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Game domination number

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Abstract

The game domination number of a (simple, undirected) graph is defined by the following game. Two players, A and D, orient the edges of the graph alternately until all edges are oriented. Player D starts the game, and his goal is to decrease the domination number of the resulting digraph, while A is trying to increase it. The *game domination number* of the graph G, denoted by $\gamma_g(G)$, is the domination number of the directed graph resulting from this game. This is well defined if we suppose that both players follow their optimal strategies. We determine the game domination number for several classes of graphs and provide general inequalities relating it to other graph parameters.

Keywords: Domination number; Directed graph; Orientation game

1. Introduction

A *dominating set* of a digraph $\vec{G}$ is a set S of vertices such that for every vertex $v \notin S$ there exists some $u \in S$ with $\vec{uv} \in E(\vec{G})$. The *domination number* $\gamma(\vec{G})$ of $\vec{G}$ is defined as the cardinality of the smallest dominating set.

We define a 'domination parameter' of an undirected graph G as the domination number of one of its orientations, determined by the following two player game. Players A and D orient the unoriented edges of the graph G alternately with D playing first, until all edges are oriented. Player D (frequently called the *Dominator*) is trying to minimize the domination number of the resulting digraph, while player A (*Avoider*) tries to maximize the domination number. This game gives a unique number

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depending only on G, if we suppose that both A and D play according to their optimal strategies. We call this number the *game domination number of* G and denote it by $\gamma_g(G)$.

As the domination number of any orientation of a graph is at least as large as the domination number of the graph itself, we clearly have $\gamma(G) \leqslant \gamma_g(G)$. Also $\gamma_g(G) \leqslant \mathrm{DOM}(G)$, where $\mathrm{DOM}(G)$ denotes the maximal domination number among all orientations of G. This parameter was examined in [6].

Similar orientation games with different goals for the players were introduced and discussed in [1,3–5].

In Section 2 we determine the game domination number for several classes of graphs including complete graphs, complete bipartite and tripartite graphs, paths and cycles. Then we obtain sharp lower and upper bounds for the game domination number of trees in terms of the smallest degree that is at least three.

Finally, in Section 4 we prove several inequalities, relating the game domination number to other graph parameters such as the number of vertices and edges, independence number and two-domination number. We establish a Nordhaus–Gaddum-type upper bound for the sum of the game domination number of a graph and its complement.

For additional results on related domination parameters we refer the reader to two excellent books [8,9].

2. Examples

In this section we determine the game domination number for a few classes of graphs. These elementary examples enable the reader to gain a feel for the parameter; also, some of the examples will be needed in the sequel.

Example 2.1. For the complete graph K_n on $n \geqslant 4$ vertices, we have $\gamma_g(K_n) = 2$.

Proof. Let us see first why $\gamma_g(K_n) \geqslant 2$, i.e. why one vertex cannot dominate the oriented graph. Player A can clearly avoid a source in K_4, and if $n \geqslant 5$ then there exists a collection of n edge-disjoint paths of length 2, one centered at each vertex (see [4]). Whenever D orients one of these edges from the central vertex, A can orient the other edge of the corresponding path towards the central vertex. Thus the in-degree of each vertex becomes at least one.

On the other hand, $\gamma_g(K_n) \leqslant 2$ since the dominator can pick two vertices, u and v, say and then reply to each move $\vec{wu}$ by $\vec{vw}$, and to each move $\vec{wv}$ by $\vec{uw}$. This strategy ensures that $\{u, v\}$ becomes a dominating set of the resulting digraph. $\square$

Note, that in Example 2.1 the dominator made use of very few edges. The same idea can be applied for much sparser graphs.

Example 2.2. Let G be a graph on $n \geqslant 4$ vertices containing all but one edges of a copy of $K_{2,n-2}$. Then $\gamma_g(G) = 2$. Also, if G contains a set S of s vertices such that every vertex not in S has at least 2 neighbors in S, then $\gamma_g \leqslant s$.

Example 2.3. Let $K_{n,m}$ denote a complete bipartite graph with $n \leqslant m$ vertices in the two partite sets, then

$$\gamma_g(K_{n,m}) = \begin{cases} \lceil (m+1)/2 \rceil & \text{if } n = 1, \\ 2 & \text{if } n = 2, \\ 3 & \text{if } n = 3, 4 \text{ or } 5, \\ 4 & \text{otherwise.} \end{cases}$$

Example 2.4. If G is a complete k-partite graph ($k \geqslant 3$) with at least three vertices in each partite class, then $\gamma_g(G) = 3$.

Now, we turn to some sparser graphs that (as expected) have larger game domination numbers. They also show that the game domination number can be much larger than the domination number. The reader is encouraged to verify the statement of the next example.

Example 2.5. For the three-dimensional cube Q_3 and the Petersen graph P we have $\gamma_g(Q_3) = 3$ and $\gamma_g(P) = 4$.

Example 2.6. For a path P_n on n vertices we have $\gamma_g(P_n) = \lceil n/2 \rceil$.

Proof. The vertices of the path can be partitioned into $\lceil n/2 \rceil$ sets of disjoint edges and possibly one single vertex. Each of these sets can be dominated by one of its vertices regardless of the orientation of the edges, showing $\gamma_g(P_n) \leqslant \lceil n/2 \rceil$.

For the lower bound, A would like to prevent D creating many vertices that dominate both their neighbors. Although A cannot do this, he can easily achieve that no even numbered vertex (with the vertices of the path labeled with $1, 2, \ldots, n$ from left to right) dominates three vertices (including itself). Indeed, whenever D orients an edge out of an even vertex, A immediately orients the other edge in.

This strategy results in an oriented path, where to dominate the $\lceil n/2 \rceil$ odd vertices we must choose $\lceil n/2 \rceil$ vertices, as no vertex will dominate two odd vertices.

Example 2.7. For a cycle C_n on n vertices, $\gamma_g(C_n) = \lfloor n/2 \rfloor$.

Proof. We show first that D can achieve an orientation with domination number $\lfloor n/2 \rfloor$. Indeed, following his second move D can make sure that there is a vertex dominating both of its neighbors. The remaining $n-3$ vertices can be partitioned into $\lceil (n-3)/2 \rceil$ independent edges (with possibly one single vertex), and these vertices will be dominated by $\lceil (n-3)/2 \rceil$ vertices regardless of the orientation. Thus $\gamma_g(C_n) \leqslant n/2$.

On the other hand, player A can force the dominating set to be as big as $\lfloor n/2 \rfloor$ using the same idea as in case of paths: he labels the vertices by $1, 2, \ldots, n$, and ensures that no even vertex dominates both of its neighbors. Then, to dominate the $\lceil n/2 \rceil$ odd vertices we need at least $\lfloor n/2 \rfloor$ vertices, giving $\gamma_{\mathrm{g}} \geqslant \lfloor n/2 \rfloor$. □

Our next example is a family of less natural graphs, this result will be used later.

Example 2.8. Let G be a 'lollipop' on n vertices formed by an even cycle with a tail (a single path) attached to one of its vertices. Then $\gamma_{\mathrm{g}}(G) = \lfloor n/2 \rfloor$.

Proof. To prove that $\gamma_{\mathrm{g}}(G) \leqslant \lfloor n/2 \rfloor$, write v for the vertex of degree 3, and u for its neighbor on the path. The dominator D starts the game with $\vec{vu}$, and in his second move also orients an edge away from v. Thus, v dominates 3 vertices (including itself), and as the cycle is even, the remaining vertices can be partitioned into a matching (with possibly a singleton), showing that $\gamma_{\mathrm{g}}(G) \leqslant \lfloor n/2 \rfloor$. The lower bound can be shown as in the previous examples. □

First, it seems that by adding edges to a graph we cannot increase its game domination number. Indeed, this is clearly the case if we add an even number of edges to a graph. However, rather surprisingly, this does not hold if we add exactly one edge to our graph.

Example 2.9. Let G be obtained from the complete bipartite graph $K_{t,4}$ ($t \geqslant 6$) as follows. Let $K_{t,4} = K(M,N)$ with $M = \{e, f, g, \ldots, z\}$, $N = \{a, b, c, d\}$), and $G = K_{t,4} + ab + cd - dz$. Then $\gamma_{\mathrm{g}}(G) = 2$, while $\gamma_{\mathrm{g}}(G + dz) = 3$.

Proof. Player D has an easy strategy to finish the game with a two-element dominating set: he starts with $\vec{az}$, then he ensures that every vertex in M is dominated by at least one vertex of $\{a, d\}$ and one vertex of $\{b, c\}$. Whenever A plays ab or cd, the dominator orients the other so that either $\{a, d\}$ or $\{b, c\}$ becomes the dominating set. Hence $\gamma_{\mathrm{g}}(G) \leqslant 2$.

What about $\gamma(G + dz)$? Playing the game on $G + dz$, the Avoider can force the Dominator to be the first to orient an edge in N. Clearly, the only way D could end up with a dominating set of size 2 is to use two independent vertices of N: $\{a, c\}, \{a, d\}, \{b, c\}$ or $\{b, d\}$. The strategy of A will be to try to dominate some of these pairs from vertices of M, making them impossible to become dominating sets. He cannot 'kill' all of the four possible dominating pairs, but at least two disjoint pairs he can. Whenever D orients the first edge of N, A can orient the other so that none of the four possible pairs could dominate the graph. We leave it to the reader to verify that D cannot do any better by orienting an edge of N before A 'kills' two of the possible dominating pairs. □

The 'jump' can be larger than one, as a little modification of the previous example shows us.

Example 2.10. If G is the same as in the previous example, $(k-1)(G+dz)+G$ has game domination number $2k$ and adding only one edge to the graph, $k(G+dz)$ has game domination number $3k$.

We believe, that this is the biggest possible jump: if $\gamma_g(G) \leqslant 2k$ then $\gamma_g(G+ab) \leqslant 3k$.

3. Trees

First, we shall derive a sharp lower bound for the game domination number of trees, then we look for upper bounds in terms of different parameters.

Theorem 3.1. *For any tree T on n vertices*

$$\gamma_g(T) \geqslant \left\lceil \frac{n}{2} \right\rceil.$$

Proof. We apply induction on the number of vertices. We clearly need $\lceil n/2 \rceil$ vertices to dominate if $n \leqslant 3$.

To proceed with the induction, if $n \geqslant 4$ we need to find either a vertex with at least two leaves attached to it or a leaf attached to a vertex of degree two. One of these always exists, we might for example consider a vertex v and the longest path starting from v. This path ends in a leaf and the previous vertex has either degree two, or another leaf adjacent with it. The two cases are essentially the same, and we shall discuss only one in detail.

Suppose that there are two leaves u and v attached to a vertex w. Player A could play the game in $T-\{u,v\}$ according to his strategy resulting in a domination number at least $\lceil (n-2)/2 \rceil$. Whenever D orients one of the edges adjacent to u or v, A immediately orients the other edge from the leaf. Thus two of the three vertices u, v, w are needed in the dominating set. At least another $\lceil (n-2)/2 \rceil - 1$ vertices are needed from the rest of the graph, giving no dominating set smaller than $\lceil n/2 \rceil$. □

Theorem 3.2. *Let T denote a tree on n vertices that is not a path, and let d denote the smallest degree in T that is at least three. Then*

$$\gamma_g(T) \leqslant \min\left\{ \left\lfloor \frac{n}{2} + \frac{n-2}{2(d-1)} \right\rfloor, \left\lfloor \frac{2}{3}n \right\rfloor \right\},$$

where the $\lfloor \frac{2}{3}n \rfloor$ bound takes over the other only if $d=3$.

Proof. We need to provide a strategy for player D resulting in a digraph with small domination number. Suppose that the tree has k vertices of degree at least d. An easy counting argument shows that $k \leqslant \lfloor (n-2)/(d-1) \rfloor$. We orient the edges so that the digraph we obtain has a small dominating set containing these k vertices.

Note, that the remaining $n-k$ vertices of T can be partitioned into paths attached to the k vertices of large degree. We claim that these vertices can be dominated by

$\lfloor (n-k)/2 \rfloor$ vertices in addition to the k vertices of degree at least d. Indeed, a path of even length $2l$ contains l independent edges and a dominating set of size l regardless of the orientation. Thus all D needs to take care of are the odd paths. But a path of length $2l+1$ starting from a dominating vertex v can easily be dominated by another l vertices if D takes the first move (he dominates the first vertex from v), and by $l+1$ vertices otherwise. As D starts the game, he is able to make the first move in at least half of the odd paths dominating the vertices on odd paths by at most half of them.

Thus, we have a dominating set of the resulting digraph of size

$$\gamma_g(T) \leqslant k + \left\lfloor \frac{n-k}{2} \right\rfloor = \left\lfloor \frac{n+k}{2} \right\rfloor \leqslant \left\lfloor \frac{n}{2} + \frac{n-2}{2(d-1)} \right\rfloor .$$

If $d=3$, then we do not put every vertex of degree three or larger into the dominating set. Instead, we partition the vertices of T into stars of at least two vertices (the existence of such a partition is obvious by induction). Player D can easily dominate a star $K_{1,r}$ with $\lfloor \frac{2}{3}(r+1) \rfloor$ vertices even if A starts the orientation, unless $r=3$. In $K_{1,3}$ two vertices will dominate if D starts the game and three if A does. Thus, the strategy of D is to make use of a star-partition: he starts in a $K_{1,3}$ (if there is any) then plays in the same star as A, except if he chooses another $K_{1,3}$. Then D does the same, ensuring that at least half of the three-stars will be dominated by two vertices, which in average gives a dominating set of size at most $5/8 < 2/3$ of the vertices in these stars. This strategy provides a domination number at most $\lfloor \frac{2}{3} n \rfloor$ in the resulting digraph. $\square$

Note, that both bounds in the previous theorem are sharp, for example if T is constructed from a path of k vertices with $d-1$ or $d-2$ leaves attached to each vertex such that each of the k vertices has degree $d \geqslant 4$. Then $\gamma_g(T) = \lfloor n/2 + (n-2)/2(d-1) \rfloor$ as A can always orient edges from leaves to the central vertices.

For $d=3$ consider a tree of three levels. The first level has only one vertex of degree $k+2$, the next level has k vertices of degree 3 and two leaves, and the third level contains $2k$ leaves. It is easy to check, that this tree has game domination number $2n/3$, showing that the second part of the theorem is also sharp.

Let us spell out one of the inequalities in the previous proof, which gives a slightly stronger version of the theorem.

Corollary 3.3. *If T is a tree on n vertices with $k \geqslant 1$ vertices of degree at least 3, then*

$$\gamma_g(T) \leqslant \left\lfloor \frac{n+k}{2} \right\rfloor .$$

We summarize our results for trees in the following inequalities. Note, that the general upper bound could have been improved a lot for special trees.

Corollary 3.4. *For any tree T we have*

$$\left\lceil \frac{1}{2}n \right\rceil \leqslant \gamma_g(T) \leqslant \left\lfloor \frac{2}{3}n \right\rfloor.$$

Corollary 3.5. *For any connected G we have*

$$\gamma_g(G) \leqslant \left\lfloor \frac{2}{3}n \right\rfloor.$$

4. Inequalities

First, we shall give a lower bound for the game domination number of a graph in terms of its maximal degree. This corresponds to the easiest basic inequality on the domination number: $\gamma(G) \geqslant n/(\Delta+1)$.

During our game A orients half of the edges and he might succeed in decreasing the out-degree of each vertex to about $\Delta/2$. This prompts us to make the following conjecture.

Conjecture 4.1. *For any graph G with n vertices and maximal degree Δ, we have $\gamma_g(G) \geqslant 2n/(1+o(1))\Delta$.*

We have not been able to prove this conjecture, but the following somewhat weaker result is an improvement on the trivial lower bound $\gamma_g(G) \geqslant \gamma(G) \geqslant n/(\Delta+1)$.

Theorem 4.2. *If G is a graph with n vertices and maximal degree Δ, then*

$$\gamma_g(G) \geqslant \left\lceil \frac{4n}{3\Delta+7} \right\rceil.$$

Proof. The goal of A is to ensure that the out-degree of any vertex is at most $(3\Delta+3)/4$. We can add edges to the graph, until it becomes a $2k$-regular multigraph G', with $2k=\Delta$ or $\Delta+1$ depending on the parity of Δ.

As shown by Tarsi [13], this graph G' has a k-system, i.e. n edge-disjoint k-stars, one centered at each vertex of the graph. Corresponding to these, there are n edge-disjoint stars in G, each with at most k edges, centered at different vertices, and at any vertex v, at most k of the incident edges do not belong to the star centered at v.

The strategy of A is to orient an edge of the same star, in which D made his previous move, into the central vertex. Hence, each vertex will have out-degree at most $\lfloor (3\Delta+3)/4 \rfloor$, and dominate at most $\lfloor (3\Delta+7)/4 \rfloor$ of the n vertices. □

Note, that when we provided strategies for D to obtain upper bounds for the game domination number we usually did that by finding a small set S of vertices dominating every other vertex at least twice, and thus ensuring that at least one of those edges can be oriented by D out of S, making S a dominating set of the resulting digraph.

The concept of multiple domination was introduced by Fink and Jacobson [7]. They call a set S *k-dominating* if every vertex of $V-S$ is adjacent to at least k vertices in S. The *k-domination number*, $\gamma_k(G)$, is the minimal cardinality of a k-dominating set. Our argument above shows that $\gamma_g(G) \leqslant \gamma_2(G)$. This gives the following immediate bounds, by some of the results on two-domination numbers in [7,12].

Theorem 4.3. *For any graph G*: $\gamma_g(G) \leqslant \gamma_2(G) \leqslant \beta_2(G) \leqslant 2\beta_0(G)$.

In the inequality above, $\beta_0(G)$ denotes the independence number of G, and $\beta_2(G)$ is the two-independence number, i.e. the maximal cardinality of a set I of the vertices such that the graph spanned by I has maximal degree at most 1. The complete graph shows that the inequality is sharp: $\gamma_g(K_n) = 2\beta_0(K_n) = 2$.

Theorem 4.4. *If the minimal degree* $\delta(G) \geqslant 3$, *then* $\gamma_g(G) \leqslant \gamma_2(G) \leqslant n/2$.

For $G = tK_4$ we have $\gamma_g(G) = 2t$, $v(G) = 4t$, so both inequalities are sharp in Theorem 4.4. We have seen the game domination number of trees to fall between $n/2$ and $2n/3$. Clearly, the proof implies that this upper bound holds for any connected graph, as player D can concentrate his attention on a spanning tree of the graph (if player A moves outside of the spanning tree, D continues to orient tree edges according to his strategy). The following theorem improves the upper bound for graphs with minimal degree at least two.

Theorem 4.5. *If a graph G has minimal degree at least* 2, *then* $\gamma_g(G) \leqslant \lfloor n/2 \rfloor$.

Proof. Our goal is to find a large one-factor in the graph and use those edges to dominate the pairs of vertices by one of them regardless of their orientation. If G has a complete matching, this gives us a dominating set.

Suppose first that $n = 2k+1$ odd, and there is a matching of size k, containing the edges $(v_1, u_1), \ldots, (v_k, u_k)$, leaving only one more vertex for v to dominate. From the minimal degree condition, v is connected to a vertex of the matching, say v_1. If u_1 is also connected to v, then the proof is done, as the resulting triangle vv_1u_1 can easily be dominated by one vertex if D starts the game. Otherwise u_1 is connected to another vertex of the matching, say v_2. Following this algorithm we build an alternating path $vv_1u_1v_2u_2 \ldots u_i$ until u_i is connected to a previous vertex on this alternating path. If this vertex is v or u_j, then we end up with an odd cycle and a matching (we might need to change the matching edges along the alternating path up to the cycle) and finish with a dominating set of size at most $n/2$ as before in Example 2.7. Finally, if u_i is attached to a vertex v_j on the path, then we have an even cycle with an odd path attached to it, and some independent edges of the original matching, and we can easily get the desired dominating set by Example 2.8.

We shall call a component odd (even), if its order is odd (even). Now we suppose that a maximal matching of G covers all but $t \geqslant 2$ vertices. By the extended version of

Tutte's theorem there is a set S of s vertices such that after deleting S from the graph we shall get $s+t$ odd components. Choose S to be maximal among all such sets. Note, that S might be empty, if G was not connected and had exactly t odd components. We shall use this S to dominate the graph.

First note, that there exists a complete matching in every even component of $G-S$ (as there is a matching covering all but t vertices of the graph). We claim that if U is an odd component of $G-S$ and $u \in U$, then $U-\{u\}$ also has a complete matching. Otherwise it has a cut set K with k vertices leaving at least $k+2$ odd components in $U-\{u\}$, since U had an odd number of vertices. But then $S \cup \{u\} \cup K$ would be a cut-set of order $s+k+1$ giving $s+t+k+1$ odd components, contradicting the maximality of S.

The strategy of D is simple: he ensures that the $s+t$ odd components of $G-S$ will be dominated 'efficiently', i.e. by less than half of their vertices plus the vertices of S. To show how this can be done we need to distinguish three types of odd components in $G-S$ depending on the numbers of edges connecting them to S.

First, there may be some isolated components, which must be odd components of G with all but one vertex covered by the matching. It is easy to see that these can be dominated by half of their vertices if D manages to start the game in at least half of them. He starts in one, and starts another isolated component every time A does so, achieving his goal.

Second, there are odd components attached to S with at least two edges, but these can be dominated from S by orienting one of those edges out of S, and using the complete matching on the remaining part of it.

Third, there are odd components attached to S by only one edge between S and a vertex v in the odd component. In this case either D is able to orient the bridge from S toward v or if A has done this, he starts the game in the component and succeeds exactly like above when we had an almost complete matching missing only vertex v. Note, that in that argument we did not use the fact that v has degree at least two (that might not hold here), this was only needed for the vertices of the matching.

Even if we have to choose every vertex of S into the dominating set, it cannot be larger than

$$s+\frac{n-s-s-t}{2}=\frac{n-t}{2}<\left\lfloor\frac{n}{2}\right\rfloor,$$

completing the proof. □

This theorem is sharp as the examples of cycles with n vertices show. The upper bound for the game domination number can be further strengthened if the minimal degree is larger using a probabilistic argument.

Theorem 4.6. *For every graph $G=(V,E)$ with n vertices and minimum degree $\delta \geqslant 2$ and for every real number p between 0 and 1, $\gamma_g(G) \leqslant np+2n(1-p)^{\delta}+1+n\delta p(1-p)^{\delta}$. Therefore, $\gamma_g(G) \leqslant (1+o(1))n\ln(\delta+1)/\delta+1$, where the $o(1)$-term tends to zero as δ tends to infinity, and the above the estimate is tight, up to the $o(1)$ error term.*

Proof. By Theorem 4.3 it suffices to prove that there is a set S of at most $np+2n(1-p)^{\delta+1}+n\delta p(1-p)^{\delta}$ vertices of G, such that each vertex not in S has at least two neighbors in S. To prove the existence of such an S let X be a random set of vertices of G obtained by choosing each vertex $v \in V$, randomly and independently, to be a member of X with probability p. Let us fix arbitrarily some set $N(v)$ of precisely δ neighbors of each vertex $v \in V$, let Y denote the set of all vertices v such that neither v nor any member of $N(v)$ lies in X, and Z will denote the set of vertices u such that precisely one member of $N(u)$ is in X. The expected cardinalities of X, Y and Z are, respectively, np, $n(1-p)^{\delta+1}$ and $n\delta p(1-p)^{\delta}$. Moreover, by adding to X two arbitrarily chosen neighbors of u for each $u \in Z$, we obtain a set S of cardinality $|X|+2|Y|+|Z|$ such that each vertex not in S has at least 2 neighbors in S. By linearity of expectation the expected cardinality of S is $np+2n(1-p)^{\delta+1}+n\delta p(1-p)^{\delta}$ and hence there is such a set of cardinality at most this quantity. For large δ we can choose $p=(\ln\delta+\ln\ln\delta)/\delta$ and check that for this choice of p the resulting set S is of cardinality at most $n\ln(\delta+1)/(\delta+1)+\mathrm{O}(n\ln\ln\delta/\delta)$, as needed.

The tightness of the estimate follows easily from the fact that the game domination number is always at least as large as the domination number of the graph and the well-known fact that there are undirected graphs with n vertices and domination number $(1+\mathrm{o}(1))n\ln(\delta+1)/(\delta+1)$ where the o(1)-term tends to zero as δ tends to infinity (see for example the discussion following Theorem 2.2 on p. 7 in [2]). □

In 1956 Nordhaus and Gaddum [11] established sharp bounds on the sum and product of the chromatic numbers of a graph and its complement. Similar results have been found for several parameters, including the following due to Jaeger and Payan [10].

Theorem 4.7. *If G is a graph of order n, then $\gamma(G)+\gamma(\bar{G}) \leqslant n+1$ and this bound is sharp.*

We establish a sharp Nordhaus–Gaddum-type inequality for the game domination number of a graph and its complement.

Theorem 4.8. *For a graph G with n vertices, $\gamma_{\mathrm{g}}(G)+\gamma_{\mathrm{g}}(\bar{G}) \leqslant n+2$. Furthermore, the bound is sharp.*

Proof. If the minimum degrees of G and $\bar{G}$ are at least two then by Theorem 4.5, $\gamma_{\mathrm{g}}(G)+\gamma_{\mathrm{g}}(\bar{G}) \leqslant n$. Hence, we may assume that $\delta(\bar{G}) \leqslant 1$. Then $\delta(G) \geqslant 2$, otherwise we have a vertex of degree at least $n-2$ in the complement as well, and the dominator D can use these two vertices to dominate almost half of both the graph and its complement, which results in dominating sets whose sum of sizes is at most $n+2$ even if we have chosen every remaining vertex into the corresponding dominating set.

Suppose now that $\delta(\bar{G}) \leqslant 1$ and $\delta(G) \geqslant 2$. Thus there is a vertex v in G with degree at least $n-2$. If another vertex u of G has degree at least $n-2$, then $\gamma_{\mathrm{g}}(G) \leqslant 3$ as using u, v and possibly one more vertex as dominating set, D can dominate G. On the other hand, if there exists an edge in $\bar{G}$, then $\gamma_g(\bar{G}) \leqslant n-1$, otherwise we need n vertices to dominate $\bar{G}$, but two vertices suffice to dominate G, providing the desired bound in either case.

We remain with the case when $d(v) \geqslant n-2$ and $d(u) \leqslant n-3$ for every $u \neq v$. Then by Theorem 4.5 player D can dominate in $\bar{G}$ with $\gamma_g(\bar{G}) \leqslant \lceil (n-1)/2 \rceil + 1$ vertices by adding v to the dominating set. Also, using the star of G centered at v, player D can easily dominate G by $\lfloor (n-2)/2 \rfloor + 2$ vertices. Hence $\gamma_g(G) + \gamma_g(\bar{G}) \leqslant n+2$.

The complete graph K_n shows that this bound is sharp: $\gamma_g(K_n) = 2$ and $\gamma_g(\bar{K}_n) = n$. □

We believe that the inequality in Theorem 4.8 can be strengthened for connected graphs.

Conjecture 4.9. *If both G and $\bar{G}$ are connected graphs with n vertices, then* $\gamma_g(G) + \gamma_g(\bar{G}) \leqslant \frac{2}{3}n + 3$.

If true, this inequality is sharp, as shown by a tree of order n with game domination number $\frac{2}{3}n$ (see Section 3).

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