 = 0$

c x-intercepts

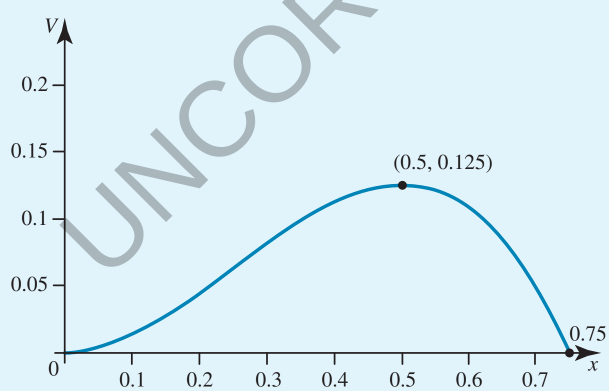
EXERCISE 4.6

1 a $h = \dfrac{3 - 4x}{2}$

b Proof required — check with your teacher.

c $0 \le x \le \frac{3}{4}$

d



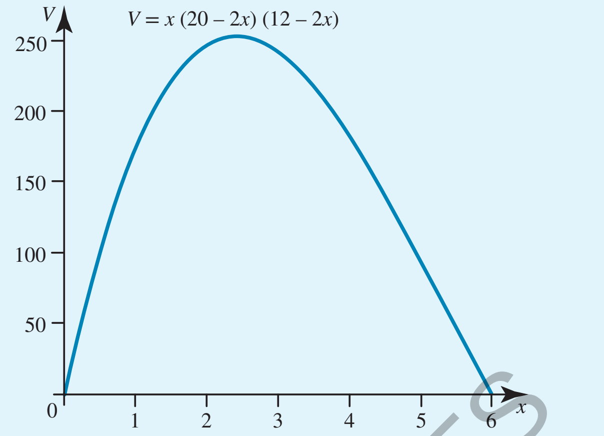
x-intercepts at $x = 0$ (touch), $x = 0.75$ (cut); shape of a negative cubic

e Cube of edge 0.5 m

2 a $l = 20 - 2x$; $w = 12 - 2x$; $V = (20 - 2x)(12 - 2x)x$

b $0 \le x \le 6$

c



x-intercepts at $x = 10, x = 6, x = 0$ but since $0 \le x \le 6$, the graph won't reach $x = 10$; shape is of a positive cubic.

d Length 15.14 cm; width 7.14 cm; height 2.43 cm; greatest volume 263 cm^3

3 a Loss of \$125; profit of \$184

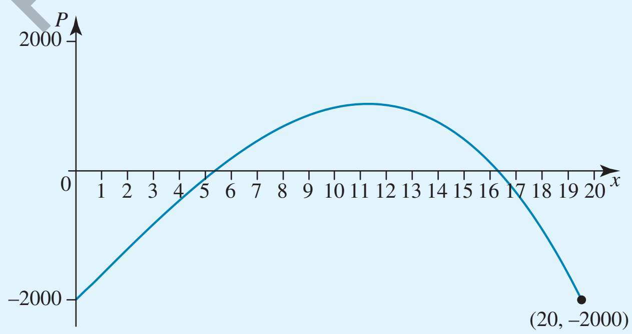
b Proof required — check with your teacher.

c Too many and the costs outweigh the revenue from the sales. A negative cubic tends to $-\infty$ as x becomes very large.

d **i** Profit \$304

ii Loss \$113

e



x-intercepts lie between 5 and 6 and between 16 and 17.

f Between 6 and 16

4 a 54

b 2 hours

c 210

d 5 pm

5 a $y = -\dfrac{1}{32}x^2(x - 6)$

b $\dfrac{81}{32}$ km

6 $-4, 1$

7 a $2\sqrt{2}x$

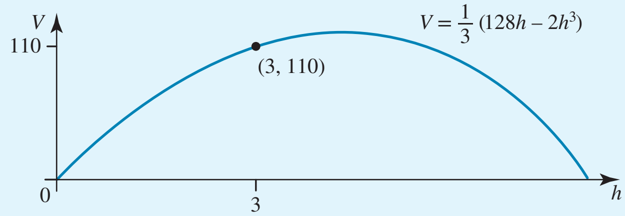
b Proof required — check with your teacher.

c Proof required — check with your teacher.

d **i** 110 m^3

ii Mathematically $0 \le h \le 8$

e



Max volume when h is between 4 and 5 (estimates will vary).

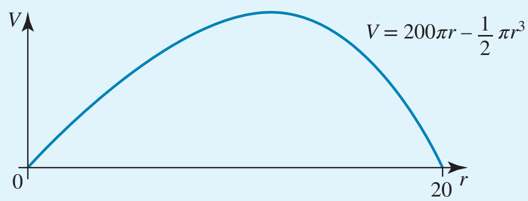
f Height $\frac{8}{\sqrt{3}} \approx 4.6$ m

8 a Proof required — check with your teacher.

b Proof required — check with your teacher.

c $0 \le r \le 20$

d



e Radius 2 cm, height 99 cm or radius 18.9 cm, height 1.1 cm

f 4837 cm^3

9 a (0, 2.1), (1.25, 1), (2.5, 1.1), (4, 0.1)

b $d = 2.1$

c $125a + 100b + 80c = -70.4$

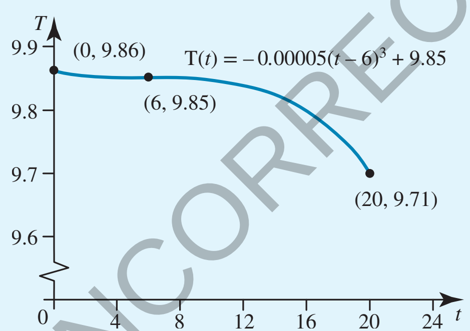
$125a + 50b + 20c = -8$

$64a + 16b + 4c = -2$

d 0.77 m

10 a $T(3) = 9.85$, $T(20) = 9.71$

b



c $T(28) = 9.32$; unlikely, but not totally impossible. Model is probably not a good predictor.

11 a $x = 0, x = 6$

b Length $2x - 6$; width $6x - x^2$

c Proof required

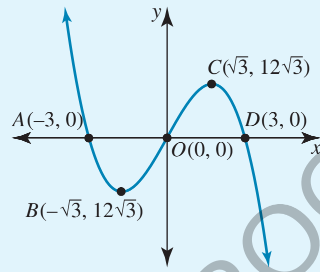
d $3 \le x \le 6$

e $x = 4, x = \frac{5 + \sqrt{33}}{2}$

12 a 3 x-intercepts; 2 turning points

b $y = -2x(x^2 - 9)$

c



d $y = 12x$

e $y = 4x^3$

13 a 4.62 m

b Proof required — check with your teacher.

c Greatest volume of 4836.8 cm^3; base radius 11.5 cm; height 11.5 cm

14 a $y = -0.164x^3 + x^2 - 1.872x + 2.1$

b Length 3.464 units; width 6.000 units