A Note on Model Reduction of Large Scale Systems

O. IBIDAPO-OBE *

Department of Civil Engineering, State University of New York at Buffalo, Buffalo, New York 14214

Submitted by T. T. Soong

1. INTRODUCTION

In an earlier paper [1], results on model reduction of large scale systems were given using a multivariate linear regression scheme. The purpose of this note is to present these results in another form, based upon a singular value decomposition approach, which is more efficient computationally.

Consider an nth-order linear system S_1 defined by

$$S_1 : \dot{x} = Ax + Bu \tag{1}$$

where x is the n-dimensional state vector, A is an $n \times n$ system matrix, and u is a p-dimensional input vector. Let z be an m-vector $(m < n)$ related to x by

$$z = Cx. \tag{2}$$

In model reduction, it is desirable to find an mth-order system S_2 described by

$$S_2 : \dot{z} = Fz + Gu. \tag{3}$$

The $m \times n$ matrix C in Eq. (2) is the aggregation matrix and S_2 is the aggregated system or the reduced model. It is easy to show that $G = CD$ and that F must satisfy the matrix equation

$$FC = CA. \tag{4}$$

Equation (4) defines an overspecified system of equations for the unknown matrix F, and hence F must be approximated. In [1], a multivariate linear regression scheme is used to yield a "best" approximation for F in the form

$$\hat{F} = CAC^{\mathrm{T}}(CC^{\mathrm{T}})^{-1}, \tag{5}$$

---

* Visiting Fulbright Research Scholar from the University of Lagos, Nigeria.

where T and -1 denote matrix transpose and matrix inverse, respectively. The rank of C is assumed to be m. The result given by Eq. (5) is interpreted as a linear, unbiased, minimum-variance estimate of F and its form agrees with that given by Aoki [2], following an ad hoc procedure.

In addition, the covariance of $\hat{F}$ is found to be

$$\text{cov}(\text{vec } \hat{F}) = \sigma^2 [(CC^T)^{-1} \otimes I_m] \tag{6}$$

and it is shown in [1] that this covariance matrix can be used for model reduction error assessment. In Eq. (6), the Kronecker product $\otimes$ and the "vec" operator are defined as

$$P \otimes Q = [p_{ij} \quad Q] \tag{7}$$

$$\text{vec}(P) = [P_1 \quad P_2 \cdots]^T, \tag{8}$$

where P and Q are matrices of arbitrary dimensions and P_k is the kth column of matrix P.

2. SINGULAR VALUE DECOMPOSITION

From the computational point of view, it is desirable to circumvent the use of matrix inverses in Eqs. (5) and (6), particularly for systems having large aggregation matrices. In what follows, this is accomplished through the use of matrix singular value decomposition (SVD) [3, 4], which has found useful application in linear least-squares problems.

The SVD concept gives the Moore–Penrose pseudo-inverse of C as

$$C^+ = V\Lambda U^T \tag{9}$$

where U and V are unitary matrices whose columns are the eigenvectors of matrices DD^T and $D^T D$, respectively, and

$$\Lambda = \begin{bmatrix} \sigma_1^{-1} & & & 0 & \vdots & \\ & \sigma_2^{-1} & & & \vdots & \\ & & \ddots & & \vdots & \\ 0 & & & \sigma_m^{-1} & \vdots & \\ \hline & & & & 0 & \\ & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix}_{n \times n} \tag{10}$$

where $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_m \geqslant 0$, called singular values, are the nonnegative square roots of the eigenvalues of $D^T D$. A discussion of this decomposition and its properties can be found in Stewart [5].

Now, Eq. (4) gives

$$\hat{F} = CAC^+ \tag{11}$$

and, using SVD, we can write

$$\hat{F} = CA(V\Lambda U^T). \tag{12}$$

The matrix C can also be written in the form

$$D = U\Sigma V^T \tag{13}$$

where

$$\Sigma = \begin{bmatrix} \sigma_1 & & & \vdots & \\ & \sigma_2 & & 0 & \vdots & \\ & & \ddots & & \vdots & \\ & 0 & & \sigma_m & \vdots & \\ \hline & & & & 0 & \\ & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix}_{n \times n} \tag{14}$$

and we have

$$\hat{F} = U\Sigma V^T A V\Lambda U^T. \tag{15}$$

Compared with Eq. (5), either Eq. (12) or Eq. (15) provides a more efficient method of computation for $\hat{F}$ due to elimination of the matrix inverse.

Similarly, advantages are realized in the calculation of $\text{cov}(\text{vec } \hat{F})$. Following the SVD scheme,

$$(DD^T)^{-1} = (D^T)^+ D^+$$
$$= (U\Lambda V^T)(V\Lambda U^T)$$
$$= U\Lambda^2 U^T. \tag{16}$$

Equation (6) now takes the form

$$\text{cov}(\text{vec } \hat{F}) = \sigma^2 [U\Lambda^2 U^T \otimes I_m], \tag{17}$$

which is clearly of a simpler structure than Eq. (6).

REFERENCES

1. T. T. SOONG, On model reduction of large-scale systems, *J. Math. Anal. Appl.* **60** (1977), 477–482.
2. M. AOKI, Control of large scale dynamic systems by aggregation, *IEEE Trans. Automat. Control* **AC-13** (1968), 246–253.
3. G. H. GOLUB AND C. REINSCH, Singular value decomposition and least squares solutions, *Numer. Math.* **14** (1970), 403–420.
4. G. H. GOLUB AND C. VAN LOAN, Total least squares, *in* "Smoothing Techniques for Curve Estimation" (T. Gasser and M. Rosenblatt, Eds.), pp. 62–76, Springer-Verlag, New York, 1979.
5. G. W. STEWART, "Introduction to Matrix Computations" Academic Press, New York, 1973.