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Weakly distance-regular digraphs

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Abstract

We consider the following generalization of distance-regular digraphs. A connected digraph Γ is said to be weakly distance-regular if, for all vertices x and y with $(\partial(x,y),\partial(y,x))=\tilde{h}$, $|\{z\in V\Gamma \mid (\partial(x,z),\partial(z,x))=\tilde{i}$ and $(\partial(z,y),\partial(y,z))=\tilde{j}\}|$ depends only on $\tilde{h}$, $\tilde{i}$ and $\tilde{j}$. We give some constructions of weakly distance-regular digraphs and discuss the connections to association schemes. Finally, we determine all commutative weakly distance-regular digraphs of valency 2.

Keywords: Weakly distance-regular digraph; Association scheme; Cayley digraph

1. Introduction

A *digraph* Γ is a pair (X,E) where X is a finite set of *vertices* and $E\subseteq X^2$ is a set of *arcs*. Throughout this paper we use the term 'digraph' to mean a finite directed graph with no loops. We often write $V\Gamma$ for X and $E\Gamma$ for E. An arc (u,v) of Γ is *undirected* if $(v,u)\in E\Gamma$. A *path* of length r joining u and v is a finite sequence of vertices $(u=w_0,w_1,\ldots,w_r=v)$ such that $(w_{t-1},w_t)\in E\Gamma$ for $t=1,2,\ldots,r$. A path $(w_0,w_1,\ldots,w_{r-1})$ with distinct vertices is called a *circuit* of length r if $(w_{r-1},w_0)\in E\Gamma$. A shortest circuit is called a *minimal circuit*. The *girth* g of Γ is the length of a minimal circuit. If a digraph contains an undirected arc, its girth is 2 by the definition. The number of arcs traversed in a shortest path joining u and v is called the *distance* from u

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to v in Γ, denoted by $\partial(u,v)$. The maximum value of the distance function in Γ is called the *diameter* d of Γ. For any two vertices $x,y\in V\Gamma$, define $\tilde{\partial}(x,y)=(\partial(x,y),\partial(y,x))$.

Let $\Gamma=(X,E)$ and $\Gamma'=(X',E')$ be two digraphs. A bijection σ from X to X' is an *isomorphism* from Γ to Γ' if $(x,y)\in E$ if and only if $(\sigma(x),\sigma(y))\in E'$. We do not distinguish between two isomorphic digraphs. An isomorphism from Γ to Γ is called an *automorphism* of Γ. The set of all automorphisms of Γ forms a group under the operation of composition. The group is called the *automorphism group* and denoted by $\mathrm{Aut}(\Gamma)$. A digraph Γ is *vertex transitive* if $\mathrm{Aut}(\Gamma)$ is transitive on $V\Gamma$.

Lam [5] introduced a concept of *distance-transitive digraphs* (by requiring that there exists an automorphism σ taking x to x' and y to y' whenever $\partial(x,y)=\partial(x',y')$), and gave some elementary properties and examples. Damerell [4] generalized this concept to that of distance-regular digraphs. He proved that the girth g of a distance-regular digraph of diameter d is either 2, d or $d+1$, and that the one with $d=g$ is a coclique extension of a distance-regular digraph with $d=g-1$. Bannai, Cameron and Kahn [1] proved that a distance-transitive digraph of odd girth is a Paley tournament or a directed cycle. Leonard and Nomura [6] proved that except directed cycles all distance-regular digraphs with $d=g-1$ have girth $g\leqslant 8$. In order to find 'good' classes of digraphs with unbounded diameter, the condition of distance-regularity seems to be too strong. Damerell [4] suggested a more natural definition of distance-transitivity, i.e., weakly distance-transitivity. In this paper, we introduce weakly distance-regular digraphs. In Section 2, we give some constructions of weakly distance-regular digraphs. In Section 3, connections to association schemes are discussed. In the last section, we determine all commutative weakly distance-regular digraphs of valency 2.

Definition 1.1 (Damerell [4]). A connected digraph Γ is said to be *weakly distance-transitive* if, for any vertices x, y, x' and y' of Γ satisfying $\tilde{\partial}(x,y)=\tilde{\partial}(x',y')$, there exists an automorphism $\sigma\in\mathrm{Aut}(\Gamma)$ such that $x'=\sigma(x)$ and $y'=\sigma(y)$.

Definition 1.2. A connected digraph Γ is said to be *weakly distance-regular* if

$$p^{\tilde{h}}_{\tilde{i},\tilde{j}}(x,y)=|\{z\in V\Gamma \mid \tilde{\partial}(x,z)=\tilde{i} \text{ and } \tilde{\partial}(z,y)=\tilde{j}\}|$$

depends only on $\tilde{i}$, $\tilde{j}$, $\tilde{h}$ and does not depend on the choices of x and y with $\tilde{\partial}(x,y)=\tilde{h}$. The numbers $p^{\tilde{h}}_{\tilde{i},\tilde{j}}$ are called the *intersection numbers* of Γ.

It is easy to see that a weakly distance-transitive digraph is weakly distance-regular.

A weakly distance-regular digraph Γ is *commutative* if $p^{\tilde{h}}_{\tilde{i},\tilde{j}}=p^{\tilde{h}}_{\tilde{j},\tilde{i}}$ for all $\tilde{i}$, $\tilde{j}$, $\tilde{h}$. Let $\Gamma_i(x)=\{y\in V\Gamma \mid \partial(x,y)=i\}$. $k_i=|\Gamma_i(x)|$ does not depend on the choice of $x\in V\Gamma$ and $k=|\Gamma_1(x)|$ is called the *valency* of Γ. Clearly, a weakly distance-regular digraph of valency 1 is a directed cycle.

Let Γ be a connected digraph of diameter d. Let $A_{i,j}$ be a square matrix of size $|V\Gamma|$, whose rows and columns are indexed by vertices of Γ such that

$$(A_{i,j})_{x,y}=\begin{cases}1 & \text{if } \tilde{\partial}(x,y)=(i,j),\\ 0 & \text{otherwise.}\end{cases}$$

$A_{i,j}$ is called the (i,j)th *adjacency matrix* of Γ. Γ is a weakly distance-regular digraph, if the span of the set $\{A_{i,j} \mid 0 \leqslant i,j \leqslant d\}$ is closed under multiplication.

Definition 1.3. Let G be a finite group and S a subset of G not containing the identity element. We define the *Cayley digraph* $\Gamma = \mathrm{Cay}(G,S)$ of G with respect to S by

$$V\Gamma = G \quad \text{and} \quad E\Gamma = \{(x,sx) \mid x \in G,\ s \in S\}.$$

A Cayley digraph $\Gamma = \mathrm{Cay}(G,S)$ is connected if and only if $G = \langle S \rangle$. It is obvious that $\mathrm{Aut}(\Gamma)$ contains the right regular representation of G, and so Γ is vertex transitive.

The following is our main result:

Theorem 1.1. *If Γ is a commutative weakly distance-regular digraph of valency* 2 *and girth g, then Γ is isomorphic to one of the following*:

(1) $\mathrm{Cay}(Z_{2g}, \{\bar{1}, \bar{2}\})$.
(2) $\mathrm{Cay}(Z_2 \times Z_q, \{(\bar{0},\bar{1}),(\bar{1},\bar{0})\})$, $q \geqslant 3$.
(3) $\mathrm{Cay}(Z_n, \{\bar{1}, \overline{n-1}\})$, $n \geqslant 3$.
(4) $\mathrm{Cay}(Z_{2g}, \{\bar{1}, \overline{g+1}\})$.
(5) $\mathrm{Cay}(Z_3^2, \{(\bar{0},\bar{1}),(\bar{1},\bar{0})\})$.

2. Constructions

Now we give another characterization of weakly distance-transitive digraphs. Let Γ be a weakly distance-regular digraph. For each vertex x of Γ, we define

$$\Gamma_{i,j}(x) = \{y \in V\Gamma \mid \tilde{\partial}(x,y) = (i,j)\}.$$

It is easy to see that $k_{i,j} = |\Gamma_{i,j}(x)|$ does not depend on the choice of x. For vertices x and y of Γ, let

$$P_{\tilde{i},\tilde{j}}(x,y) = \{z \in V\Gamma \mid \tilde{\partial}(x,z) = \tilde{i} \text{ and } \tilde{\partial}(z,y) = \tilde{j}\}.$$

If $\tilde{\partial}(x,y) = \tilde{h}$, then $|P_{\tilde{i},\tilde{j}}(x,y)| = p^{\tilde{h}}_{\tilde{i},\tilde{j}}(x,y)$.

The proof of next proposition is similar to the one in the undirected case. (see [3].)

Proposition 2.1. *A connected digraph Γ with diameter d is weakly distance-transitive if and only if it is vertex transitive and the stabilizer of a fixed vertex v is transitive on the set $\Gamma_{i,j}(v)$ for each $i,j \in \{0,1,\ldots,d\}$.*

Proposition 2.2. *Let G be a finite abelian group and S a subset of G not containing the identity element. If $\Gamma = \mathrm{Cay}(G,S)$ is a weakly distance-regular digraph, then Γ is commutative.*

Proof. Since G is abelian, we write the operation additively. Thus

$$\tilde{\partial}(\bar{x}, \bar{y}) = (t_1, t_2) \text{ if and only if } \tilde{\partial}(-\bar{y}, -\bar{x}) = (t_1, t_2).$$

It is easy to see that

$$\bar{z} \in P_{(i_1,i_2),(j_1,j_2)}(\bar{x}, \bar{y}) \text{ if and only if } -\bar{z} \in P_{(j_1,j_2),(i_1,i_2)}(-\bar{y}, -\bar{x}).$$

So we have

$$p^{(t_1,t_2)}_{(i_1,i_2),(j_1,j_2)} = p^{(t_1,t_2)}_{(j_1,j_2),(i_1,i_2)}.$$

Hence, the desired result follows. □

Remarks. This result can also be obtained as a corollary to the following well-known result: Let (G, X) be a transitive group with an abelian subgroup H acting regularly on X. Then the group-type association scheme associated with it is commutative.

Proposition 2.3. *Let Z_{2g} be a cyclic group of order $2g$. Then*

$$\Gamma = \mathrm{Cay}(Z_{2g}, \{\bar{1}, \bar{2}\})$$

is a commutative weakly distance-transitive digraph.

Proof. Let $0 \leqslant x, y \leqslant 2g-1$. If $\tilde{\partial}(\bar{0}, \bar{x}) = \tilde{\partial}(\bar{0}, \bar{y})$, then

$$\left\lceil \frac{x}{2} \right\rceil = \left\lceil \frac{y}{2} \right\rceil \quad \text{and} \quad \left\lceil \frac{2g-x}{2} \right\rceil = \left\lceil \frac{2g-y}{2} \right\rceil,$$

where $\lceil z \rceil$ denotes the minimal integer not less than z. We conclude $\bar{x} = \bar{y}$. Suppose not. Without loss of generality, let x be even and y odd. Thus $y = x-1$. So

$$\left\lceil \frac{2g-y}{2} \right\rceil = \left\lceil \frac{2g-x+1}{2} \right\rceil = \left\lceil \frac{2g-x}{2} \right\rceil + 1,$$

which is impossible. Hence $|\Gamma_{i,j}(\bar{0})| \leqslant 1$ for all i, j. Γ is vertex transitive, so it is a weakly distance-transitive digraph by Proposition 2.1. The commutativity follows from Proposition 2.2. □

Definition 2.1. Let Γ be a digraph, and let t be an integer at least 2. Γ' is said to be a *t-coclique extension* of Γ if

$$V\Gamma' = \{(u, i) \mid u \in V\Gamma \text{ and } 0 \leqslant i \leqslant t-1\}$$

and

$$E\Gamma' = \{((u, i), (v, j)) \mid (u, v) \in E\Gamma\}.$$

Γ'' is said to be a *t-clique extension* of Γ if

$$V\Gamma'' = \{(u, i) \mid u \in V\Gamma \text{ and } 0 \leqslant i \leqslant t-1\}$$

and

$$E\Gamma'' = \{((u,i),(v,j)) \mid (u,v) \in E\Gamma \text{ or } u = v \text{ and } i \neq j\}.$$

Proposition 2.4. *Let Γ be a weakly distance-regular digraph of diameter d and girth g with intersection numbers $p^{\tilde{h}}_{\tilde{i},\tilde{j}}$. Then a t-coclique extension Γ' of Γ is weakly distance-regular if and only if one of the following holds*:

(i) $k_{g,g} = 0$.
(ii) $k_{g,g} = p^{(g,g)}_{(g,g),(g,g)} + 1$ *and* $p^{(0,0)}_{\tilde{i},\tilde{j}} = p^{(g,g)}_{\tilde{i},\tilde{j}}$ *for all* $\tilde{i},\tilde{j} \notin \{(0,0),(g,g)\}$.

Proof. Let $A_{i,j}$ be the (i,j)th adjacency matrix of Γ, and let $A'_{i,j}$ be the (i,j)th adjacency matrix of Γ'. It is easy to check that

$$A'_{i,j} = \begin{cases} A_{0,0} \otimes I_t & \text{if } (i,j) = (0,0), \\ A_{i,j} \otimes J_t & \text{if } (i,j) \notin \{(0,0),(g,g)\}, \\ A_{g,g} \otimes J_t + A_{0,0} \otimes (J_t - I_t) & \text{if } (i,j) = (g,g), \end{cases}$$

where I_t is the identity matrix of size t and J_t is the all one's matrix of size t. For any $\tilde{i} \notin \{(0,0),(g,g)\}$,

$$A'_{\tilde{i}} A'_{g,g} = t A_{\tilde{i}} A_{g,g} \otimes J_t + (t-1) A_{\tilde{i}} \otimes J_t$$

can be written as a linear combination of $A'_{s,t}$ $(0 \leqslant s,t \leqslant \max\{d,g\})$ if and only if $k_{g,g} = 0$ or $k_{g,g} = p^{(g,g)}_{(g,g),(g,g)} + 1$. Note that the latter equation is equivalent to the following:

$$A_{g,g} A_{g,g} = k_{g,g} A_{0,0} + (k_{g,g} - 1) A_{g,g}.$$

Moreover, if it is the case, then

$$A'_{g,g} A'_{g,g} = (t \cdot k_{g,g} + t - 1) A'_{0,0} + (t \cdot k_{g,g} + t - 2) A'_{g,g}.$$

For all $\tilde{i},\tilde{j} \notin \{(0,0),(g,g)\}$, $A'_{\tilde{i}} A'_{\tilde{j}} = t A_{\tilde{i}} A_{\tilde{j}} \otimes J_t$ can be written as a linear combination of $A'_{s,t}$ $(0 \leqslant s,t \leqslant \max\{d,g\})$ if and only if $p^{(0,0)}_{\tilde{i},\tilde{j}} = p^{(g,g)}_{\tilde{i},\tilde{j}}$. Hence the desired result follows. $\square$

Corollary 2.5. *Let Z_n be a cyclic group of order n. Then the following hold*:

(1) $\Gamma = \mathrm{Cay}(Z_{2tg}, \{\bar{1}, \bar{2}, \overline{2g+1}, \overline{2g+2}, \ldots, \overline{2(t-1)g+1}, \overline{2(t-1)g+2}\})$ *is a commutative weakly distance-transitive digraph*, *where* $t,g \geqslant 2$.
(2) $\Gamma = \mathrm{Cay}(Z_{tg}, \{\bar{1}, \overline{g+1}, \ldots, \overline{(t-1)g+1}\})$ *is a commutative weakly distance-transitive digraph*, *where* $t,g \geqslant 2$.

Proof. (1) It is easy to see that Γ is isomorphic to a t-coclique extension of $\mathrm{Cay}(Z_{2g}, \{\bar{1}, \bar{2}\})$. By Proposition 2.3 and Theorem 2.4, Γ is a commutative weakly distance-transitive digraph.

(2) It is easy to see that Γ is isomorphic to a t-coclique extension of a directed cycle of length g. By Propositions 2.3 and 2.4, the desired result follows. $\square$

Proposition 2.6. *Let Γ be a weakly distance-regular digraph with intersection numbers $p^{\tilde{h}}_{\tilde{i},\tilde{j}}$. Then a t-clique extension of Γ is weakly distance-regular if and only if one of the following holds*:

(i) $k_{1,1}=0$.
(ii) $k_{1,1}=p^{(1,1)}_{(1,1),(1,1)}+1$ *and* $p^{(0,0)}_{\tilde{i},\tilde{j}}=p^{(1,1)}_{\tilde{i},\tilde{j}}$ *for all* $\tilde{i},\tilde{j}\notin\{(0,0),(1,1)\}$.

Proof. The proof is similar to that of Proposition 2.4 and will be omitted. $\square$

Theorem 2.7. *Let $\Gamma_1=(X,E_1),\dots,\Gamma_n=(X,E_n)$ be distance-regular digraphs of girth 3 and diameter 2 with the same intersection numbers. Let $\tilde{X}=X^n$, i.e., the direct product of n copies of X. Two vertices $\tilde{x}=(x_1,x_2,\dots,x_n), \tilde{y}=(y_1,y_2,\dots,y_n)\in\tilde{X}$ are adjacent if there exists some j such that*

$$\partial_{\Gamma_j}(x_j,y_j)=1 \quad \text{and} \quad x_i=y_i \quad \text{for all } i\neq j.$$

Then the digraph $\tilde{\Gamma}$ defined above is a commutative weakly distance-regular digraph.

Proof. Take $\tilde{x}=(x_1,\dots,x_n)$, $\tilde{y}=(y_1,\dots,y_n)\in\tilde{X}$ with $\tilde{\partial}(\tilde{x},\tilde{y})=\tilde{h}$. If

$$|\{i\mid\partial_{\Gamma_i}(x_i,y_i)=1\}|=s \quad \text{and} \quad |\{j\mid\partial_{\Gamma_j}(x_j,y_j)=2\}|=t$$

then $\tilde{h}=(s+2t,t+2s)$. For any two vertices $\tilde{x}'=(x'_1,\dots,x'_n)$, $\tilde{y}'=(y'_1,\dots,y'_n)\in\tilde{X}$ with $\tilde{\partial}(\tilde{x}',\tilde{y}')=\tilde{h}$, if

$$|\{i\mid\partial_{\Gamma_i}(x'_i,y'_i)=1\}|=s' \quad \text{and} \quad |\{j\mid\partial_{\Gamma_j}(x'_j,y'_j)=2\}|=t'$$

then $(s+2t,t+2s)=(s'+2t',t'+2s')$ and so $s=s'$ and $t=t'$. Let $\tilde{A}_{i,j}$ be the (i,j)th adjacency matrix of $\tilde{\Gamma}$. It is easy to check that the span of the set $\{\tilde{A}_{i,j}\mid 0\leqslant i,j\leqslant 2n\}$ is closed under multiplication, so $\tilde{\Gamma}$ is weakly distance-regular. $\square$

Proposition 2.8. *For integers n and m with $2\leqslant n\leqslant m$, let*

$$\Gamma=\mathrm{Cay}(Z_n\times Z_m,\{(\bar{1},\bar{0}),(\bar{0},\bar{1})\}).$$

Then Γ is weakly distance-regular if and only if $n=2$ or $n=m=3$. Moreover, if Γ is weakly distance-regular, then it is weakly distance-transitive.

Proof. Let Γ be a weakly distance-regular digraph. Suppose $n\neq 2$. We will prove $n=m=3$. If $m\neq n$, then

$$p^{(3,m+n-3)}_{(2,m-2),(1,n-1)}((\bar{0},\bar{0}),(\bar{1},\bar{2}))\neq 0 \quad \text{and} \quad p^{(3,m+n-3)}_{(2,m-2),(1,n-1)}((\bar{0},\bar{0}),(\bar{2},\bar{1}))=0.$$

This is impossible. If $m=n\geqslant 4$, then

$$p^{(4,2n-4)}_{(2,n-2),(2,n-2)}((\bar{0},\bar{0}),(\bar{2},\bar{2}))\neq 0 \quad \text{and} \quad p^{(4,2n-4)}_{(2,n-2),(2,n-2)}((\bar{0},\bar{0}),(\bar{3},\bar{1}))=0.$$

This is impossible. Hence, $n=m=3$. Conversely, if $n=m\leqslant 3$, then it is easy to check that Γ is weakly distance-transitive. Now we consider the case $2=n<m$. If $\tilde{\partial}((\bar{0},\bar{0}),(\bar{x},\bar{y}))=\tilde{\partial}((\bar{0},\bar{0}),(\bar{x}',\bar{y}'))$, then $\bar{x}=\bar{x}'$ and $\bar{y}=\bar{y}'$. Thus $|\Gamma_{i,j}(\bar{0})|\leqslant 1$ for all i,j. By Proposition 2.1, Γ is weakly distance-transitive. □

3. Connections to association schemes

In this section, we will discuss the relations between weakly distance-regular digraphs and association schemes.

Definition 3.1. Let X be a finite set. Let $\emptyset\neq R_i\subseteq X\times X$, $i=0,1,\ldots,d$ satisfy the following:

(i) $R_0=\{(x,x)\mid x\in X\}$.
(ii) $X\times X=R_0\cup\cdots\cup R_d$ and $R_i\cap R_j=\emptyset$ if $i\neq j$.
(iii) ${}^tR_i=R_{i'}$ for some $i'\in\{0,1,\ldots,d\}$, where ${}^tR_i=\{(x,y)\mid (y,x)\in R_i\}$.
(iv) For $h,i,j\in\{0,1,\ldots,d\}$ and $(x,y)\in R_h$,

$$p^h_{i,j}=|\{z\in X\mid (x,z)\in R_i,(z,y)\in R_j\}|$$

depends only on h,i,j and does not depend on the choice of $(x,y)\in R_h$.

Such a configuration $\mathscr{X}=(X,\{R_i\}_{0\leqslant i\leqslant d})$ is called an *association scheme* of class d on X. If $p^h_{i,j}=p^h_{j,i}$ for all $h,i,j\in\{0,1,\ldots,d\}$, $\mathscr{X}$ is called a *commutative association scheme*. If ${}^tR_i=R_i$ for all i, $\mathscr{X}$ is called a *symmetric association scheme*.

For more information about association schemes we would like to refer readers to [2].

Let Γ be a digraph, and let $R_{i,j}=\{(x,y)\in V\Gamma\times V\Gamma\mid \tilde{\partial}(x,y)=(i,j)\}$. Set $I=\{(i,j)\mid R_{i,j}\neq\emptyset\}$. Γ is a weakly distance-regular digraph if $\mathscr{X}=(V\Gamma,\{R_{i,j}\}_{(i,j)\in I})$ is an association scheme.

Theorem 3.1. *Let* $\mathscr{X}=(X,\{R_{0,0},R_{1,r(1,1)},\ldots,R_{1,r(1,t_1)},\ldots,R_{q,r(q,1)},\ldots,R_{q,r(q,t_q)}\})$ *be an association scheme satisfying*

$${}^tR_{i,j}=\{(x,y)\in X\times X\mid (y,x)\in R_{i,j}\}=R_{j,i} \quad \textit{and} \quad R_{0,0}=\{(x,x)\mid x\in X\},$$

where $r(j,i)$ *denotes a positive integer for each* i,j. *Let* $A_{j,r(j,i)}$ *be the adjacency matrix with respect to* $R_{j,r(j,i)}$ *and let* $A=A_{1,r(1,1)}+\cdots+A_{1,r(1,t_1)}$, $R=R_{1,r(1,1)}\cup\cdots\cup R_{1,r(1,t_1)}$. *Then the following are equivalent*:

(i) *Let* $\Gamma=(X,R)$ *denote a digraph. Then* $\tilde{\partial}(x,y)=(j,r(j,i))$ *if and only if* $(x,y)\in R_{j,r(j,i)}$, *i.e.*, Γ *is weakly distance-regular*.

(ii) *For any non-negative integer* $s \leqslant q$, *there exist numbers* $n(j,r(j,i),s)$ *such that*

$$A^s = \sum_{j=0}^{s} \sum_{i=1}^{t_j} n(j,r(j,i),s) A_{j,r(j,i)}, \tag{1}$$

where $n(s,r(s,i),s) \neq 0$ *for all* $1 \leqslant i \leqslant t_s$.

(iii) *For any non-negative integers* $j,i \leqslant q$ *and* $1 \leqslant l \leqslant t_j$, *let*

$$\tilde{p}_{i,1}^{(j,r(j,l))} = \sum_{1 \leqslant t \leqslant t_i, 1 \leqslant s \leqslant t_1} p_{(i,r(i,t)),(1,r(1,s))}^{(j,r(j,l))}.$$

Then $\tilde{p}_{i,1}^{(j,r(j,l))} = 0$ *if* $j - i \geqslant 2$ *and* $\tilde{p}_{j-1,1}^{(j,r(j,l))} \neq 0$.

Proof. (i) $\Rightarrow$ (ii): Suppose (i) holds. Let $(x,y) \in R_{j,r(j,i)}$. Then

$$(A^s)_{x,y} = \begin{cases} 0 & \text{if } j > s, \\ n(j,r(j,i),s) & \text{if } j \leqslant s. \end{cases}$$

Thus (1) holds and $n(j,r(j,i),j) \neq 0$.

(ii) $\Rightarrow$ (i): Suppose (ii) holds. Then $n(s,r(s,j),s)$ is the number of paths of length s connecting x and y with $(x,y) \in R_{s,r(s,j)}$ for all $1 \leqslant s \leqslant q$, $1 \leqslant j \leqslant t_s$. We claim that $(x,y) \in \bigcup_{i=1}^{t_l} R_{l,r(l,i)}$ if and only if $\partial(x,y) = l$. When $l = 0$ or 1, the claim is true. Suppose $(x,y) \in \bigcup_{i=1}^{t_l} R_{l,r(l,i)}$. Then there exists a positive integer p such that $(x,y) \in R_{l,r(l,p)}$. $n(l,r(l,p),l) \neq 0$ implies $\partial(x,y) \leqslant l$. If $\partial(x,y) < l$, then by induction $(x,y) \notin \bigcup_{i=1}^{t_l} R_{l,r(l,i)}$, which contradicts our assumption. Thus $\partial(x,y) = l$. Conversely, if $\partial(x,y) = l$, then $(A^l)_{x,y} \neq 0$. Hence, by (1) there exist $l_1 \leqslant l$ and i_1 such that

$$(x,y) \in R_{l_1,r(l_1,i_1)}.$$

If $l_1 < l$, by induction $\partial(x,y) = l_1 < l$, which is impossible. Thus $(x,y) \in \bigcup_{i=1}^{t_l} R_{l,r(l,i)}$, and so our claim holds. Since $(x,y) \in R_{l,r(l,j)}$ if and only if $(y,x) \in R_{r(l,j),l}$, it is clear that $\tilde{\partial}(x,y) = (l,r(l,j))$ if and only if $(x,y) \in R_{l,r(l,j)}$.

(i) $\Rightarrow$ (iii): Suppose (i) holds. By the triangle inequality, $\tilde{p}_{i,1}^{(j,r(j,l))} = 0$ if $j - i \geqslant 2$, and by the connectivity of Γ, $\tilde{p}_{j-1,1}^{(j,r(j,l))} \neq 0$.

(iii) $\Rightarrow$ (i): Suppose (iii) holds. We claim that if $(x,y) \in R_{j,r(j,i)}$ then $\partial(x,y) = j$. We use induction on j. If $j = 0,1$, our claim holds. Now suppose $j \geqslant 2$. Let $R_{j,r(j,i)}(x) = \{y \mid (x,y) \in R_{j,r(j,i)}\}$. Then for $y \in R_{j,r(j,i)}(x)$, there exist i_1 and i_2 such that $R_{j-1,r(j-1,i_1)}(x) \cap R_{r(1,i_2),1}(y) \neq \emptyset$ and so $\partial(x,y) \leqslant j$. Moreover, for all $m \leqslant j-2$, we have

$$\bigcup_{l=1}^{t_m} R_{m,r(m,l)}(x) \cap \bigcup_{n=1}^{t_1} R_{r(1,n),1}(y) = \emptyset,$$

which implies $\partial(x,y) \geqslant j$. Thus $\partial(x,y) = j$. Since $(x,y) \in R_{j,r(j,i)}$ if and only if $(y,x) \in R_{r(j,i),j}$, $(x,y) \in R_{j,r(j,i)}$ if and only if $\tilde{\partial}(x,y) = (j,r(j,i))$. $\square$

As an obvious corollary we have the following result:

Corollary 3.2. *Let Γ be a weakly distance-regular digraph with adjacency matrices $A_{0,0}, A_{1,r(1,1)}, \dots, A_{1,r(1,t_1)}, \dots, A_{q,r(q,1)}, \dots, A_{q,r(q,t_q)}$ and let $\mathscr{A}(\Gamma)$ be the Bose–Mesner algebra of Γ. Then*

$$d+1 \leqslant \dim \mathscr{A}(\Gamma) \leqslant 1+t_1+\cdots+t_q. \tag{2}$$

Moreover, if both equalities hold in (2), *then Γ is distance-regular.*

4. Proof of Theorem 1.1

Throughout this section, we assume that Γ is a commutative weakly distance-regular digraph of valency 2 and girth g. We firstly prove the following result:

Proposition 4.1. *If there exists an arc (u,v) with $\partial(v,u)=q-1 \geqslant g$, then Γ is isomorphic to one of the following*:

(1) $\mathrm{Cay}(Z_{2g}, \{\bar{1}, \bar{2}\})$.
(2) $\mathrm{Cay}(Z_2 \times Z_q, \{(\bar{0}, \bar{1}), (\bar{1}, \bar{0})\})$.

We need the following lemma to prove the proposition:

Lemma 4.2. *Assume the hypothesis in Proposition* 4.1. *For any $x \in V\Gamma$, there exist the following two circuits with only one common vertex*

$$(x=x_0, x_1, \dots, x_{g-1}) \quad \textit{and} \quad (x=z_0, z_1, \dots, z_{q-1}).$$

Proof. Let $(x=x_0, x_1, \dots, x_{g-1})$ be a minimal circuit and let $(x=y_0, y_1, \dots, y_{q-1})$ be a circuit with $\tilde{\partial}(y_0, y_1)=(1, q-1)$. If the two circuits have another common vertex y_j, then $y_j = x_{g-q+j}$ by $\partial(y_j, x)=q-j$. Let i be the minimal index such that $y_i = x_{g-q+i}$. Thus we have $\tilde{\partial}(y_i, y_{i+1}) = \tilde{\partial}(y_i, x_{g-q+i+1}) = (1, g-1)$. Since $k_{1,g-1}=1$, $y_{i+1}=x_{g-q+i+1}$. By induction, we have $y_j = x_{g-q+j}$ for all $i \leqslant j \leqslant q$. Since $y_i \in P_{(1,q-1),(1,g-1)}(y_{i-1}, y_{i+1})$, there exists $y_i' \in P_{(1,g-1),(1,q-1)}(y_{i-1}, y_{i+1})$ by the commutativity of Γ. By induction, there exists a path $(y_{i-1}=y_{i-1}', y_i', \dots, y_{q-1}', x)$ satisfying $y_j' \in P_{(1,g-1),(1,q-1)}(y_{j-1}', y_{j+1})$, where $i \leqslant j \leqslant q-1$ and $y_q = x$. It is clear that $y_j' \neq y_j$ for all $i \leqslant j \leqslant q-1$. Take $z_0 = x$, $z_1 = y_1, \dots, z_{i-1} = y_{i-1}$, $z_i = y_i', \dots, z_{q-1} = y_{q-1}'$, then $(x_0, x_1, \dots, x_{q-1})$ and $(z_0, z_1, \dots, z_{q-1})$ are the desired circuits. □

In the rest of this section, we always assume that all first subscriptions of x are taken modulo g.

Proof of Proposition 4.1. Let $(x_{0,0}, x_{1,0}, \dots, x_{g-1,0})$ be a minimal circuit. By Lemma 4.2, for any i, we can take a circuit $(x_{i,0}, x_{i,1}, \dots, x_{i,q-1})$ with $x_{i,1} \neq x_{i+1,0}$ and $x_{i,g-1} \neq x_{i-1,0}$. Note that $\tilde{\partial}(x_{i,0}, x_{i,1}) = \tilde{\partial}(x_{i,q-1}, x_{i,0}) = (1, q-1)$. Then $x_{i,0} \in P_{(1,g-1),(1,q-1)}(x_{i-1,0}, x_{i,1})$ for all i. By the commutativity of Γ, for any i, there exists $x_i' \in P_{(1,q-1),(1,g-1)}(x_{i-1,0}, x_{i,1})$.

Since $k_{1,q-1}=1$, we get $x_i'=x_{i-1,1}$. So $\tilde{\partial}(x_{i-1,1},x_{i,1})=(1,g-1)$ for all i. Let $r=|\{x_{i,1}, x_{i,q-1}\mid 0\leqslant i\leqslant g-1\}|$. First we consider the case $r<2g$. Since $k_{1,q-1}=1$, without loss of generality, we assume there exist two non-negative integers $j<i$ at most $g-1$ such that $x_{i,1}=x_{j,q-1}$. In this case, $(x_{i,1},x_{j,0},\dots,x_{i,0})$ is a circuit of length $i-j+2$. Since $\tilde{\partial}(x_{i,0},x_{i,1})=(1,q-1)$, we have $i-j>g-2$. Thus $i=g-1$, $j=0$, and so $p^{(1,g-1)}_{(1,q-1),(1,q-1)}=1$. By $k_{1,q-1}=1$ and $\tilde{\partial}(x_{i-1,0},x_{i-1,1})=(1,q-1)$, we have $x_{i-1,1}\in P_{(1,q-1),(1,q-1)}(x_{i-1,0},x_{i,0})$ for all i. Hence $\Gamma\simeq \mathrm{Cay}(Z_{2g},\{\bar{1},\bar{2}\})$.

Now assume that $r=2g$. Suppose all second subscriptions of x are taken modulo q. We claim that

$$\tilde{\partial}(x_{i,j},x_{i,j+1})=(1,q-1) \quad\text{and}\quad \tilde{\partial}(x_{i-1,j+1},x_{i,j+1})=(1,g-1) \text{ for all } i,j.$$

We will prove our claim by induction on j. Our claim holds for $j=q-1,0$. Suppose our claim holds for all integers $q-1,0,1,\dots,j$. By the argument of first paragraph, we know that $|\{x_{i,j-1},x_{i,j+1}\mid 0\leqslant i\leqslant g-1\}|=2g$. So

$$\tilde{\partial}(x_{i,j-1},x_{i,j+1})=\tilde{\partial}(x_{i,j},x_{i,j+2})=(2,q-2).$$

Since $x_{i,j}\in P_{(1,q-1),(1,q-1)}(x_{i,j-1},x_{i,j+1})$, $x_{i,j+1}\in P_{(1,q-1),(1,q-1)}(x_{i,j},x_{i,j+2})$ by $k_{1,q-1}=1$, and so $\tilde{\partial}(x_{i,j+1},x_{i,j+2})=(1,q-1)$. Since $x_{i,j+1}\in P_{(1,g-1),(1,q-1)}(x_{i-1,j+1},x_{i,j+2})$, by $k_{1,q-1}=1$ and the commutativity of Γ, $x_{i-1,j+2}\in P_{(1,q-1),(1,g-1)}(x_{i-1,j+1},x_{i,j+2})$, and so $\tilde{\partial}(x_{i-1,j+2},x_{i,j+2})=(1,g-1)$. Thus our claim is valid. We claim that all vertices $x_{i,j}$ with $0\leqslant i\leqslant g-1$, $0\leqslant j\leqslant q-1$ are distinct. Suppose not. Without loss of generality we may assume that $x_{0,0}=x_{i,j}$ with $1\leqslant i\leqslant g-1$ and $2\leqslant j\leqslant q-1$. Since there is a circuit $(x_{0,0},x_{1,0},\dots,x_{i,0},x_{i,1},\dots,x_{i,j-1})$ with $x_{i,j}=x_{0,0}$ of length $i+j$ which includes an arc $(x_{i,0},x_{i,1})$, we have $i+j\geqslant q$. On the other hand, there is a path $(x_{i,0},x_{i+1,0},\dots,x_{g-1,0},x_{0,0}=x_{i,j})$ of length $g-i$. Since $\tilde{\partial}(x_{i,h},x_{i,h+1})=(1,q-1)$ for $h=0,1,j-1$, $\partial(x_{i,0},x_{i,j})=j$. Thus $g-i\geqslant j$ or $g\geqslant i+j$. Since $g<q$, this is a contradiction. This proves the claim. Now it is clear that $\Gamma\simeq\mathrm{Cay}(Z_g\times Z_q,\{(\bar{0},\bar{1}),(\bar{1},\bar{0})\})$. Hence, $\Gamma\simeq\mathrm{Cay}(Z_2\times Z_q,\{(\bar{0},\bar{1}),(\bar{1},\bar{0})\})$ by Proposition 2.8. □

Proposition 4.3. *If every arc is contained in a minimal circuit, then Γ is isomorphic to one of the following*:

(1) $\mathrm{Cay}(Z_n,\{\bar{1},\overline{n-1}\})$.

(2) $\mathrm{Cay}(Z_{2g},\{\bar{1},\overline{g+1}\})$.

(3) $\mathrm{Cay}(Z_3^2,\{(\bar{0},\bar{1}),(\bar{1},\bar{0})\})$.

If $g=2$, $\Gamma\simeq\mathrm{Cay}(Z_n,\{\bar{1},\overline{n-1}\})$. So we only need to consider the case $g\geqslant 3$. We need the following lemma to prove the proposition:

Lemma 4.4. *Assume the hypothesis in Proposition* 4.3. *For any $x\in V\Gamma$, there exist the following two minimal circuits*:

$$(x=x_0,x_1,\dots,x_{g-1}) \quad\textit{and}\quad (x=y_0,y_1,\dots,y_{g-1})$$

satisfying $|\{x_1,x_{g-1},y_1,y_{g-1}\}|=4$.

Proof. Let $(x=x_0,x_1,\ldots,x_{g-1})$ be a minimal circuit, and let $\partial(y_{g-1},x)=\partial(x,y_1)=1$ with $x_1\neq y_1$ and $x_{g-1}\neq y_{g-1}$. If a path (y_{g-1},x_0,y_1) is not contained in any minimal circuit, then every minimal circuit containing the arc (y_{g-1},x_0) [resp. (x_0,y_1)] must contain the arc (x_0,x_1) [resp. (x_{g-1},x_0)]. So we have

$$p^{(1,g-1)}_{(g-1,1),(2,g-2)}(x_0,x_1)=|\{y_{g-1},x_{g-1}\}|=2$$

and

$$p^{(1,g-1)}_{(g-1,1),(2,g-2)}(x_0,y_1)=|\{x_{g-1}\}|=1.$$

This is impossible. □

Proof of Proposition 4.3. Let $(x_{0,0},x_{1,0},\ldots,x_{g-1,0})$ be a minimal circuit. By Lemma 4.4, we can take a circuit $(x_{i,0},x_{i,1},\ldots,x_{i,g-1})$ such that $|\{x_{i-1,0},x_{i+1,0},x_{i,1},x_{i,g-1}\}|=4$ for each i. $x_{g-1,0}\in P_{(g-1,1),(1,g-1)}(x_{0,0},x_{g-1,1})$, so $P_{(1,g-1),(g-1,1)}(x_{0,0},x_{g-1,1})\neq\emptyset$ by the commutativity of Γ. $k_1=2$ implies that $\partial(x_{g-1,1},x_{1,0})=1$ or $\partial(x_{g-1,1},x_{0,1})=1$. First we consider the case $\partial(x_{g-1,1},x_{1,0})=1$. Then $p^{(2,g-2)}_{(1,g-1),(1,g-1)}=2$ and $\partial(x_{i,1},x_{i+2,0})=1$ for all i. It is clear that $(x_{i,0},x_{i,1},x_{i+2,0},\ldots,x_{i-1,0})$ is a minimal circuit for any i. Thus $\tilde{\partial}(x_{i-1,0},x_{i,1})=(2,g-2)$, and so $\partial(x_{i-1,1},x_{i,1})=1$ for all i. Hence $\Gamma\simeq\mathrm{Cay}(Z_{2g},\{\bar{1},\overline{g+1}\})$.

Now assume that $\partial(x_{g-1,1},x_{0,1})=1$ and $\partial(x_{g-1,1},x_{1,0})\neq 1$. Suppose all second subscriptions of x are taken modulo g. Then $p^{(2,g-2)}_{(1,g-1),(1,g-1)}=1$ and $\partial(x_{i,1},x_{i+2,0})\neq 1$ for all i. We claim that

$$\partial(x_{i,j},x_{i+1,j})=1 \quad\text{and}\quad x_{i,j+1}\neq x_{i+1,j} \quad \text{for all } i,j.$$

We prove our claim by induction on j. If $j=0$, our claim is valid. Now suppose our claim holds for $0,1,\ldots,j$. Since $x_{i,j}\in P_{(g-1,1),(1,g-1)}(x_{i,j+1},x_{i+1,j})$, there exists $x'_{i,j}\in P_{(1,g-1),(g-1,1)}(x_{i,j+1},x_{i+1,j})$ by the commutativity of Γ. Since $k_1=2$, we get $x'_{i,j}=x_{i+1,j+1}$ or $x_{i+2,j}$. If $x'_{i,j}=x_{i+2,j}$, then $p^{(2,g-2)}_{(1,g-1),(1,g-1)}=2$, which is impossible. So $x'_{i,j}=x_{i+1,j+1}$ and $\partial(x_{i,j+1},x_{i+1,j+1})=1$. If there exists i such that $x_{i,j+2}=x_{i+1,j+1}$, then $\tilde{\partial}(x_{i,j},\ x_{i+1,j+1})=(2,g-2)$ and $x_{i+1,j},x_{i,j+1}\in P_{(1,g-1),(1,g-1)}(x_{i,j},x_{i+1,j+1})$. So $p^{(2,g-2)}_{(1,g-1),(1,g-1)}=2$, which is impossible. Thus $x_{i,j+2}\neq x_{i+1,j+1}$. So our claim is valid. We claim that all vertices $x_{i,j}$ with $0\leqslant i,j\leqslant g-1$ are distinct. Suppose not. Without loss of generality we may assume that $x_{0,0}=x_{i,j}$ with $1\leqslant i,j\leqslant g-1$. Since there is a circuit $(x_{0,0},x_{1,0},\ldots,x_{i,0},x_{i,1},\ldots,x_{i,j-1})$ with $x_{i,j}=x_{0,0}$ of length $i+j$, we have $i+j\geqslant g$. On the other hand, there is a path $(x_{i,0},x_{i+1,0},\ldots,x_{g-1,0},x_{0,0}=x_{i,j})$ of length $g-i$. Hence $g-i\geqslant j$ or $g\geqslant i+j$. Thus $i+j=g$. Now there is a circuit of length g containing $x_{i-1,0}$ and $x_{i,1}$, so we have $p^{(2,g-2)}_{(1,g-1),(1,g-1)}=2$, which is a contradiction. This proves the claim. Thus $\Gamma\simeq\mathrm{Cay}(Z_g\times Z_g,\{(\bar{0},\bar{1}),(\bar{1},\bar{0})\})$. By Proposition 2.8, $\Gamma\simeq\mathrm{Cay}(Z_3^2,\{(\bar{0},\bar{1}),(\bar{1},\bar{0})\})$. □

Combining Propositions 4.1 and 4.3, we complete the proof of Theorem 1.1. We also note that Theorem 1.1 also holds for a weakly distance-transitive digraph.

5. Concluding remarks

(1) We do not know any example of non-commutative weakly distance-regular digraphs, so we think it is important to find such examples.
(2) In Theorem 2.7, the base graph of $\tilde{\Gamma}$ is a Hamming graph. It seems interesting that when an orientation of a distance-regular graph defines a weakly distance-regular digraph.
(3) Recently, A. Hanaki searched weakly distance-regular digraphs with small number of vertices using the data of the classification of association schemes of small size (joint work with I. Miyamoto). He kindly uploaded the data of his search on his homepage at: URL: http://kissme.shinshu-u.ac.jp/as/data/wdrdg.

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