



## Note

## Concise proofs for adjacent vertex-distinguishing total colorings

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## ABSTRACT

Let  $G = (V, E)$  be a graph and  $f : (V \cup E) \rightarrow [k]$  be a proper total  $k$ -coloring of  $G$ . We say that  $f$  is an adjacent vertex-distinguishing total coloring if for any two adjacent vertices, the set of colors appearing on the vertex and incident edges are different. We call the smallest  $k$  for which such a coloring of  $G$  exists the adjacent vertex-distinguishing total chromatic number, and denote it by  $\chi_{at}(G)$ . Here we provide short proofs for an upper bound on the adjacent vertex-distinguishing total chromatic number of graphs of maximum degree three, and the exact values of  $\chi_{at}(G)$  when  $G$  is a complete graph or a cycle.

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## 1. Introduction

Let  $G = (V, E)$  be a simple graph. A proper total  $k$ -coloring of  $G$  is a mapping  $f : (V \cup E) \rightarrow [k]$  such that no two adjacent vertices receive the same color, no two incident edges receive the same color, and no vertex and incident edge receive the same color. Given such a coloring  $f$ , for any vertex  $v \in V$  let  $C(v) = \{f(v)\} \cup \{f(uv) : uv \in E(G)\}$ . For every pair of adjacent vertices  $uv \in E$ , if  $C(u) \neq C(v)$  then we say that  $f$  is an adjacent vertex distinguishing total coloring (AVDTC). We call the smallest  $k$  for which such a coloring of  $G$  exists the adjacent vertex-distinguishing total chromatic number, denoted by  $\chi_{at}(G)$ .

This coloring is related to vertex-distinguishing proper edge colorings of graphs, first examined by Burr and Schelp [1] and further discussed by many others, including Bazgan, et al. [2] and Balister, et al. [3]. This type of coloring was later extended to require only adjacent vertices to be distinguished by Zhang et al. [4], which was in turn extended to proper total colorings [5].

Zhang et al. in [5] determined  $\chi_{at}(G)$  for many basic families of graphs, including cycles, complete graphs, complete bipartite graphs, and trees. Additionally, the following bound on  $\chi_{at}(G)$  in terms of the maximum degree of a graph  $\Delta(G)$  was conjectured.

**Conjecture 1** ([5]). *Let  $G$  be a simple graph. Then  $\chi_{at}(G) \leq \Delta(G) + 3$ .*

Wang in [6] was able to prove this bound for the case  $\Delta(G) = 3$  by employing subdivision graphs and intricate case analysis. In this note we present a more elementary proof of this bound as well as more concise proofs for determining  $\chi_{at}(G)$  when  $G$  is a complete graph or a cycle.

## 2. Complete graphs and cycles

Zhang et al. in [5] determined  $\chi_{at}(G)$  for complete graphs and cycles. Here we give concise proofs for these results. The first is due to A. Gyárfás.

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**Proposition 2.**  $\chi_{at}(K_n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n+2 & \text{if } n \text{ is odd} \end{cases}$  for  $n \geq 2$ .

**Proof.** An optimal AVDTC for complete graphs can be easily obtained from the standard near-factorization of  $K_{2m+1}$ , defined as follows: on vertex set  $[2m+1]$ , for every  $i \in [2m+1]$  let  $F_i$  be the set of edges  $(i-x, i+x)$  for  $x = 1, 2, \dots, m$  according to modulo  $2m+1$  arithmetic (here we denote 0 by  $2m+1$  for notational convenience). If vertices are colored with their vertex labels and edges in  $F_i$  are colored with  $i$ , then we have a proper total coloring of  $K_{2m+1}$  with  $2m+1$  colors. If one vertex is removed, an AVDTC of  $K_{2m}$  is obtained and if two vertices are removed, an AVDTC of  $K_{2m-1}$  is obtained.

The first statement holds for any near-factorization, and the second follows from the fact that in the standard near-factorization above, there are no four-cycles in the union of two color classes. Suppose there exists a four-cycle with edges colored  $a$  and  $b$ . This means the set of labels of vertices involved can be expressed as both  $\{a-i, a+i, a-j, a+j\}$  and  $\{b-k, b+k, b-\ell, b+\ell\}$  for some  $i, j, k, \ell \in \{1, 2, \dots, m\}$ . Since these two sets must be equal modulo  $2m+1$ , their sums must also be equal. Therefore  $4a \equiv 4b \pmod{2m+1}$ ; but since  $(4, 2m+1) = 1$ , this implies  $a = b$ , a contradiction.

It is clear that  $2m+1$  colors are needed for  $K_{2m}$ . To show that  $2m+1$  colors are necessary for  $K_{2m-1}$ , suppose an AVDTC with  $2m$  colors is possible. If a color is absent from the color set of one vertex, it must be present at every other vertex because all color sets must be distinct. Additionally, every vertex must be colored distinctly, and so by the pigeonhole principle there exists a color  $i$  that appears only on one vertex. It follows that every remaining vertex is incident to an edge colored by  $i$ . However, an odd number of vertices remain and obviously no perfect matching exists between them, a contradiction.  $\square$

**Proposition 3.**  $\chi_{at}(C_n) = 4$  for  $n \geq 4$ .

**Proof.** If  $n$  is even, alternately color the vertices of the cycle 1 and 2, and alternately color the edges 3 and 4. This is an AVDTC because it is clearly proper and the color sets of adjacent vertices are distinguished by their vertex color. If  $n$  is odd, again alternately color the vertices and edges of the cycle as in the even case except for one vertex, say  $v_1$ , and its incident edges. Thus we assume  $v_2$  is colored 1,  $v_n$  is colored 2, and both have an incident edge colored 3. Then color  $v_1$  by 4,  $v_1v_2$  by 2, and  $v_nv_1$  by 1.  $\square$

### 3. Graphs with $\Delta(G) = 3$

Wang in [6] provides a case analysis detailing an AVDTC in six colors for graphs with maximum degree three. Here we give a much shorter proof using a different approach.

**Definition 4.** Let  $G = (V, E)$  be a graph and  $f : (V \cup E) \rightarrow [k]$  a proper total coloring. Define  $C_E \subseteq [k]$  to denote the set of colors that appear on the edges of  $G$  and  $C_V \subseteq [k]$  to denote the set of colors that appear on the vertices of  $G$ . We say that  $f$  is an almost disjoint total coloring if  $|C_E \cap C_V| \leq 1$ .

**Lemma 5.** An almost disjoint proper total coloring of a graph is an AVDTC.

**Proof.** Observe that two adjacent vertices have identical color sets only if the color appearing on each vertex is used to color an incident edge of the other. However, this cannot happen since the colors used to color two adjacent vertices cannot both be used to also color edges.  $\square$

**Theorem 6.** If  $G$  is a graph with  $\Delta(G) = 3$  and  $G \neq K_4$ , then there exists an almost disjoint total 6-coloring of  $G$ .

**Proof.** We claim there exists a partial almost-disjoint 6-coloring  $f$  of  $G$  with the following properties:

- (1) The vertices of  $G$  are colored 1, 2, and 3.
- (2) The edges incident to the 3 color class are colored 4, 5, and 6.
- (3) The edges incident to the 1 and 2 color classes are colored 3, 4, 5, 6, or remain uncolored.

The graph  $G$  has a proper vertex coloring with colors 1, 2, and 3 by Brooks' Theorem. Consider the bipartite graph formed by all edges with one endpoint in color class 3, and the other endpoint in color classes 1 or 2; by König's Theorem, these edges can be 3-colored with colors 4, 5, and 6. Therefore there exist partial 6-colorings of  $G$  that satisfy Properties (1) and (2). Consider the collection  $F$  of colorings of  $G$  with these two properties. We claim there exists a coloring in  $F$  that satisfies Property (3) with no uncolored edges.

Suppose this is not the case; choose a coloring  $f \in F$  with the fewest number of uncolored edges. Given such an  $f$ , we can create another partial 6-coloring  $f' \in F$  such that the number of uncolored edges is one less, thus deriving a contradiction.

Consider an edge  $uv$  that is left uncolored by  $f$ . By Property (3),  $uv$  must be incident to a 1 vertex and a 2 vertex. If  $C(u)$  and  $C(v)$  have a common color then exactly three of the colors 3, 4, 5, and 6 are found at  $u$  or  $v$ , and so we may choose the fourth color with which to color  $uv$  in  $f'$ .

Suppose  $C(u)$  and  $C(v)$  have no common color. Without loss of generality, suppose  $u$  has a 4 edge and a 3 edge, call it  $uw$ , and  $v$  has a 5 edge and a 6 edge. Now consider  $C(w)$ . If  $C(w) \neq C(v)$ , in  $f'$  we can color  $uv$  3 and recolor  $uw$  with either 5 or 6, whichever is not present at  $w$ .

Suppose  $C(w) = C(v)$ . Consider the longest path  $P$  consisting of edges alternately colored 4 and 5 originating from  $u$  and switch the colors of each edge along it. If  $P$  does not terminate at  $v$ , we may now color  $uv$  4 in  $f'$  since the color 4 no longer appears at  $u$ . If  $P$  does terminate at  $v$ , it obviously cannot terminate at  $w$  and so in  $f'$  we may color  $uv$  3 and recolor  $uw$  4. This exhausts all possibilities, and therefore there exists an almost disjoint total 6-coloring of  $G$ .  $\square$

**Corollary 7.** Let  $G$  be a graph with  $\Delta(G) = 3$ . Then  $\chi_{at}(G) \leq 6$ .

**Proof.** By Proposition 2,  $\chi_{at}(K_4) = 5$ . If  $G \neq K_4$ , by Theorem 6 there exists an almost disjoint total 6-coloring of  $G$ , which is an AVDTC by Lemma 5.  $\square$

These results above confirm Conjecture 1 of Zhang et al. [5] for  $\Delta(G) = 3$ . Although complete graphs of odd order show the conjectured bound is sharp for even maximum degree, many maximum degree three graphs, including the  $K_4$ ,  $K_{3,3}$ , and Petersen graphs, have an AVDTC with only 5 colors. Therefore, we propose the following problem.

**Problem 8.** For a graph  $G$  with  $\Delta(G) = 3$ , is the bound  $\chi_{at}(G) \leq 6$  sharp?

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