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Triangle-free eulerian tours in graphs with maximum degree at most 4

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Abstract

In this paper we completely characterise the family of eulerian simple graphs G with maximum degree at most 4 which admit a *triangle-free eulerian tour*, i.e., a sequence $v_1 v_2 \cdots v_m v_1$ such that each v_i is a vertex, the pairs v_i, v_{i+1}, $i = 1, 2, \ldots, m$ are the m distinct edges in G and finally, $v_{i+3} \neq v_i$ for all $i = 1, 2, \ldots, m$, with indices counted modulo m.

1. Introduction

In [1, Problem 3.3] Roland Häggkvist formulated the following problem:

Let G be an eulerian graph with minimum degree δ. Show that there exists a function $f(\delta)$ such that G admits an eulerian tour $v_1 v_2 \cdots v_m v_1$ such that every segment $v_i v_{i+1} \cdots v_{i+j}$ induces a path or cycle if $j = 1, 2, \ldots, f(\delta)$.

A simple variant of this problem asks for a characterisation of the eulerian graphs which admit an triangle-free eulerian tour as defined in the abstract. In this paper it is shown that an eulerian graph of maximum degree at most 4 admits a triangle-free eulerian tour if it does not contain any of a small number of subgraphs. In Section 2 we list all these subgraphs and in Section 3 we prove that the list is complete. The method used is a case-by-case analysis where we pick a triangle T in the graph, then look at its neighbour set and try to reroute some given eulerian tour so as to avoid conflict locally around T.

A *trail* is a sequence $v_1 v_2 \cdots v_m$ such that each v_i is a vertex and the pairs v_i, v_{i+1}, $i = 1, 2, \ldots, m-1$ are $m-1$ distinct edges in G. To denote a subset of a trail we write P, i.e., in $v_1 v_2 P v_3$, P is a non-determined subtrail of the trail. The *length*, $l(P)$, of a trail P is defined to be the number of interior vertices in it $+1$, i.e., the number of interior edges plus the two connecting edges. A *triangle-free trail* is defined in an obvious way. If $v_1 P v_2$ is a triangular-free trail, $v_2 P v_1$ is the same trail traversed in the opposite direction. It is of course also triangle-free. For simplicity's sake we do not use any notation to indicate that P is traversed in the opposite direction. A graph is *split along*

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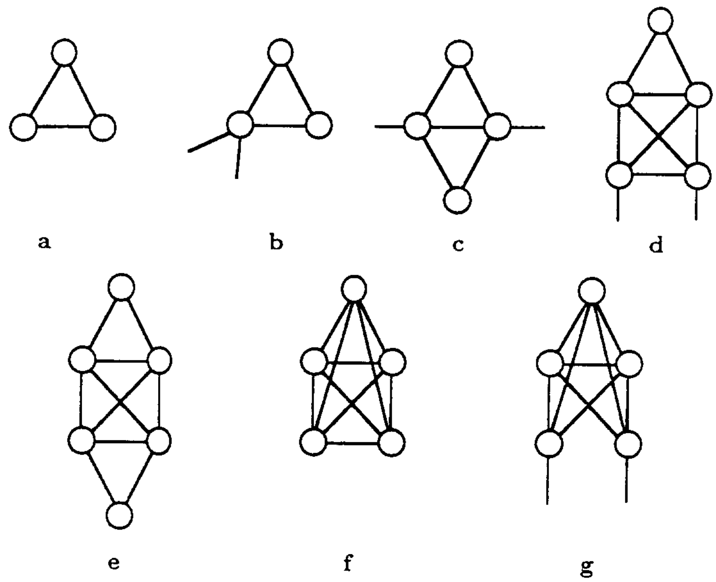

Fig. 1

a trail $v_1 v_2 \cdots v_{m-1}, v_m$ when we do the following operation. For each $i = 2, \ldots, m-1$ where $d(v_i) = 4$, v_i is split into v_i' and v_i'', v_i' taking the edges $v_{i-1}v_i$ and $v_i v_{i+1}$, and v_i'' taking the rest of v_i's edges.

The conventions for the figures used in this paper are the following. When a figure is said to depict a graph, then it is not necessarily the whole graph that is displayed. Edges in a figure that have only one vertex explicitly drawn are meant to indicate that the edge is connected to a part of the graph without significance to the argument. Trails are depicted with bold lines. Edges printed in dotted lines may or may not exist.

2. Forbidden subgraphs

There exists a number of graphs that makes it impossible for an eulerian graph containing them as subgraphs to have a triangle-free eulerian tour. In this section we list these graphs and in our main theorem we shall show that this is the complete list.

Definition. A subgraph that makes it impossible to find a triangle-free eulerian tour in a simple eulerian graph G with $\delta \geqslant 2$ and $\Delta \leqslant 4$ is called a *forbidden* subgraph. Let $\mathscr{F}$ be a set containing the graphs as defined by Fig. 1. For convenience we give them the following names:

(a) 3-cycle C_3, also called a triangle,

(b) jutting triangle,

(c) double triangle,
(d) lantern,
(e) double lantern,
(f) complete graph on 5 vertices, denoted K_5,
(g) $K_5 - e$, so named since it is a K_5 lacking one edge.

Although tedious, it is a routine matter to check that all the graphs in $\mathscr{F}$ are forbidden subgraphs.

3. Results and Proofs

Theorem. *The graphs in $\mathscr{F}$ are the only ones to obstruct a triangle-free euler tour for graphs with maximum degree at most 4. I.e., every simple eulerian graph G, with $\delta \geqslant 2$ and $\Delta \leqslant 4$, that does not contain any subgraph belonging to $\mathscr{F}$, has a triangle-free eulerian tour.*

If $\Delta = 2$ then G does not contain any triangle since $|V(G)| > 3$, henceforth we will assume that $\Delta = 4$. First a remark which will simplify our proof. Let G be a graph as in the theorem. Let $v \in G$, $d(v) = 4$. Split v into 2 vertices α and β, each of degree 2. Split α into α_1 and α_2 and add an edge between these 2 vertices. Do the same with β. Call the resulting graph G'. If G' contains a subgraph that belongs to $\mathscr{F}$ then (since the only difference between the graphs G' and G is the vertices α_i and β_i, $i = 1, 2$), this subgraph must contain one or more of the α_i and β_i. Since these vertices are of degree 2 but do not belong to any triangle (unlike all vertices of degree 2 in the graphs in $\mathscr{F}$), this cannot be the case. Hence, G' does not contain any subgraphs belonging to $\mathscr{F}$.

Proof of Theorem. The proof is by induction on $t + N_4$, where t is the number of triangles in G and N_4 is the number of vertices of degree 4 in G. If $t + N_4 = 1$ the theorem is trivially true (the case $N_4 = 0$ is trivial). Assume that G is a simple eulerian graph with $\Delta = 4$ and with $t + N_4 = \mu$ and that the theorem is true for all graphs with $t + N_4 < \mu$. Consider a triangle T on vertices v_1, v_2, v_3. Since $\Delta = 4$ we have 6 cases according to the number of triangles adjacent to T. The cases are described below and depicted in Fig. 2.

1. 1 triangle on each side of T and the three triangles have their top vertex in common, i.e. T has its vertices in a 4-clique,
2. 1 triangle on each side of T and these three triangles have three different top vertices,
3. one triangle on each of two sides of T,
4. two triangles on one side of T,
5. one triangle adjacent to T,
6. no triangle adjacent to T.

Since we use induction to prove the theorem, we can assume that when we investigate each case, G does not contain any subgraphs that apply to any preceeding cases. This

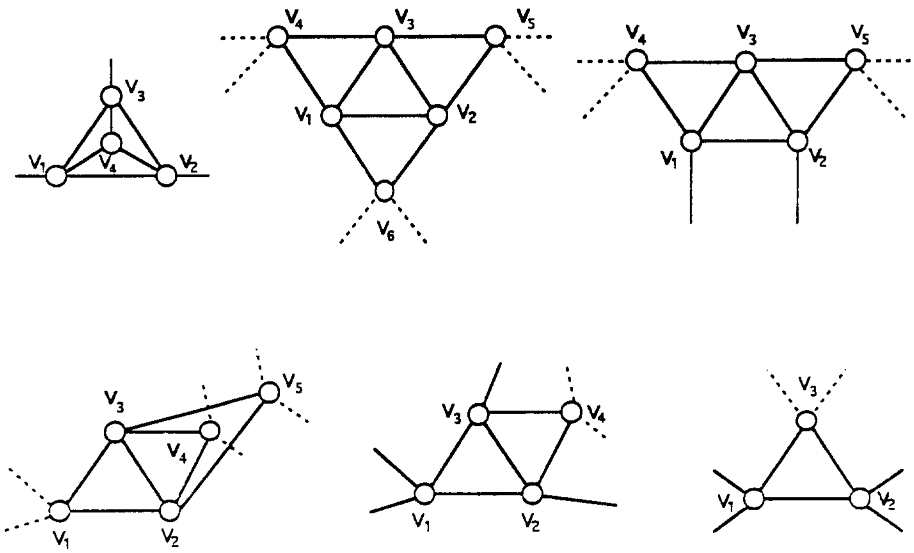

Fig. 2

is because in our inductive step, we may pick any triangle we like, and hence we pick them in case order, i.e., pick all the triangles that applies to case 1, then those applying to case 2, and so on. This is of importance when case 5 is studied. Before we continue, we prove a lemma.

Lemma. *Assume G has the following properties*:

(i) $\exists S \subset V(G)$ *such that all vertices are independent in S and that G_1, G_2 are two subgraphs such that $G_1 \cup G_2 = G$ and $V(G_1) \cap V(G_2) = S$.*

(ii) *For every $v \in S$ it holds that $d_{G_1}(v) = d_{G_2}(v) = 2$. In other words, S separates G such that every vertex in S has two edges to each component.*

(iii) *G_1 and G_2 do both have an eulerian tour which is triangle-free with a possible exception for the vertices in S.*

Then G has a triangle-free eulerian tour.

Proof. (By induction on the size of the minimal set S). If $|S| = 1$ and $v \in S$, split v into α_1, α_2 and β_1, β_2, where α_1, α_2 take the edges from G_1 to v and β_1, β_2 take the edges from G_2 to v. Add an edge between α_1 and α_2 and an edge between β_1 and β_2. These new graphs do both have triangle-free eulerian tours which we may assume are $P_1\alpha_1\alpha_2$ and $P_2\beta_1\beta_2$. Then P_1vP_2v is a triangle-free eulerian tour for G. If $|S| \geqslant 2$, repeat the splitting procedure for one vertex v in S. By induction, the resulting graph has a triangle-free eulerian tour $P_1\alpha_1\alpha_2P_2\beta_1\beta_2$ say and then P_1vP_2v is a triangle-free eulerian tour for G. This proves the lemma. □

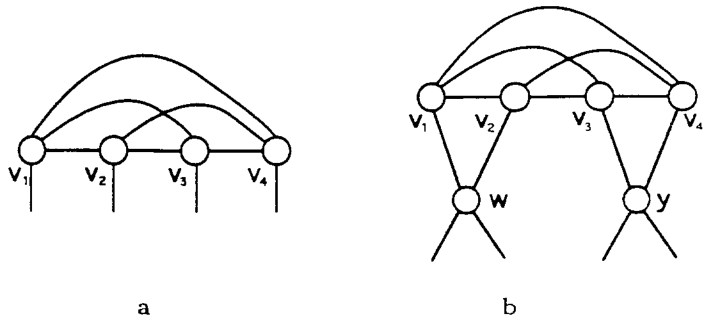

Fig. 3

Proof of Theorem (continued). Since we use induction over $t + N_4$ in the proof, the lemma implies that if the requirements (i) and (ii) are fulfilled by G then (iii) also is. This is because when splitting vertices as in the proof, we reduce N_4. Hence, if G contains a set S as in the lemma we can conclude that G has a triangle-free eulerian tour. Note that in an eulerian graph, a cutvertex is a special case of the set S.

We now turn to the task of examining the six different cases.

Case 1: In case 1, G has a subgraph like that in Fig. 3(a).

Let v_1, v_2, v_3, v_4 be the vertices shown in the figure and note that they form a 4-clique in G. Case 1 has different subcases according to the neighborhood of the 4-clique. If all 4 of the vertices in the clique are adjacent to a single vertex, then $G = K_5$ and if 3 of the vertices are adjacent to a single vertex, then $K_5 - e \subset G$. Hence, we have only to consider the three subcases when the clique has 2, 1 or 0 pair(s) of vertices with a distinct vertex in common.

Subcase 1: In the first subcase, we have the situation in Fig. 3(b) where w and y are adjacent to two distinct pairs of vertices in the 4-clique.
Note that neither w nor y has degree 2 since otherwise G contains a lantern or a double lantern. Construct G' by deleting v_1, v_2, v_3, v_4 and split w into w_1 and w_2, adding an edge between them and doing the same with y (getting y_1 and y_2) getting the graph in Fig. 4a. The graph G' has no forbidden subgraph and since w and y are not cut vertices, it has, by induction, a triangle-free eulerian tour $P_1 w_1 w_2 P_2 y_1 y_2$. This induces a triangle-free eulerian tour in G, namely, $P_1 w v_2 v_1 v_4 v_3 y P_2 w v_1 v_3 v_2 v_4 y$. See Fig. 4(b).

Subcase 2: In the second subcase, depicted in Fig. 5(a), only one vertex, w say, has edges to 2 of the vertices in the clique, say to v_1 and v_2. If $d(w) = 2$ then G contains the lantern, hence this is not the case. Moreover, by the lemma, w is not a cut vertex. Now construct G' by removing the vertices v_1 and v_2. Then split w into w_1 and w_2, each vertex taking one edge from w each. Add an edge between w_1 and w_2. G' has no forbidden subgraphs and is depicted in Fig. 5(b). By induction, we have a triangle-free eulerian tour which we, without loss of generality, may assume looks like $E' = P_1 v_3 v_4 P_2 w_1 w_2$, where $l(P_i) \geqslant 2$ for each $i = 1, 2$ (since $v_3 w \notin E(G)$, $v_4 w \notin E(G)$).

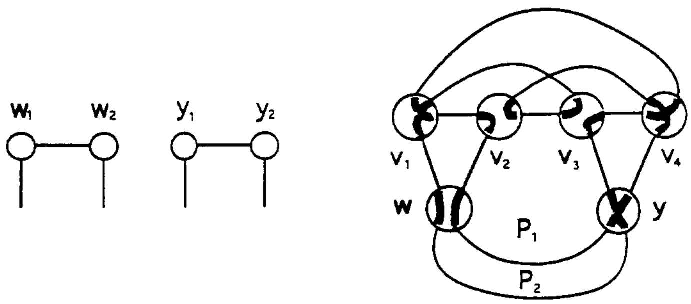

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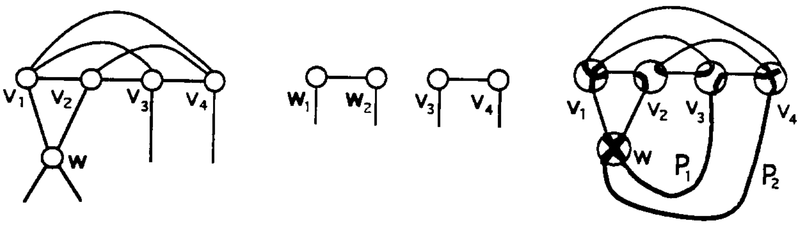
Fig. 4



a b c

Fig. 5

Using E' we construct a triangle-free eulerian tour in G, $P_1 v_3 v_4 v_1 v_2 w P_2 v_4 v_2 v_3 v_1 w$. See Fig. 5(c).

Subcase 3: In the third subcase, no pair of vertices have a vertex in common (outside the clique) and since G is eulerian it is possible to select two vertices that do not separate the graph. We may assume that v_1 and v_2 have this property. Let G' be the graph obtained by deleting the edges $v_1 v_3, v_1 v_4, v_2 v_3$ and $v_2 v_4$ as in Fig. 6(b). By induction we can find a triangle-free eulerian tour $E' = v_1 v_2 P_1 v_3 v_4 P_2$ in G'. Note that, since no pair of vertices have a common neighbour (outside the clique), $l(P_i) \geqslant 3$ for $i = 1, 2$. By using the eulerian tour E' we can construct

$$E = v_1 v_2 v_4 v_3 P_1 v_2 v_3 v_1 v_4 P_2$$

which is triangle-free in G. See Fig. 6(c). This proves the theorem in case 1.

Case 2: In case 2, G has one of the subgraphs depicted in Fig. 7. We have four different subcases according to the degree of the vertices v_4, v_5 and v_6.

(a) $d(v_4) = d(v_5) = d(v_6) = 2$,

(b) $d(v_4) = d(v_5) = 2, d(v_6) = 4$,

(c) $d(v_4) = d(v_5) = 4, d(v_6) = 4$,

(d) $d(v_4) = d(v_5) = d(v_6) = 4$.

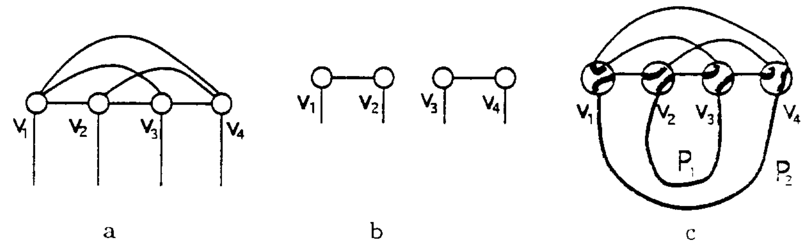


Fig. 6

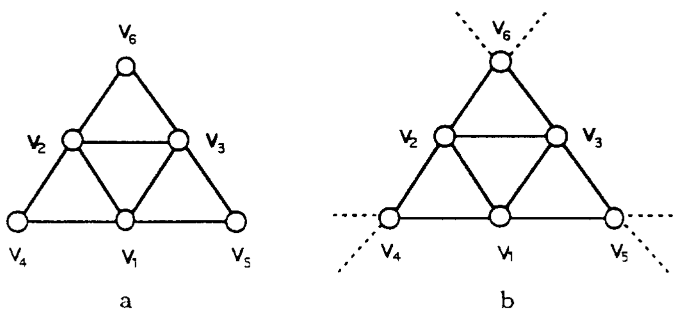


Fig. 7

In subcase (a), G looks like the graph in Fig. 7(a) and has a triangle-free eulerian tour $v_1v_3v_6v_2v_1v_5v_3v_2v_4$.

In the remaining cases we see that the lemma can be applied with an appropiate selection of v_4, v_5, v_6, forming the set S.

Case 3: In case 3, we have a subgraph as depicted in Fig. 8(a). We have 3 subcases.

(1) $d(v_4) = d(v_5) = 2$,

(2) $d(v_4) = d(v_5) = 4$,

(3) $d(v_4) = 4$, $d(v_5) = 2$.

In subcase 1, $G - \{v_4, v_3, v_5\}$ has a triangle-free eulerian tour $w_1v_1v_2w_2P$ and $w_1v_1v_3v_5v_2v_1v_4v_3v_2w_2P$ is a triangle-free eulerian tour for G. In subcase 2, split G along $w_1v_1v_2w_2$ and call this graph G'. See Fig. 8b. By the lemma, G' is connected and by induction, it has a triangle-free eulerian tour. The same eulerian tour is triangular-free in G unless it contains any of the sequences $v_1''v_4w_1v_1'$ and $v_2''v_5w_2v_2'$. Suppose the tour contains $v_1''v_4w_1v_1'$. Then there is a vertex x, such that v_3v_4x also is in the tour. It is evident that an interchange between the edges "out from" v_4 do not produce any triangles in the eulerian tour. Hence, there exists a triangle-free eulerian tour where $v_1''v_4x$ and $v_3v_4w_1v_1'$ exist as sequences. By repeating the argument, if necessary, for the sequence $v_2''v_5w_2v_2'$, we know that there exists a triangle-free eulerian tour in G' with such features that it is also triangle-free in G. Subcase 3 is treated in the same way as subcase 2, with the exception that the graph is instead splitted along $w_1v_1v_3v_5v_2w_2$.

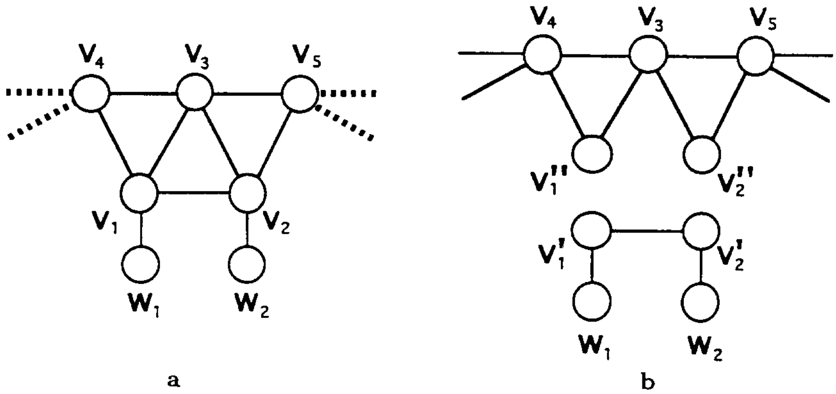


Fig. 8

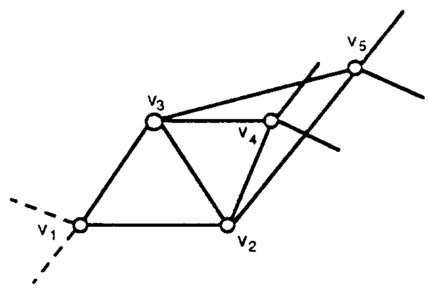


Fig. 9

Case 4: In case 4, exactly one edge in T has 2 adjacent triangles. G then has a subgraph as in Fig. 9.

At least two of the vertices v_1, v_4, v_5 have degree 4. (Or else G has a forbidden subgraph.) We can apply the lemma with a suitable selection of v_1, v_4, v_5 as the set S.

Case 5: In case 5, only one edge in T is adjacent to another triangle, and G will look as in Fig. 10.

We assume that none of w_2, w_3 is adjacent to none of v_1, v_4, since otherwise we would have case 3. Split G along $w_2v_2v_3w_3$ and call the resulting graph G'. By the lemma, G' is connected and has by induction a triangle-free eulerian tour. By the assumption above this eulerian tour is also triangle-free in G.

Case 6: In case 6, G has a subgraph as in Fig. 11(a). We have 2 subcases:

1. $d(v_3) = 2$,
2. $d(v_3) = 4$.

However, subcase 1 is treated in virtually the same way as subcase 2, so we will only show this latter case.

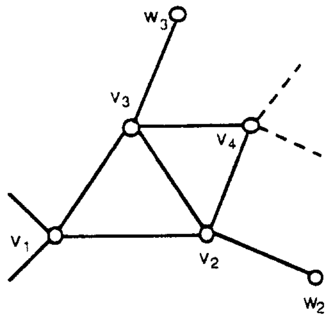


Fig. 10

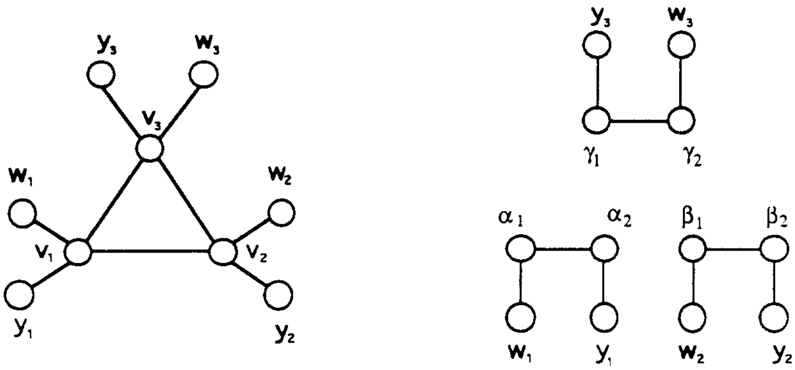


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Fig. 11

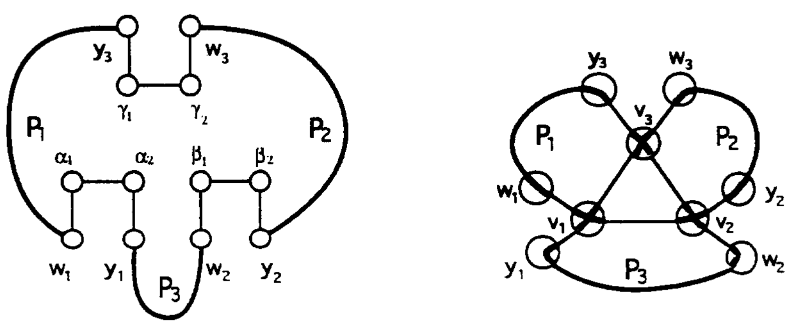

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Fig. 12

Let the neighbours to v_i be w_i, y_i for $i = 1, 2, 3$ as in the Fig. 11(a). Construct a new graph G' by deleting the edges v_1v_2, v_2v_3 and v_1v_3. Then split v_1 into α_1, α_2, each vertex taking one edge from v_1. Repeat the procedure for v_2 and v_3, letting the new vertices be β_1, β_2 and γ_1, γ_2, respectively. See Fig. 11(b). G' does not have any forbidden subgraph and since none of v_1, v_2 or v_3 is a cut vertex, G' has, by induction, a triangle-free eulerian tour, which we, without loss of generality, may assume is $\alpha_1 w_1 P_1 y_3 \gamma_1 \gamma_2 w_3 P_2 y_2 \beta_2 \beta_1 w_2 P_3 y_1 \alpha_2$. See Fig. 12(a). Then

$$v_1 w_1 P_1 y_3 v_3 v_2 w_2 P_3 y_1 v_1 v_3 w_3 P_2 y_2 v_2$$

is a triangle-free eulerian tour in G. See Fig. 12(b). With this the last case is examined and the theorem is proved. □

References

[1] R. Häggkvist, Decompositions of regular bipartite graphs, In: J. Siemons, ed., Surveys in Combinatorics, LMS Notices 141 (Cambridge University Press, Cambridge, 1989) 115–147.