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LINEAR ALGEBRA AND ITS APPLICATIONS

On digraphs and forbidden configurations of strong sign nonsingular matrices

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Abstract

A square real matrix A is called a strong sign nonsingular matrix (or "S^2NS" matrix) if all matrices with the same sign pattern as A are nonsingular and the inverses of these matrices all have the same sign pattern. A digraph which is the underlying digraph of the signed digraph of an S^2NS matrix (with a negative main diagonal) is called an S^2NS digraph. In [9], Thomassen gave a characterization of strongly connected S^2NS digraphs in terms of the forbidden subdigraphs. In [2], Brualdi and Shader constructed minimal forbidden configurations for S^2NS digraphs for the general cases where the digraphs considered are not necessarily strongly connected. They also proposed the problem about the existence of new minimal forbidden configurations other than those found in [2,9]. In this paper, we construct infinitely many new (basic) minimal forbidden configurations and thus obtain the answer to this problem. We also obtain several necessary conditions for minimal forbidden configurations and give a generalization of Thomassen's Theorem. © 1998 Elsevier Science Inc. All rights reserved.

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1. Introduction

The sign of a real number a, denoted by sgn a, is defined by

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$$\operatorname{sgn} a = \begin{cases} 1 & a > 0, \\ 0 & a = 0, \\ -1 & a < 0. \end{cases}$$

Let $A = (a_{ij})$ be a real matrix, the (0,1,−1) matrix $\operatorname{sgn} A = (\operatorname{sgn} a_{ij})$ is called the sign pattern of A. The set of real matrices with the same sign pattern as A is called the qualitative class of A, and is denoted by $Q(A)$. Qualitative matrix theory involves the study of "qualitative properties" which depend only on the sign patterns of the matrices (and do not depend on the magnitudes of the entries of the matrices), that is, properties which hold for all matrices in a qualitative class of matrices. Qualitative matrix theory has been extensively studied, see for examples, [1–10].

A square real matrix A is called a sign nonsingular matrix (abbreviated SNS matrix), if each matrix with the same sign pattern as A is nonsingular.

An SNS matrix A is called a strong SNS matrix (abbreviated S^2NS matrix), if the inverses of the matrices in $Q(A)$ all have the same sign pattern.

A signed digraph S is a digraph where each arc of S is assigned a sign +1 or −1. The sign of a subdigraph (for example, a path or a cycle) S_1 of S is defined as the product of the signs of all the arcs of S_1, denoted by $\operatorname{sgn}(S_1)$.

The digraph $D(A)$ of a square real matrix $A = (a_{ij})$ of order n is the digraph with the vertex set $V = \{1, 2, \ldots, n\}$ and arc set $E = \{(i, j) \mid a_{ij} \neq 0\}$. The signed digraph $S(A)$ of the matrix A is obtained from the digraph $D(A)$ by assigning the sign $\operatorname{sgn} a_{ij}$ to each arc (i,j) in $D(A)$. Clearly, $S(A)$ completely determines the sign pattern of A. Thus the study of the qualitative properties of A (which depend only on the sign pattern of A) can be turned into the study of the graph theoretical properties of the signed digraph $S(A)$.

It is well known that a necessary condition for a square real matrix A to be an S^2NS matrix is that it can be transformed into a matrix with a negative main diagonal by successive applications of the following operations:

(1.1) Permuting the rows or columns.

(1.2) Multiplying a row or a colomn by −1,

while these operations preserve the property of being an S^2NS matrix.

An S^2NS matrix A with a negative main diagonal can be characterized in terms of its signed digraph $S(A)$ in the following way.

Theorem 1.A ([1,2,9]). *Let A be a square real matrix with a negative main diagonal. Then A is an S^2NS matrix if and only if the signed digraph $S(A)$ satisfies the following two conditions:*

(1.3) *Every cycle of $S(A)$ is negative.*

(1.4) *Any two paths in $S(A)$ with the same initial vertex and the same terminal vertex have the same sign.*

In view of Theorem 1.A, we make the following definitions.

Definition 1.1. A signed digraph S is called an S^2NS signed digraph if it satisfies the two conditions (1.3) and (1.4) in Theorem 1.A.

2. A digraph D is called an S^2NS digraph if the arcs of D can be suitably assigned the signs so that the resulting signed digraph is an S^2NS signed digraph. (Namely, if D is the underlying digraph of an S^2NS signed digraph.)

A digraph D which is not an S^2NS digraph is also called a 'forbidden configuration'.

It is clear from the definitions that any signed subdigraph of an S^2NS signed digraph is an S^2NS signed digraph, so any subdigraph of an S^2NS digraph is again an S^2NS digraph. In other words, any digraph containing a forbidden configuration as its subdigraph is also a forbidden configuration.

The following concept of "minimal forbidden configurations" was first introduced by Brualdi and Shader in [2].

Definition 1.2. A digraph D is called a minimal forbidden configuration (abbreviated MFC) if D is a forbidden configuration (i.e., D is not an S^2NS digraph), but any proper subdigraph of D is not a forbidden configuration.

Definition 1.3. Let D be a digraph.

1. "Splitting an arc (x,y)" of D means deleting the arc (x,y) and then inserting a new vertex x_1 and two new arcs (x,x_1) and (x_1,y). A subdivision of the digraph D is a digraph obtained from D by a sequence of arc splittings.
2. "Splitting a vertex x" of D means inserting a new vertex x_1, a new arc (x,x_1), and replacing each arc of D of the form (x,v) by the arc (x_1,v). A splitting of the digraph D is a digraph obtained from D by a sequence of arc splittings and vertex splittings.
3. A pair of oppositely directed arcs (x,y) and (y,x) in D is called (in this paper) an (undirected) edge of the digraph D, denoted by $[x,y]$. An "even edge splitting" on an edge $[x,y]$ means deleting the edge $[x,y]$ and then inserting an even number of new vertices $x_1,x_2,\dots,x_{2k}$ and the new edges $[x,x_1]$, $[x_1,x_2],\dots,[x_{2k-1},x_{2k}]$ and $[x_{2k},y]$.

It is not difficult to verify from the definitions that if D_1 is a splitting or an even edge splitting of D, then D is an S^2NS digraph if and only if D_1 is. And if D_1 is a MFC, then so is D. In view of this, we make the following definition.

Definition 1.4. A digraph D is called a "basic MFC" if D is a MFC and D cannot be obtained by vertex splittings, or arc splittings, or even edge splittings from other digraphs.

Example 1.1 ([9]). Let D_3 be the digraph with three vertices v,x,y and four arcs (x,y), (y,x), (x,v) and (y,v). Let D_3' be the digraph obtained from D_3 by reversing the direction of each of its arcs.

It is not hard to see ([9]) that neither of D_3, D_3' is an S^2NS digraph. It follows that any splitting of D_3 or D_3' is not an S^2NS digraph. We also notice that any splitting of D_3 (or D_3') actually contains a subdivision of D_3 (or D_3'). In fact all the subdivisions of D_3 or D_3' are MFC's.

From Example 1.1 we see that a necessary condition for a digraph D to be an S^2NS digraph is that D contains no subdivisions of D_3 or D_3'. Thomassen ([9]) showed that this necessary condition is also sufficient in the strongly connected case.

Theorem 1.B ([9]). *Let D be a strongly connected digraph. Then the following conditions are equivalent:*
1. *D is an S^2NS digraph.*
2. *D contains no subdivisions of D_3.*
3. *D contains no subdivisions of D_3'.*

The following digraph Γ_1 (see Fig. 1) constructed by Brualdi and Shader ([2], p. 188) shows that in the general case when D is not necessarily strongly connected, then not containing a subdivision of D_3 or D_3' is only a necessary condition, but not a sufficient condition, for D to be an S^2NS digraph. This means that there exist MFC's other than the subdivisions of D_3 and D_3'. In fact, the digraph Γ_1 in Fig. 1 is one of such MFC's.

Brualdi and Shader also pointed out that all splittings of even edge splittings of Γ_1 are MFC's (except splittings on the vertices of indegree zero or outdegree zero). They further pointed out in their book ([2], p. 188) that: "it is unknown whether there are other minimal forbidden configurations", and thus proposed the problem about the existence of new MFC's other than those subdivisions of D_3 or D_3', and those splittings of even edge splittings of Γ_1.

Note that among the above known MFC's, only D_3, D_3', and Γ_1 are the *basic* MFC's. So any *basic* MFC other than D_3, D_3', and Γ_1 will be a new MFC.

In Section 2, we will construct a family of infinitely many *basic* MFC's different from D_3, D_3' and Γ_1, thus obtain the answer to the above mentioned

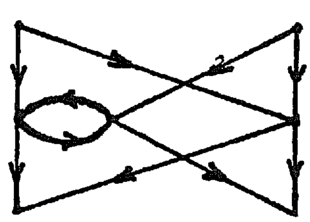

Fig. 1. The digraph Γ_1.

problem. Indeed, we will show that for any even number $m \geqslant 6$, there exists a basic MFC with exactly m strong components.

In Section 3, we first generalize Thomassen's Theorem 1.B (for the characterizations of S^2NS digraphs) from the strongly connected case to more general cases which include strongly connected case as a special case. In fact, this generalization also provides an alternate proof of Thomassen's Theorem. We then give [illegible] necessary conditions for MFC's in Theorem 3.2. From these necessary conditions we can see that the MFC Γ_1 (Fig. 1) constructed in [2] is in fact a MFC with the smallest number of vertices, the smallest number of arcs and the smallest number of strong components, except those subdivisions of D_3 and D_3'.

2. The constructions of new MFC'S

In this section we construct (in Theorem 2.1) a family of infinitely many new MFC's, which includes infinitely many *basic* MFC's.

A doubly directed path of length t is a digraph obtained from the graph of an undirected path of length t by replacing each edge by a pair of oppositely directed arcs. The two end vertices of the undirected path are also called the two end vertices of the doubly directed path.

Theorem 2.1. *Let $k \geqslant 2$ be a positive integer, $t_1, t_2, \ldots, t_k$ be nonnegative integers such that $t_1 + t_2 + \cdots + t_k$ is odd. Let D_i be a doubly directed path of length t_i with two end vertices v_i and u_i $(i = 1, 2, \ldots, k)$. Let $D = D(t_1, \ldots, t_k)$ be a digraph (see Fig. 2) obtained by adding to the disjoint union of $D_1, D_2, \ldots, D_k$ the new vertices y_1, y_2; $x_1, x_2, \ldots, x_k$ and the following new arcs (where the notation $\equiv$ means that the subscripts are read module k):*

$(x_i, v_i) \quad (i = 1, \ldots, k),$

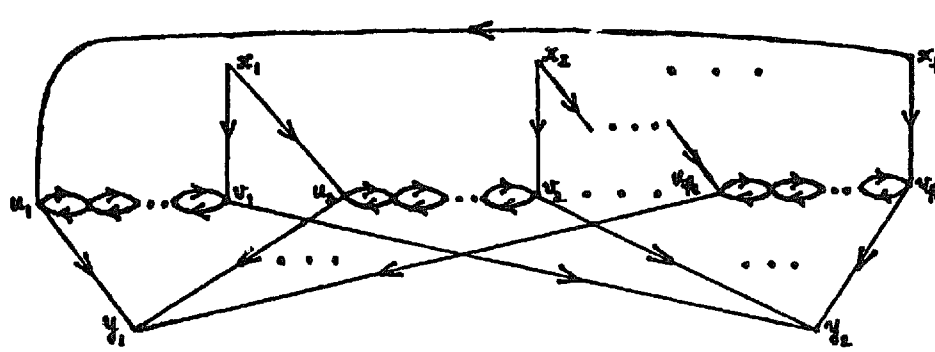

Fig. 2. The digraph $D(t_1, \ldots, t_k)$.

$$(x_i, u_{i+1}) \quad (i \equiv 1, \dots, k \bmod k),$$
$$(u_i, y_1) \quad (i = 1, \dots, k),$$
$$(v_i, y_2) \quad (i = 1, \dots, k),$$

Then D is a MFC.

Proof. First we show that D is not an S^2NS digraph. Suppose not, let S be an S^2NS signed digraph with D as its underlying digraph. For any two vertices x and y in D, if there is an unique path from x to y in D, then we denote this path by $P(x,y)$ and denote the sign of this path in (the signed digraph) S by $s(x,y)$. Now let

$$P_{i1} = P(x_i, v_i) + P(v_i, u_i) + P(u_i, y_1) \quad (i = 1, \dots, k), \tag{2.1}$$

$$Q_{i1} = P(x_i, u_{i+1}) + P(u_{i+1}, y_1) \quad (i \equiv 1, \dots, k \bmod k), \tag{2.2}$$

$$P_{i2} = P(x_i, v_i) + P(v_i, y_2) \quad (i = 1, \dots, k), \tag{2.3}$$

$$Q_{i2} = P(x_i, u_{i+1}) + P(u_{i+1}, v_{i+1}) + P(v_{i+1}, y_2) \quad (i \equiv 1, \dots, k \bmod k), \tag{2.4}$$

then P_{ij} and Q_{ij} are two paths in (the S^2NS signed digraph) S from x_i to y_j, so

$$\operatorname{sgn}(P_{ij}) \operatorname{sgn}(Q_{ij}) = 1 \quad (i = 1, \dots, k;\ j = 1, 2). \tag{2.5}$$

Also we have

$$s(v_i, u_i) s(u_i, v_i) = (-1)^{t_i} \quad (i = 1, \dots, k) \tag{2.6}$$

since $P(v_i, u_i) + P(u_i, v_i)$ is a union of t_i cycles in S.

From Eqs. (2.1)–(2.6) we have

$$\begin{aligned}
1 &= \prod_{i=1}^{k} (\operatorname{sgn}(P_{i1})\operatorname{sgn}(Q_{i1}))(\operatorname{sgn}(P_{i2})\operatorname{sgn}(Q_{i2})) \\
&= \prod_{i=1}^{k} s(x_i, v_i)^2 \prod_{i=1}^{k} s(x_i, u_{i+1})^2 \prod_{i=1}^{k} s(u_i, y_1) \prod_{i=1}^{k} s(u_{i+1}, y_1) \prod_{i=1}^{k} s(v_i, y_2) \prod_{i=1}^{k} s(v_{i+1}, y_2) \\
&\quad \prod_{i=1}^{k} s(v_i, u_i) \prod_{i=1}^{k} s(u_{i+1}, v_{i+1}) \\
&= \left(\prod_{i=1}^{k} s(u_i, y_1)\right)^2 \left(\prod_{i=1}^{k} s(v_i, y_2)\right)^2 \left(\prod_{i=1}^{k} s(v_i, u_i) s(u_i, v_i)\right) = \prod_{i=1}^{k} (-1)^{t_i}.
\end{aligned}$$

This contradicts to the assumption that $t_1 + \cdots + t_k$ is odd.

Secondly we show that D-e is an S^2NS digraph for any arc e of D (this will imply that any proper subdigraph of D is an S^2NS digraph since D contains no isolated vertices).

Case 1: $e = (u_i, y_1)$ for some $1 \leqslant i \leqslant k$. Without loss of generality we may assume that $e = (u_1, y_1)$.

We use the following procedure to sign (the arcs of) the digraph *D-e* into an S^2NS signed digraph.

Step 1: We assign the negative sign to all arcs of $P(v_i, u_i)$ and assign the positive sign to all arcs of $P(u_i, v_i)$. Then every cycle of *D-e* is negative.

Step 2: For $i = 1, \ldots, k$, we give arbitrary signs to the arcs (v_i, y_2).

Step 3: For $i \equiv 1, \ldots, k \pmod k$, we give suitable signs to the arcs (x_i, v_i) and (x_i, u_{i+1}), so that the two paths P_{i2} and Q_{i2} (defined in Eqs. (2.3) and (2.4)) in *D-e* from x_i to y_2 have the same sign.

Step 4: For $i = 2, \ldots, k$, we successively sign the arcs (u_i, y_1), such that the arc (u_i, y_1) is given the sign so that for $i = 3, \ldots, k$, the two paths $P_{(i-1)1}$ and $Q_{(i-1)1}$ in *D-e* from x_{i-1} to y_1 have the same sign.

Since $\{P_{i2}, Q_{i2}\}$ $(i = 1, \ldots, k)$ and $\{P_{i1}, Q_{i1}\}$ $(i = 2, \ldots, k-1)$ are all the pairs of paths in *D-e* with the same initial vertex and the same terminal vertex, we see that the digraph *D-e* signed in this way is an S^2NS signed digraph.

Case 2: $e = (v_i, y_2)$ for some $1 \leqslant i \leqslant k$.

This is similar to Case 1.

Case 3: $e \in E(P(v_i, u_i))$ for some $1 \leqslant i \leqslant k$. Without loss of generality, we may assume that $e \in E(P(v_k, u_k))$.

We first use the same procedure as in Case 1 to sign the arcs of *D-e* (except that we do not need to sign the arc e in Step 1). Then we assign the suitable sign to the arc (u_1, y_1) so that the two paths P_{11} and Q_{11} from x_1 to y_1 have the same sign.

Since $\{P_{i2}, Q_{i2}\}$ $(i = 1, \ldots, k)$ and $\{P_{i1}, Q_{i1}\}$ $(i = 1, \ldots, k-1)$ are all the pairs of paths in *D-e* with the same initial vertex and the same terminal vertex, we see that the digraph *D-e* signed in this way is an S^2NS signed digraph.

Case 4: $e \in E(P(u_i, v_i))$ for some $1 \leqslant i \leqslant k$.

This is similar to Case 3.

Case 5: $e = (x_i, v_i)$ for some $1 \leqslant i \leqslant k$. Without loss of generality we may assume that $e = (x_k, v_k)$.

We first use the same procedure as in Case 1 to sign the arcs of *D-e* (except that in Step 3 we do not sign the arc (x_k, v_k) and do not need the two paths P_{k2} and Q_{k2} have the same sign). Then we assign the suitable sign to the arc (u_1, y_1) so that the two paths P_{11} and Q_{11} from x_1 to y_1 have the same sign.

Since $\{P_{i2}, Q_{i2}\}$ $(i = 1, \ldots, k-1)$ and $\{P_{i1}, Q_{i1}\}$ $(i = 1, \ldots, k-1)$ are all the pairs of paths in *D-e* with the same initial vertex and the same terminal vertex, we see that the digraph *D-e* signed in this way is an S^2NS signed digraph.

Case 6: $e = (x_i, u_{i+1})$ for some $1 \leqslant i \leqslant k$.

This is similar to Case 5.

Combining Cases 1–6 we see that *D-e* is an S^2NS digraph for any arc e of D, so D is a MFC. $\square$

Remark. The digraphs $D'(t_1, \ldots, t_k)$ obtained by reversing the directions of all the arcs of $D(t_1, \ldots, t_k)$ are also MFC's (when $k \geqslant 2$ and $t_1 + \cdots + t_k$ is odd), and they are different from those $D(t_1, \ldots, t_k)$ in the case $k \geqslant 3$.

We notice that in the above digraph $D(t_1, \ldots, t_k)$, there is no vertex with out-degree one, so $D(t_1, \ldots, t_k)$ is not the splitting of any digraph. This implies that in the case $k \geqslant 2$, $t_1 + \cdots + t_k$ odd and

$$t_i \in \{0, 1, 2\} \quad (i = 1, \ldots, k)$$

the digraph $D(t_1, \ldots, t_k)$ is a *basic* MFC. In this way, we have actually constructed infinitely many basic MFC's, hence obtain the answer to the problem mentioned in Section 1. Indeed, the basic MFC Γ_1 in Fig. 1 is a special case of $D(t_1, \ldots, t_k)$ with $k = 2$, $t_1 = 1$ and $t_2 = 0$.

Note that the number of strong components of $D(t_1, \ldots, t_k)$ is the even number $2k + 2$ $(k \geqslant 2)$. So we have actually proved that for any even number $m \geqslant 6$, there is a basic MFC with exactly m strong components. (We will show in Theorem 3.2 that a basic MFC different from D_3 and D_3' must contain at least six strong components.)

It is unknown whether there exist basic MFC's with odd number of strong components.

3. A generalization of Thomassen's theorem and some necessary conditions for minimal forbidden configurations

In this section, we first generalize Thomassen's Theorem 1.B (for the characterizations of S^2NS digraphs) from the strongly connected case to more general (not necessarily strongly connected) cases which include strongly connected case as a special case (Theorem 3.1). Indeed, this generalization also gives an alternate proof of Thomassen's Theorem. We then use Theorem 3.1 to give several necessary conditions for MFC's in Theorem 3.2. These necessary conditions provide useful information in the study and constructions of MFC's.

We adopt the following notation in this section: if x and y are two vertices of a path P in a digraph D, and if x preceeds y in P, then we use xPy to denote the subpath of P from x to y.

Before proving Theorem 3.1, we first prove the following Lemma 3.1 which gives an equivalent condition for a digraph to contain no subdivisions of D_3 or D_3'.

Lemma 3.1. *Let D be a digraph. Then the following two conditions are equivalent:*
1. *D contains no subdivisions of D_3 (or D_3').*
2. *For any strong subdigraph D_1 of D and a vertex x not in D_1, any two paths from D_1 to x (or from x to D_1) have a common vertex in D_1.*

Proof. It is obvious that (2) implies (1). Now we prove that (1) implies (2).

Suppose that (2) is not true, then there exists two paths P_1 and P_2 from some strong subdigraph D_1 of D to a vertex x not in D_1 having no common vertex in D_1. Let y_i be the last vertex of P_i in D_1 $(i = 1, 2)$, then $y_1 \neq y_2$. Let Q_1 (and Q_2) be a path from y_1 to y_2 (and from y_2 to y_1) in D_1 (since D_1 is strong), and let u be the first vertex of $V(Q_2) \setminus \{y_2\}$ which is also on Q_1. Then $C = y_2Q_2u + uQ_1y_2$ is a cycle in D_1 and $C + y_2P_2x + uQ_2y_1 + y_1P_1x$ contains a subdivision of D_3, a contradiction. □

A terminal (or initial) component of a digraph D is a strong component H of D such that there is no arc of D from a vertex in H (or outside H) to a vertex outside H (or in H).

We now prove the following generalization of Thomassen's Theorem 1.B.

Theorem 3.1. *Let D be a digraph with a unique terminal component (or unique initial component) G. Then D is an S^2NS digraph if and only if D contains no subdivisions of D_3 and no subdivisions of D_3'.*

Proof. The necessity part is obvious and we now prove the sufficiency part.

Take a vertex w in the unique terminal component G. Then for any vertex x in D, there is a path from x to w.

Let D_1 be the subdigraph of D consisting of all those arcs e such that there is a path containing e which terminates at the vertex w, we claim that D_1 is an acyclic digraph. Suppose not, then D_1 contains a cycle C (clearly w is not on C since D_1 contains no arc starting from w). Let P be a shortest path from the cycle C to the vertex w and let x be the initial vertex of P. Let (x,y) be the arc on C starting from x. Then there is a path Q in D containing the arc (x,y) and terminating at w since $(x,y) \in E(C) \subseteq E(D_1)$. Now the subpath yQw does not pass through x, so yQw and P are two paths in D from the strong subdigraph C to the vertex w with no common vertex in C, thus D contains a subdivision of D_3 by Lemma 3.1, contradicting to our hypothesis. So D is an acyclic digraph.

We now sign the arcs of D in such a way that all the arcs in D_1 are assigned the positive sign and all the remaining arcs are assigned the negative sign. We claim that the resulting signed digraph S is an S^2NS signed digraph.

It is clear that every cycle of S contains at least one negative arc (since D_1 is acyclic). Now suppose that some cycle C of S contains two negative arcs e_1 and e_2. Let P be a shortest path from the cycle C to the vertex w and let x be the initial vertex of P. Let y be the initial vertex of the arc e_1 and suppose that $y \neq x$ (otherwise we can replace e_1 by e_2). Then $yCx + P$ is a path containing the arc e_1 and terminating at w, so e_1 is an arc of D_1, contradicting that e_1 is a negative arc of S. This shows that every cycle of S contains at most one negative arc, hence contains exactly one negative arc. So every cycle of S is negative.

We now prove that any two paths in S with the same initial vertex and the same terminal vertex have the same sign. Suppose not, let P_1 and P_2 be a pair of paths with the minimal total length such that they have the same initial vertex (say, u) and the same terminal vertex (say, v), but they have the different signs. Then P_1 and P_2 are internally vertex disjoint by the minimality of their total length. Now take a path Q from v to w. If v is the unique common vertex of Q and $P_1 \cup P_2$, then one of the two paths $P_1 + Q$ and $P_2 + Q$ contains a negative arc and terminates at w, a contradiction. Otherwise, let y be the first vertex of Q different from v which is also on $P_1 \cup P_2$. If $y = u$, then the two cycles $C_1 = P_1 + vQu$ and $C_2 = P_2 + vQu$ have different signs, a contradiction. If $y \neq u$, then $P_1 + P_2 + vQy$ is a subdivision of D_3', also a contradiction.

Combining the above two aspects, we see that S is an S^2NS signed digraph and so D is an S^2NS digraph.

(If D has a unique initial component, then the digraph D' obtained by reversing the directions of all the arcs of D has a unique terminal component, so D' is an S^2NS digraph, and hence D is also an S^2NS digraph.) □

Note that a strongly connected digraph does have a unique terminal component, so Theorem 3.1 is a generalization of Thomassen's Theorem.

Theorem 3.1 can also be used to derive some necessary conditions for MFC's. First we notice that a MFC D is necessarily not strongly connected, for otherwise D would contain a subdivision H of D_3 by Thomassen's Theorem, and $H \neq D$ since H is not strongly connected, but D is. This contradicts the minimality of D (as a forbidden configuration). Secondly a MFC D is connected in the undirected sense (when we ignore the directions of the arcs of D and then view D as an undirected graph), for otherwise one of the undirected component of D would be a smaller forbidden configuration, again contradicting the minimality of D. These properties also imply that a MFC D contains no isolated (strong) components.

The following Theorem 3.2 gives some further necessary conditions for MFC's.

Theorem 3.2. *Let D be a MFC which is not a subdivision of D_3 or D_3', then:*

1. *D contains at least two initial components and at least two terminal components.*
2. *There is no arc from an initial component of D to a terminal component of D.*
3. *For any terminal (or initial) component G of D, there are at least two different components G_1 and G_2 (different from G) of D such that there exist arcs from each G_i to G (or from G to each G_i), $(i = 1, 2)$.*
4. *D contains at least 6 components, 7 vertices and 10 arcs.*

Proof. It is clear that a MFC D which is not a subdivision of D_3 or D_3' also contains no subdivisions of D_3 or D_3'.

(1) This follows directly from Theorem 3.1.

(2) Assuming that there is an arc $e=(x,y)$ from some initial component D_1 to some terminal component D_2. By the minimality of D (as a forbidden configuration) we know that D-e is an S^2NS digraph, so there is an S^2NS signed digraph S_e with D-e as its underlying digraph. Now we assign a sign to the arc e so that this sign of e is the same as the sign of all the paths in (the S^2NS signed digraph) S_e from x to y (if any). We claim that the resulting signed digraph S (with D as its underlying digraph) is an S^2NS signed digraph.

Clearly every cycle of S is negative since the arc e is not contained in any cycle of D. Now let P and Q be any two paths in S with the same initial vertex (say, u) and the same terminal vertex (say, v). If neither of P or Q contains the arc e, then both P and Q are paths in S_e and so they have the same sign. Otherwise we assume that P contains the arc e. Then $P = uPx + e + yPv$. Now uPx is a path entirely contained in the component D_1 since D_1 is an initial component and x is in D_1. Similarly yPv is a path entirely contained in the terminal component D_2. Since D contains no subdivisions of D_3 or D_3', the path Q (from u to v) must also pass through the vertex x and the vertex y by Lemma 3.1. So we have $Q = uQx + xQy + yQv$. Since S_e is an S^2NS signed digraph, we have

$$\operatorname{sgn}(uPx) = \operatorname{sgn}(uQx), \qquad \operatorname{sgn}(yPv) = \operatorname{sgn}(yQv).$$

Now if $xQy = e$, then clearly $\operatorname{sgn}(xQy) = \operatorname{sgn}(e)$. If $xQy \neq e$, then xQy is a path in S_e from x to y and we also have $\operatorname{sgn}(xQy) = \operatorname{sgn}(e)$ by the choice of the sign of e. Therefore P and Q have the same sign and S is an S^2NS signed digraph. So D is an S^2NS digraph, a contradiction.

(3) There is at least one arc (say, $e_1 = (x_1, y_1)$) coming into the terminal component G since G cannot be an isolated component. Assuming x_1 is in the component G_1 (y_1 is clearly in the component G). Now if there is no arc other than e_1 coming into the component G, then we sign the S^2NS digraph D-e_1 (by the minimality of D as a forbidden configuration) into an S^2NS signed digraph and then assign any sign to the arc e_1. The resulting signed digraph S (with D as its underlying digraph) is then an S^2NS signed digraph (since any two paths in D with the same initial vertex and the same terminal vertex either both contain the arc e_1 or both do not contain the arc e_1), a contradiction. This argument shows that there exists another arc e_2 different from e_1 coming into the terminal component G. Suppose the initial vertex of e_2 is in the component G_2, then $G_1 \neq G_2$. For otherwise D would contain a subdivision of D_3 or D_3' by Lemma 3.1, again a contradiction. This proves (3).

(4) By (1) D contains at least two initial components (say, C_1 and C_2) and at least two terminal components (say, C_3 and C_4). By (3) there are at least two different components (say, C_5 and C_6) from which there are arcs coming into the terminal component C_3. By (2) C_5 or C_6 is different from C_1 or C_2. So $C_1, C_2, C_3, C_4, C_5, C_6$ are different from each other and thus D contains at least six components.

On the other hand, D contains at least one nontrivial component, since otherwise D would be an acyclic digraph and then D would also be an S^2NS digraph (an acyclic digraph can be signed into an S^2NS signed digraph by assigning the positive sign to each of its arcs), a contradiction. Now a nontrivial component contains at least two vertices, so D contains at least seven vertices.

Now by (3) there are at least two arcs coming out of each initial component, so by (1) there are at least four arcs coming out of some initial component. Similarly there are at least four arcs coming into some terminal component. By (2) these arcs are different. Also a nontrivial component of D contains at least two arcs. So altogether D contains at least 10 arcs. □

The necessary conditions for MFC's given in Theorem 3.2 provide useful information in the study and constructions of MFC's. In particular, we know from result (4) of Theorem 3.2 that the MFC Γ_1 (Fig. 1) constructed in [2] is in fact a MFC with the smallest number of components (6 components), the smallest number of vertices (7 vertices) and the smallest number of arcs (10 arcs), except those subdivisions of D_3 and D_3'.

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