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Construction of Balanced Doubles Schedules

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A recursive construction for v-player balanced doubles schedules based on resolvable $B[4, 1; v]$ designs is presented. The schedules are resolvable when $v \equiv 4 \pmod{12}$. Combined with special constructions for $v = 54$ and 55 and previously known constructions, this gives schedules for all v.

Sportsmen wanting to test individual performance in a team sport need a tournament schedule that will shuffle the players among the teams instead of shuffling the teams. To ensure fairness to each player, the schedule should assign each pair of players to the same team exactly λ_1 times and to opposing teams exactly λ_2 times. A *balanced doubles schedule* (BDS) for v players (S_v) accomplishes this for doubles, while minimizing the number of matches. The following are BD schedules for four, five, six, and seven players.

S_4:

1,2	vs	3,4
2,3		1,4
3,1		2,4

S_5:

1,2	vs	3,5
2,3		4,1
3,4		5,2
4,5		1,3
5,1		2,4

S_6:

1,4	vs	2,3	1,4	vs	2,6	2,3	vs	1,6
2,5		3,4	2,5		3,6	3,4		2,6
3,1		4,5	3,1		4,6	4,5		3,6
4,2		5,1	4,2		5,6	5,1		4,6
5,3		1,2	5,3		1,6	1,2		5,6

S_7:

1,2	vs	3,4	2,4	vs	6,1	4,1	vs	5,2
2,3		4,5	3,5		7,2	5,2		6,3
3,4		5,6	4,6		1,3	6,3		7,4
4,5		6,7	5,7		2,4	7,4		1,5
5,6		7,1	6,1		3,5	1,5		2,6
6,7		1,2	7,2		4.6	2,6		3,7
7,1		2,3	1,3		5,7	3,7		4,1

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Constructing BD schedules lacks the illustrious mathematical pedigree of the Kirkman schoogirl problem, but has a similar history, and a similar solution: A hard puzzle [4, 7] posed in the 19th century that is extended in the 20th into an interesting combinatoric problem with applications in statics and calibration, and a solution which depends on block designs.

Until the sixties, when Bose and Cameron [1] picked up the problem, most published results consisted of listings for schedules with fewer than 17 players. In their 1965 paper, Bose and Cameron restated the problem in terms of the criteria given on p. 1, constructed schedules for between 4 and 50 players using a variety of techniques, and showed that several infinite classes of BDS existed. In the follow-up paper 2 years later [2], they looked at tournaments involving larger teams and demonstrated the schedules' value as statistical designs for high-precision calibration.

This paper offers constructions which produce BDS for any number of players greater than 50, thus completing the work done on the problem by Bose Cameron.

The parameters needed to discuss BD schedules are borrowed from block designs.

- v Number of players
- b Number of matches
- r Number of matches played by each player
- λ_1 Number of matches in which each pair appears as a team
- λ_2 Number of matches in which a pair appears as opponents

A careful consideration of the definition reveals much about the parameters. Since a player must team with each of $v - 1$ other players λ_1 times, $r = \lambda_1(v - 1)$. Further, he opposes two players in each match and must face every other player λ_2 times, so $2r = \lambda_2(v - 1)$. Eliminating r and $v - 1$ yields $\lambda_2 = 2\lambda_1$.

Each of b matches has four players, and each of v players is in r matches, so

$$4b = vr = v\lambda_1(v - 1),$$
$$b = \lambda_1 v(v - 1)/4,$$

and since b must be an integer, if $\lambda_1 = 1$, then $v \equiv 0,1 \pmod 4$, $r = v - 1$ and $\lambda_2 = 2$. Call these schedules type 1; for examples see S_4 and S_5 above.

Type 2 schedules will have $\lambda_1 = 2$, $v \equiv 2, 3 \pmod 4$, $b = v(v - 1)/2$, $r = 2(v - 1)$ and $\lambda_2 = 4$; S_6 and S_7 represent this category.

At this point some definitions are required because the construction given for the BD schedules depends on pairwise balanced and resolvable incomplete block designs.

When v and λ are positive integers and K is a finite set of integers, a pairwise balanced design $B[K, \lambda; v]$ is a design on a finite set of points X, consisting of blocks B_i (subsets of X) so that

1. $|X| = v$;
2. $|B_i| \in K$;
3. each unordered pair of points $\{x, y\}$ is in exactly λ blocks, making it pairwise balanced.

Since blocks of various sizes can occur, b_k is defined as the number of blocks of size k; of course, the total number of blocks $b = \sum b_k$.

When the blocks of a pairwise balanced design are all one size, $B[k, \lambda; v]$ is a *balanced incomplete block design* (BIBD). For this design, $b = \lambda v(v - 1)/k(k - 1)$ and each point occurs in $r = \lambda(v - 1)/(k - 1)$ blocks.

A *parallel class* in a BIBD is a class of v/k blocks that partition X. A BIBD is *resolvable* if its blocks can be partitioned into $\lambda(v - 1)/(k - 1)$ parallel classes.

As an example of a resolvable BIBD [4, 1; 16]:

1,2,3,4	5,6,7,8	9,10,11,12	13,14,15,16
1,5,9,13	2,6,10,14	3,7,11,15	4,8,12,16
1,6,11,16	2,5,12,15	3,8,9,14	4,7,10,13
1,7,12,14	2,8,11,13	3,5,10,16	4,6,9,15
1,8,10,15	2,7,9,16	3,6,12,13	4,5,11,14

(Kraitchik [7, p. 236])

Here, each of the five lines forms a parallel class of four blocks.

Hanani *et al.* [6] showed that there exists a resolvable BIBD $B[4, 1; v]$ iff $v \equiv 4 \pmod{12}$. Label any of these resolvable designs D_t. D_t is a $B[4, 1; 12t + 4]$ design with parameters $b = (3t + 1)(4t + 1)$ and $r = 4t + 1$. It has $4t - 1$ parallel classes each with $4/v = 3t + 1$ blocks. Kraitchik's design is D_1.

A schedule can be constructed from the blocks of a pairwise balanced design, $B[k, 1; v]$. When v and the block sizes are congruent to 0, 1 (mod 4), the players assigned to each block play a tournament among themselves using schedule S_k, resulting in a type 1 schedule. When $v \equiv 2, 3 \pmod 4$ a type 2 schedule is needed, so players assigned to a block with $k \equiv 0, 1 \pmod 4$ must follow S_k twice, while those assigned to one with $k \equiv 2, 3 \pmod 4$ follow S_k just once. In either case, the pairwise balance of the design assigns each pair to exactly one block and then the k-player tournament based on that block satisfies the partner–opponent requirements.

Using Kraitchik's resolvable $B[4, 1; 16]$ BIBD D_1 and S_4, this process gives a resolvable BD schedule for 16 players.

	1 2 vs 3 4	5 6 vs 7 8	9 10 vs 11 12	13 14 vs 15 16
	1 3 2 4	5 7 6 8	9 11 10 12	13 15 14 16
	1 4 2 3	5 8 6 7	9 12 10 11	13 16 14 15
	1 5 9 13	2 6 10 14	2 7 11 15	4 8 12 16
	1 9 5 13	2 10 6 14	3 11 7 15	4 12 8 16
	1 13 5 9	2 14 6 10	3 15 7 11	4 16 8 12
	1 6 11 16	2 5 12 15	3 8 9 14	4 7 10 13
$S_{1,0}$	1 11 6 16	2 13 5 15	3 9 8 14	4 10 7 13
	1 16 6 11	2 15 5 12	3 14 8 9	4 13 7 10
	1 7 12 14	2 8 11 13	3 5 10 16	4 6 9 15
	1 12 7 14	2 11 8 13	3 10 5 16	4 9 6 15
	1 14 7 12	2 13 8 11	3 16 5 10	4 15 6 9
	1 8 10 15	2 7 9 16	3 6 12 13	4 5 11 14
	1 10 8 15	2 9 7 16	3 12 6 13	4 11 5 14
	1 15 8 10	2 16 7 9	3 13 6 12	4 14 5 11

Clearly a schedule based directly on D_t is resolvable giving a resolvable BDS whenever $v \equiv 4 \pmod{12}$.

A more powerful construction is to add new points to the resolvable design forming a pairwise balanced design and then construct the schedule.

Starting from D_t for each integer c, $0 \leqslant c \leqslant 4t + 1$, construct a pairwise balanced design $D_{t,c}$ by adjoining c new points, one to each block of c parallel classes; when $c > 1$ use the set of new points C as an additional block. $D_{t,c}$ is a $B[\{4, 5, c\}, 1; v' = 12t + 4 + c]$ design with $b_5 = c(3t + 1)$,

$$b_4 = (r - c)(3t + 1) \qquad \text{and} \qquad b_c = \begin{cases} 0 & \text{for } c = 0, 1 \\ 1 & \text{for } c > 1. \end{cases}$$

When $c = 4$ (5), C should be counted with the other four blocks (five blocks) and will raise b_4 (b_5) by one.

$D_{t,c}$ is the basis for a balanced doubles schedule $S_{t,c}$ when $c \in \{0, 1, 4, 5, 6, \ldots 4t + 1\}$. ($c = 2, 3$ are lost since it is impossible to arrange the two or three players assigned to block C into a doubles match.) If $v' \equiv 0, 1 \pmod 4$, then so is c; thus the blocks will all support type 1 schedules. If $v' \equiv 2, 3 \pmod 4$, the same is true for c, and block C will support the type 2 schedule S_c while the 4- and 5-blocks will use S_4 and S_5 twice, which will still produce the type 2 schedule needed for v'.

To illustrate the process, here is the construction for a 20-player BDS.

Kraitchik's resolvable BIBD listed above serves as D_1. Adjoining the points of the new block $C = \{17, 17, 19, 20\}$ to the first four parallel classes gives the pairwise balanced design

	1, 2, 3, 4, 17	5, 6, 7, 8, 17	9, 10, 11, 12, 17	13, 14, 15, 16, 17
	1, 5, 9, 13, 18	2, 6, 10, 14, 18	3, 7, 11, 15, 18	4, 8, 12, 16, 18
	1, 6, 11, 16, 19	2, 5, 12, 15, 19	3, 8, 9, 14, 19	4, 7, 10, 13, 19
$D_{1,4}$	1, 7, 12, 14, 20	2, 8, 11, 13, 20	3, 5, 10, 16, 20	4, 6, 9, 15, 20
	1, 8, 10, 15	2, 7, 9, 16	3, 6, 12, 13	4, 5, 11, 14
	17, 18, 19, 20			

Note that $K = \{4, 5\}$, $b_4 = 5$, and $b_5 = 16$.

Using S_4 and S_5 with the appropriate blocks produces $S_{1,4}$, a balanced doubles schedule for 20 players with 95 blocks and r equal to 19.

1 2 vs 3 17	5 6 vs 7 17	9 10 vs 11 17	13 14 vs 15 17
2 3 4 1	6 7 8 5	10 11 12 9	14 15 16 13
3 4 17 2	7 8 17 6	11 12 17 10	15 16 17 14
4 17 1 3	8 17 5 7	12 17 9 11	16 17 13 15
17 1 2 4	17 5 6 8	17 9 10 12	17 13 14 16
1 5 9 18	2 6 10 18	3 7 11 18	4 8 12 18
5 9 13 1	6 10 14 2	7 11 15 3	8 12 16 4
9 13 18 5	10 14 18 6	11 15 18 7	12 16 18 8
13 18 1 9	14 18 2 10	15 18 3 11	16 18 4 12
18 1 5 13	18 2 6 14	18 3 7 15	18 4 8 16
1 6 11 19	2 5 12 19	3 8 9 19	4 7 10 19
6 11 16 1	5 12 15 2	8 9 14 3	7 10 13 4
11 16 19 6	12 15 19 5	9 14 19 8	10 13 19 7
16 19 1 11	15 19 2 12	14 19 3 9	13 19 4 10
19 1 6 16	19 2 5 15	19 3 8 14	19 4 7 13
1 7 12 20	2 8 11 20	3 5 10 20	4 6 9 20
7 12 14 1	8 11 13 2	5 10 16 3	6 9 15 4
12 14 20 7	11 13 20 8	10 16 20 5	9 15 20 6
14 20 1 12	13 20 2 11	16 20 3 10	15 20 4 9
20 1 7 14	20 2 8 13	20 3 5 16	20 4 6 15
1 8 10 15	2 7 9 16	3 6 12 13	4 5 11 14
1 10 8 15	2 9 7 16	3 12 6 13	4 11 5 14
1 15 8 10	2 16 7 9	3 13 6 12	4 14 5 11
			17 18 19 20
			17 19 18 20
			17 20 18 19

BDS for 20 players $S_{1,4}$

Combining this construction with the results of Hanani *et al.* produces a BDS for all but 26 cases, providing a simple, constructive proof of the existence of BD schedules for any number of players.

To see that this is true, let V_t be the set of integers (v) with a BDS constructed from a resolvable $B[4, 1; 12t + 4]$. Thus,

$$V_1 = \{16, 17, 20, 21\},$$
$$V_2 = \{28, 29, 32, 33, 34, 35, 36, 37\},$$
$$V_3 = \{40, 41, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53\},$$
$$V_4 = \{52, 53, 56,..., 69\},$$
$$V_5 = \{64, 65, 68,..., 85\},$$

and, in general,

$$V_t = \{12t + 4, 12t + 5, 12t + 8,..., 12t + c,..., 16t + 5\}.$$

With $t \geqslant 3$ the sets overlap, allowing several different constructions for a BDS. Note that the BDSs for $v = 12t + 6, 7$ cannot be developed from D_t, but may be built from D_{t-1}.

Bose and Cameron's constructions [1] settle all the missing cases except 54 and 55. An easy construction for BDS on 54 and 55 players is based on a block design from $PG(2, 7)$. Hall, [5, p. 292] gives a starting block (1, 6, 7, 9, 19, 38, 42, 49) which is developed into a $B[8, 1; 57]$ BIBD by incrementing all the numbers mod 57. The design has $b = 57$ and $r = 8$.

Deleting 56 and 0 from all blocks leaves a pairwise balanced design. Since the pair 56, 0 occurred in one block, it has been shortened to 6. Fourteen other blocks lose one entry, and 42 are unchanged: thus $b_6 = 1$, $b_7 = 14$, and $b_8 = 42$. Using S_6 and S_7 with the 6- and 7-blocks, and S_8 twice with each 8-block produces the needed BDS for 55 players.

A check of the starting block guarantees that three consecutive numbers cannot appear in the same block. So deleting 55, 56 and 0 gives a pairwise design with $b_6 = 3$, $b_7 = 18$ and $b_8 = 36$. The required schedule for 54 players may be developed from this design in the same manner.

This completes the proof of the theorem: A balanced doubles schedule can be constructed for any number of players $v \geqslant 4$.

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