# Extensions of Iterative Congruences on Free Iterative Algebras 

Francesco Parisi-Presicce<br>Department of Mathematics, University of Southern California, Los Angeles, California 90007

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#### Abstract

This paper investigates special congruences on $R_{\Sigma}$ (the $\Sigma$-algebra of partial regular trees) which extend congruences on $\bar{R}_{\Sigma}$ (the $\Sigma$-algebra of total regular trees). It is proved that if a congruence on $\bar{R}_{\Sigma}$ induces an iterative factor algebra, then so does its extension on $R_{\Sigma}$. This result is used to show that if an iterative algebra admits a faithful regular extension, then that extension is again iterative.


## Introduction

The least fixed point approach [1-3] and the unique fixed point approach [8] to the semantics of programming languages have been compared by Tiuryn [12] using the framework of universal algebras. In [12], Tiuryn proves the existence of extensions of algebras with the unique fixed point property (iterative algebras) to ordered algebras with the least fixed point property (regular algebras), extensions preserving the fixed-point solutions. These regular extensions need not in general be iterative [13].

We will prove that if the extension is faithful (i.e., if no two elements of the original carrier are identified), then the extension is both iterative and regular. Then we will carry the idea of the proof one step further to prove the following:
let $K$ be a congruence on $\bar{R}_{\Sigma}$ (the $\Sigma$-algebra of total regular trees) inducing an iterative factor algebra and let $K^{\prime}$ be an "extension" (in terms made precise later) of $K$ in $R_{\Sigma}$ (the $\Sigma$-algebra of partial regular trees). Then $R_{\Sigma} / K^{\prime}$ is iterative.

The notion of extension of a congruence $K$ on $R_{\Sigma}$ is a natural one. Two (possibly partial) regular trees $t_{1}$ and $t_{2}$ are equivalent if they can be decomposed into two components, one containing the "bottom" element $\perp$ (representing the undefined parts), identical for both $t_{1}$ and $t_{2}$, and the other one consisting of an $n$-tuple of total regular trees, the $n$-tuple for $t_{1} K$-congruent to the $n$-tuple for $t_{2}$. Roughly speaking, these extensions reflect the idea that where the undefined parts occur is also a valuable piece of information.

The paper is divided into six sections. Section 1 establishes the basic notation and some preliminary lemmas on sets of words. Section 2 contains the definitions of iterative algebras, of regular algebras, and of $\Sigma$-trees. The main result (Theorem 3.4) is stated in Section 3 and a sketch of its proof is given there. Section 4 contains the details of the proof of Theorem 3.4. In Section 5, we use the main result to answer a question of Tiuryn's on faithful extensions of iterative algebras.

## 1. Preliminaries

Notation 1.0. $\omega=\{1,2, \ldots\}$ is the set of positive integers and for every $n>0$, $[n]=\{1, \ldots, n\}$.

The domain of a function $f: A \rightarrow B$ is denoted by $\operatorname{dom} f$. We define $\operatorname{ker} f=\left\{\left(a_{1}, a_{2}\right)\right.$ : $a_{1}, a_{2} \in A$ and $\left.f\left(a_{1}\right)=f\left(a_{2}\right)\right\}$. Every $f: A \rightarrow B$ determines, for every $n>0$, a map $\tilde{f}: A^{n} \rightarrow B^{n}$ given by $f(a)=\left(f\left(a_{1}\right), \ldots, f\left(a_{n}\right)\right.$, for every $a=\left(a_{1}, \ldots, a_{n}\right)$ in $A^{n}$. We will use $f$ to denote both $\hat{f}$ and $f$.

Let $A$ be a set, $n>0$ and $a \in A^{n}$. For $i \in[n]$, the $i$ th component of $a$ will be denoted by either $a_{i}$ or $a(i)$. The map that associates with each vector of $A^{n}$ its $i$ th component is denoted by $e_{i}^{n}$.

If $R$ is an equivalence relation on $X, a \in X$ and $A \subseteq X$, then $|a|_{R}=\{x \in X$ : $(a, x) \in R\}$ and $A / R=\left\{|a|_{R}: a \in A\right\}$.

We will often omit the subscript $R$ when this will cause no confusion.
DEfinition 1.1. Let $Z$ be an alphabet, i.e., a nonempty set, finite or infinite, of symbols. Denote by $Z^{+}$the set of nonempty words on $Z$ and by $\lambda$ the empty word. Let $Z^{*}=Z^{+} \cup\{\lambda\}$. Given two words $x$ and $y$ in $Z^{+}$, the catenation product of $x$ and $y$ is denoted by $x y$.

Let $A$ and $B$ be subsets of $Z^{*}: A^{\mathrm{c}}$ denotes the complement of $A$.

$$
\begin{aligned}
A B & =\left\{w \in Z^{*}: w=x y \text { for some } x \in A, y \in B\right\} \\
A^{-1} B & =\left\{w \in Z^{*}: a w \in B \text { for some } a \in A\right\} \\
A^{-m} B & =\left(A^{m}\right)^{-1} B \text { for every positive integer } m, \\
B A^{-1} & =\left\{w \in Z^{*}: w a \in B \text { for some } a \in A\right\} .
\end{aligned}
$$

Definition 1.2. Let $x, y \in Z^{*}$. We say that $x$ is an (initial) subword of $y$ (in symbols $x \leqslant y$ ) if there exists a word $z \in Z^{*}$ such that $x z=y ; x$ is a proper subword of $y$ if in addition $z \neq \lambda$, i.e., $x \leqslant y$ and $x \neq y$. In this case we write $x<y$. $\downarrow w=\left\{x \in Z^{*}: x<w\right\}=$ the set of proper subwords of $w . \nexists A=\bigcup\{\downarrow w: w \in A\}$, $\downarrow w=\downarrow w \cup\{w\}$ and $\downarrow A=\downarrow A \cup A$.

Notice that $\downarrow w$ is totally ordered.

Lemma 1.3. For any subsets $A, B$ of $Z^{*}$
(1) If $A \subseteq B$, then $\downarrow A \subseteq \downarrow B$,
(2) $\downarrow(\downarrow A)=\downarrow A$,
(3) $\downarrow(A \cup B)=(\downarrow A) \cup(\downarrow B)$,
(4) $\downarrow(A \cap B) \subseteq(\downarrow A) \cap(\downarrow B)$,

$$
\begin{equation*}
\downarrow(A B)=A(\downarrow B) \cup(\downarrow A) . \tag{5}
\end{equation*}
$$

Definition 1.4. Let $A$ be a set and $R$ a binary relation on $A$. Two elements $a, b \in A$ are said to be incomparable with respect to $R$ if $(a, b) \notin R$ and $(b, a) \notin R$. A subset $B \subseteq A$ is totally unordered (relative to $R$ ) if every two distinct elements of $B$ are incomparable. If $B \subseteq A, B^{*}$ will denote the set of all elements of $A$ which are incomparable with every element of $B$. If $B \subseteq Z^{*}$ and $R=\leqslant$, it is easy to see that $B^{*}=\left(B Z^{*}\right)^{c} \cap(\downarrow B)^{c}$ and that $\lambda \notin B^{*}$.

Lemma 1.5. Let $A \subseteq Z^{*}$. The following are equivalent:
(i) $A$ is totally unordered,
(ii) $A \subseteq\left(A Z^{+}\right)^{\mathrm{c}}$,
(iii) $(\nmid A) \subseteq A^{\mathrm{c}}$.

It follows directly from Lemma 1.5 that if $B$ is a subset of a totally unordered set, then $B$ is totally unordered.

Lemma 1.6. (i) If $A$ and $B$ are totally unordered, then $A B$ is totally unordered.
(ii) If $A$ is totally unordered and $a \in A^{n}$ for some $n>0$, then there exist unique elements $a_{1}, \ldots, a_{n} \in A$ such that $a=a_{1} \cdots a_{n}$.

Lemma 1.7. Let $A, B \subseteq Z^{*}, n>0$ and $m \neq n$. Then
(i) $A^{\#} B \subseteq A^{\#}$,
(ii) $A^{n} A^{\#} \cap A^{m} A^{\#}=\varnothing$.

Proof. (i) Let $a \in A^{*}, b \in B$, and $a^{\prime} \in A$. If $a b \leqslant a^{\prime}$, then $a \leqslant a^{\prime}$ which is impossible by Definition 1.4. If $a^{\prime} \leqslant a b$, then either $a^{\prime} \leqslant a$ or $a \leqslant a^{\prime}$, and both are impossible by 1.4 again. Hence $a b$ and $a^{\prime}$ are incomparable.

To prove (ii), assume without loss of generality that $m>n$. Suppose, for the sake of reaching a contradiction, that $w \in A^{n} A^{\#} \cap A^{m} A^{*}$. Then $A^{-n} w \in A^{\#} \cap A^{m-n} A^{*}$ contradicting $A^{*} \subseteq\left(A Z^{*}\right)^{c}$.

Let $Z$ be an alphabet and define $\operatorname{REC}(Z)$ to be the smallest class of subsets of $Z^{*}$ containing the empty set $\varnothing, Z$, all finite subsets of $Z^{*}$, and closed under finite union, catenation product and Kleene closure.

Definition 1.8. Let $Z$ be an alphabet and $L \subseteq Z^{*}$. Define $\operatorname{MIN}(L)$ to be the set of minimal words of $L$, that is,

$$
\operatorname{MIN}(L)=L \cap\left(L Z^{+}\right)^{\mathrm{c}}
$$

It is known [7] that if $L \in \operatorname{REC}(Z)$, then $\operatorname{MIN}(L) \in \operatorname{REC}(Z)$.
Remarks 1.9. (1) $\operatorname{MIN}(L)=\{\lambda\}$ if and only if $\lambda \in L$.
(2) If $L^{\prime} \subseteq L$, then $\operatorname{MIN}(L) \cap L^{\prime} \subseteq \operatorname{MIN}\left(L^{\prime}\right)$.
(3) $\operatorname{MIN}\left(L Z^{*}\right)=\operatorname{MIN}(L)$.
(4) If $\lambda \notin L$ and $x \in L$, then $\downarrow x \cap \operatorname{MIN}(L) \neq \varnothing$.
(5) $\operatorname{MIN}(L)$ is totally unordered.
(6) $\operatorname{MIN}(\operatorname{MIN}(L))=\operatorname{MIN}(L)$.

Definition 1.10. Let $R$ be any binary relation defined on a set $A$. A finite sequence $a_{1} \cdots a_{n}$ of elements of $A$ is called an $R$-word if for every $i=1, \ldots, n-1$ $\left(a_{i}, a_{i+1}\right) \in R$. Any sequence of length 1 is considered an $R$-word.

The proof of the following well-known result can be found in [6].
Theorem 1.11. If $R$ is a binary relation defined on a finite set $A$ containing a and $b$, then the set of all $R$-words beginning with $a$ and ending with $b$ is regular.

Corollary 1.12. If $R$ is a binary relation defined on a finite set $A$ and $A^{\prime} \subseteq A$, then the set of all $R$-words beginning with a word in $A^{\prime}$ is regular.

Definition 1.13. Let $\{A(i j): 1 \leqslant i, j \leqslant n\}$ be a (doubly indexed) family of finite sets of words over some alphabet. For every $r>1$ and every $w \in[n]^{r}$, define $A(w)=$ $A(w(1) w(2)) \cdots A(w(r-1) w(r))$, where $w(k)$ is the $k$ th component of $w$. For every $i=1, \ldots, n$ and every $r \geqslant 1$, define $A(i, r)=\bigcup\left\{A(i w): w \in[n]^{r}\right\}$.

Lemma 1.14. For every $i \in[n]$, both $\bigcup\{A(i, r): r \geqslant 1\}$ and $\bigcup\left\{\bigcup\left\{A(i w) A(w(r))^{\#}\right.\right.$ : $\left.\left.w \in[n]^{r}\right\}: r \geqslant 1\right\}$ are regular sets.

## 2. Trees and Algebras

In this section we summarize the main definitions and results concerning trees, iterative and regular algebras in order to establish our notation. For more details, we refer the reader to $[1,10,11]$.

DEFINITION 2.1. A signature (or ranked alphabet) $\Sigma$ is a countable collection $\left\{\Sigma_{n}: n \geqslant 0\right\}$ of mutually disjoint sets. We will use the same symbol $\Sigma$ to denote both $\left\{\Sigma_{n}: n \geqslant 0\right\}$ and $\bigcup\left\{\Sigma_{n}: n \geqslant 0\right\}$. If $\Sigma$ is a signature and $Y$ a set, $\Sigma(Y)$ denotes the
signature obtained by "adding" the elements of $Y$ to the "constant symbols" of $\Sigma$. Thus $\Sigma(Y)_{0}=\Sigma_{0} \cup Y$ and $\Sigma(Y)_{n}=\Sigma_{n}$ for $n>0$.

Definition 2.2. Let $\Sigma$ be a signature. A $\Sigma$-algebra $A$ is a set (the carrier) equipped with a collection $\left\{\sigma_{A}: \sigma \in \Sigma\right\}$ of functions such that if $\sigma \in \Sigma_{n}$, then $\sigma_{A}: A^{n} \rightarrow A$. Let $A$ and $B$ be $\Sigma$-algebras. A $\Sigma$-homomorphism $f$ from $A$ to $B$ is a function $f: A \rightarrow B$ such that for every $n \geqslant 0$, every $\sigma \in \Sigma_{n}$ and all $\left(a_{1}, \ldots, a_{n}\right) \in A^{n}$, $f\left(\sigma_{A}\left(a_{1}, \ldots, a_{n}\right)\right)=\sigma_{B}\left(f\left(a_{1}\right), \ldots, f\left(a_{n}\right)\right)$.

A congruence relation on $A$ is an equivalence relation $K$ on $A$ such that for every $n>0$ and every $\sigma \in \Sigma_{n}$, if $\left(a_{i}, b_{i}\right) \in K$ for $i=1, \ldots, n$, then $\left(\sigma_{A}\left(a_{1}, \ldots, a_{n}\right)\right.$, $\left.\sigma_{A}\left(b_{1}, \ldots, b_{n}\right)\right) \in K$.

Definition 2.3. Let $\Sigma$ be a signature. A $\Sigma$-tree is a nonempty partial function $t: \omega^{*} \rightarrow \bigcup\left\{\Sigma_{n}: n \geqslant 0\right\}$ satisfying the following conditions:
(1) $\downarrow \operatorname{dom} t \subseteq \operatorname{dom} t$ (in particular $\lambda \in \operatorname{dom} t$ ),
(2) for all $u \in \omega^{*}$ and $i \in \omega, u i \in \operatorname{dom} t$ only if there exist $n>0$ and $\sigma \in \Sigma_{n}$ such that $i \leqslant n$ and $t(u)=\sigma$.

If we replace "only if" with "if and only if" in (2), we obtain the definition of total $\Sigma$-tree. A tree $t$ is finite if dom $t$ is finite.

We denote with $C T_{\Sigma}$ (resp. $T_{\Sigma}$ ) the set of all (resp. finite total) $\Sigma$-trees.
Definition 2.4. Let $t$ be a $\Sigma$-tree and let $w \in \operatorname{dom} t$. We denote by $t \upharpoonright w$ the subtree of $t$ determined by the "path" $w$. More precisely, $\operatorname{dom}(t \upharpoonright w)=\left\{u \in \omega^{*}\right.$ : $w u \in \operatorname{dom} t\}=w^{-1}(\operatorname{dom} t)$ and for all $u \in \operatorname{dom}(t \upharpoonright w),(t \upharpoonright w)(u)=t(w u)$.

It follows easily from the definition that if $t$ is a $\Sigma$-tree and $x y \in \operatorname{dom} t$, then $(t \upharpoonright x) \upharpoonright y=t \upharpoonright x y$. A tree is said to be regular or of finite index if $\{t \upharpoonright u: u \in \operatorname{dom} t\}$ is finite. Denote by $\bar{R}_{\Sigma}$ the set of all regular $\Sigma$-trees.

If $Y$ is any set, $C T_{\Sigma(Y)}$ is a $\Sigma(Y)$-algebra and, therefore a $\Sigma$-algebra. Denote the $\Sigma$ algebra by $C T_{\Sigma}(Y)$. We use $\bar{R}_{\Sigma}(Y)$ and $T_{\Sigma}(Y)$ in a similar way.

DEFINITION 2.5. Let $X=\left\{x_{1}, x_{2}, \ldots\right\}$ be a countable set of "variables" disjoint from $\Sigma$ and for every $n>0$, let $X_{n}=\left\{x_{1}, \ldots, x_{n}\right\}$. The elements of $T_{\Sigma}\left(X_{n}\right)$ are called $n$ ary $\Sigma$-polynomials. Let $Y$ be a set and $y \in Y$. Then $y_{\mathrm{T}}$ denotes the tree in $T_{\Sigma}(Y)$ defined by dom $y_{\mathrm{T}}=\{\lambda\}$ and $y_{\mathrm{T}}(\lambda)=y$.

A polynomial $p \in T_{\Sigma}\left(X_{n}\right)$ is said to be ideal if for every $j \in[n], p \neq x_{j T}$; a vector $q \in T_{\Sigma}\left(X_{n}\right)^{m}$ is ideal if every component is ideal.

To simplify the notation, we will use $T_{\Sigma}(n)$ (resp. $C T_{\Sigma}(n), \bar{R}_{\Sigma}(n)$ ) instead of $T_{\Sigma}\left(X_{n}\right)$ (resp. $C T_{\Sigma}\left(X_{n}\right), \bar{R}_{\Sigma}\left(X_{n}\right)$ ). Since $T_{\Sigma}(Y)$ is the free $\Sigma$-algebra generated by $Y$, given a $\Sigma$ algebra $A$, we can define a derived operation $t_{A}: A^{n} \rightarrow A$ for each $t \in T_{\Sigma}(n)$. (See $[1$, 10, 11] for details.)

DEFINITION 2.6. Let $t \in C T_{\Sigma}(n)$ and $p=\left(p_{1}, \ldots, p_{n}\right) \in C T_{\Sigma}^{n}$. The result of replacing $x_{i}$ with $p_{i}$ in each occurrence of $x_{i}$ in $t$ for every $i \in[n]$, is denoted by $t[p]$. Specifically, if we let $V_{i}=t^{-1}\left(x_{i}\right)$, then $\operatorname{dom}(t[p])=\operatorname{dom} t \cup \cup\left\{V_{i} \operatorname{dom} p_{i}: i \in[n]\right\}$ and

$$
\begin{aligned}
t[p](u)=t(u) \quad & \quad \text { if } \quad u \in \operatorname{dom} t-\bigcup\left\{V_{i}: i \in[n]\right\}, \\
=p_{i}(v) \quad & \text { if } \quad u=w v \in V_{i} \operatorname{dom} p_{i} .
\end{aligned}
$$

We will sometimes use $t\left[p_{1}, \ldots, p_{n}\right]$ for $t[p]$. It is a matter of routine to check that substitution is associative.

Notice also that if $u \in V_{i}$, then $t[p] \upharpoonright u=p_{i}$ and if $u \in \operatorname{dom} t$, then $(t \upharpoonright u)[p]=$ $t[p] \upharpoonright u$.

Definition 2.7 [12]. A $\Sigma$-algebra $A$ is said to be iterative if for every $n, k>0$ and every ideal $p \in T_{\Sigma}(n+k)^{n}$
(i) for each $a \in A^{k}$, the equation $x=p_{A}(x, a)$ has a unique solution in $A^{n}$, denoted by $\left(p_{A}\right)^{+}(a)$;
(ii) there exists an $a \in A^{k}$ such that $a_{1} \neq e_{1}^{n}\left(p_{A}\right)^{+}(a)$.

It is easy to see [12] that every $\Sigma$-homomorphism between iterative algebras preserves solutions of fixed point equations.

We now give a different characterization of regular trees.
Propostrion $2.8[4,9]$. (1) Let $t \in C T_{\Sigma}$ be ideal. If $t$ is regular, then there exist $n>0$ and an ideal $p \in T_{\Sigma}(n)^{n}$ such that $t=e_{1}^{n}\left(p_{C T_{\Sigma}}\right)^{+}$.
(2) Let $t \in C T_{\Sigma}$. Then $t$ is regular if and only if
(i) $t^{-1}(\sigma)$ is empty for all but a finite number of $\sigma \in \Sigma$,
(ii) $t^{-1}(\sigma)$ is a regular set for every $\sigma \in \Sigma$.

For the remainder of this section, we will assume that all algebras $A$ have ordered carrier with a least element, denoted by $\perp_{A}$ or just $\perp$.

Definition 2.9 [10]. Let $A$ be $\Sigma$-algebra. A map $f: A^{n} \rightarrow A^{n}$ is called algebraic if $f(x)=p_{A}(x, a)$ for some $p \in T_{\Sigma}(n+k)^{n}$ and $a \in A^{k}$. Define $L_{f}=\left\{f^{n}(\perp, \ldots, \perp): n \geqslant 0\right\}$. A subset $B$ of $A$ is called an iteration if there exist $n>0$ and an algebraic map $f: A^{n} \rightarrow A^{n}$ such that $B=e_{1}^{n}\left(L_{f}\right)$.

Definition 2.10 [10]. An ordered $\Sigma$-algebra $A$ is regular if for every $n>0$ and every algebraic map $f: A^{n} \rightarrow A^{n}$ we have
(i) $f(\perp, \ldots, \perp) \leqslant f(x)$ for every $x \in A^{n}$,
(ii) $L_{f}$ has a least upper bound in $A^{n}$,
(iii) $f\left(\sup L_{f}\right)=\sup L_{f}$.
$A$ is an ordered regular $\Sigma$-algebra if (i) is replaced by the stronger requirement that every $\Sigma$-operation be monotonic.

It is easy to see that if $f(x)=p_{A}(x, a)$ for some $a$ in $A^{k}$ and $p \in T_{\Sigma}(n+k)^{n}$, then $\sup L_{f}$ is the least solution of $x=p_{A}(x, a)$. We denote $\sup L_{f}$ by $\left(p_{A}\right)^{\nabla}(a)$.

If $A$ and $B$ are regular $\Sigma$-algebras, a map $f: A \rightarrow B$ is a regular homomorphism if it is a $\Sigma$-homomorphism such that $f\left(\perp_{A}\right)=\perp_{B}$ and for every iteration $E$ in $A$ there is an iteration $E^{\prime}$ in $B$ such that $f(E) \subseteq E^{\prime}$ and $f(\sup E)=\sup f(E)$.

Definition 2.11. Let $\Sigma$ be a signature, $Y$ a set and $\perp \notin Y$. Let $R_{\Sigma}(Y)=$ $\bar{R}_{\Sigma}(Y \cup\{\perp\})$ and define on $R_{\Sigma}(Y)$ a partial order $\leqslant$ as follows:
$t \leqslant t^{\prime}$ if and only if $t^{\prime}$ is obtained from $t$ by replacing some occurrences of $\perp$ with elements of $R_{\Sigma}(Y)$.

Then $R_{\Sigma}(Y)$ is a $\Sigma$-algebra where the carrier is a poset with least element $\perp$.
Fact 2.12 [12]. Since $\bar{R}_{\Sigma}(Y)$ (resp. $R_{\Sigma}(Y)$ ) is the free iterative (resp. regular) algebra on $Y$, we can define derived operations $t_{A}$ for every $t \in \bar{R}_{\Sigma}(n)$ (resp. $t \in R_{\Sigma}(n)$ ) in each iterative (resp. regular) $\Sigma$-algebra.

## 3. Main Result and Outline of Its Proof

The next result gives a necessary condition for a congruence on regular trees to induce an iterative factor algebra. This condition will be used to prove that the extension of a congruence is, in fact, a congruence and to prove Case I of the main theorem.

Proposition 3.1 [17]. Let $Y$ be any set and $K$ a congruence relation on $\bar{R}_{\Sigma}(Y)$. If $\bar{R}_{\Sigma}(Y) / K$ is iterative, then $K$ is closed under substitution.

If $q \in \bar{R}_{\Sigma}(n)$ and $\left(r_{i}, s_{i}\right) \in K$, for $i \in[n]$, then $(q[r], q[s]) \in K$.

DEFinition 3.2. A subset $K^{\prime}$ of $R_{\Sigma}(Y) \times R_{\Sigma}(Y)$ is an extension of a $\Sigma$ congruence $K$ on $\hat{R}_{\Sigma}(Y)$ if $K^{\prime}=\left\{(q[r], q[s]): q \in R_{\Sigma}(n),\left(r_{i}, s_{i}\right) \in K\right.$ for $\left.i \in[n]\right\}$, where $r=\left(r_{1}, \ldots, r_{n}\right)$ and $s=\left(s_{1}, \ldots, s_{n}\right)$. Notice that $K \subseteq K^{\prime}$.

Proposition 3.3 [17]. Let $K^{\prime}$ be an extension of $K$. If $\bar{R}_{\Sigma}(Y) / K$ is iterative, then $\dot{K}^{\prime}$ is a $\Sigma$-congruence relation on $R_{\Sigma}(Y)$.

Proof. If $\bar{R}_{\Sigma}(Y) / K$ is iterative, then $K$ is closed under substitution. This is used to prove that $K^{\prime}$ is transitive, by a proof similar to that of Proposition 6.3 in [12]. The verification that the other conditions are satisfied is straightforward. Denote by $\|t\|$ the $K^{\prime}$-equivalence class of $t$.

Theorem 3.4. If $K^{\prime}$ is an extension of $K$ and $\bar{R}_{\Sigma}(Y) / K$ is iterative, then $R_{\Sigma}(Y) / K^{\prime}$ is iterative.

Proof. To prove that $R_{\Sigma}(Y) / K^{\prime}$ is iterative, let $p \in T_{\Sigma}(n+k)^{n}, a \in R_{\Sigma}(Y)^{k}$ and consider
(1) $x=p_{R_{2}\left(Y^{\prime}\right) / K^{\prime}}(x,\|a\|)$.

We have to prove that (1) has a solution in $R_{\Sigma}(Y) / K^{\prime}$ and that this solution is unique.

Existence. Since $R_{\Sigma}(Y)$ is iterative, $x=p_{R_{\Sigma}(Y)}(x, a)$ has a solution $t$ and therefore (1) has a solution in $R_{\Sigma}(Y) / K^{\prime}$, namely, $\|t\|$.

Uniqueness. In this section we give only a sketch of this part of the proof, deferring the details until Section 4. Let both $\left(\left\|z_{1}^{\prime}\right\|, \ldots,\left\|z_{n}^{\prime}\right\|\right)$ and $\left(\left\|z_{1}^{\prime \prime}\right\|, \ldots,\left\|z_{n}^{\prime \prime}\right\|\right)$ be solutions of (1) in $R_{\Sigma}(Y) / K^{\prime}$. Selecting representatives of each equivalence class, we have
(1) $q_{i}^{\prime}\left[s_{i}^{\prime}\right]=p_{i}\left[q_{1}^{\prime}\left[r_{i}^{\prime}\right], \ldots, q_{n}^{\prime}\left[r_{n}^{\prime}\right], a\right]$,
(2) $q_{i}^{\prime \prime}\left[s_{i}^{\prime \prime}\right]=p_{i}\left[q_{1}^{\prime \prime}\left[r_{1}^{\prime \prime}\right], \ldots, q_{n}^{\prime \prime}\left[r_{n}^{\prime \prime}\right], a\right]$,
(3) $\left(r_{i}^{\prime}, s_{i}^{\prime}\right) \in K$,
(4) $\left(r_{i}^{\prime \prime}, s_{i}^{\prime \prime}\right) \in K$,
where $q_{i}^{\prime} \in R_{\Sigma}\left(m_{i}^{\prime}\right), q_{i}^{\prime \prime} \in R_{\Sigma}\left(m_{i}^{\prime \prime}\right), r_{i}^{\prime}, s_{i}^{\prime} \in \bar{R}_{\Sigma}(Y)^{m_{i}^{\prime}}, r_{i}^{\prime \prime}, s_{i}^{\prime \prime} \in \bar{R}_{\Sigma}(Y)^{m_{i}^{\prime \prime}}$ and $\left.\| q_{i}^{\prime} \mid s_{i}^{\prime}\right] \|=$ $\left\|z_{i}^{\prime}\right\|,\left\|q_{i}^{\prime \prime}\left[s_{i}^{\prime \prime}\right]\right\|=\left\|z_{i}^{\prime \prime}\right\|$.

The proof will be divided into two parts, according to whether $a \in \bar{R}_{\Sigma}(Y)^{k}$ or not. If $a \in \bar{R}_{\Sigma}(Y)^{k}$, we will use Proposition 3.1 to show that $\left(q_{i}^{\prime}\left[s_{i}^{\prime}\right], q_{i}^{\prime \prime}\left[s_{i}^{\prime \prime}\right]\right) \in K^{\prime}$ for every $i \in[n]$. If $a \notin \bar{R}_{\Sigma}(Y)^{k}$, then, for each $i \in[n]$, we will find $M_{i} \in \omega, q_{i} \in R_{\Sigma}\left(M_{i}\right)$ and $t_{i}^{\prime}, t_{i}^{\prime \prime} \in \bar{R}_{\Sigma}(Y)^{M_{i}}$ such that
(i) $q_{i}\left[t_{i}^{\prime}\right]=q_{i}^{\prime}\left[s_{i}^{\prime}\right]$,
(ii) $q_{i}\left[t_{i}^{\prime \prime}\right]=q_{i}^{\prime \prime}\left[s_{i}^{\prime \prime}\right]$, and
(iii) $\left(t_{i}^{\prime}, t_{i}^{\prime \prime}\right) \in K$.

The details are contained in the next section.

## 4. Proof of Uniqueness

4.0. For all $i \in[n]$ and $j \in[n+k]$, let $H(i j)=p_{i}^{-1}\left(x_{j}\right)$ and $H(i)=\bigcup\{H(i j)$ : $j \in[n]\}$.

First, we can assume, for the remainder of our discussion, that for every $j$, $n<j \leqslant n+k$, there is an $i \in[n]$ such that $H(i j) \neq \varnothing$. If this is not the case, we can "relabel" the variables corresponding to the "parameters" to obtain a $p^{\prime} \in T_{\Sigma}(n+k-1)^{n}$ equivalent to $p$. Furthermore, we can restrict our attention to irreducible vectors $p$, that is, to vectors satisfying the condition that for every $i, j \in[n]$, there exists a $w \in[n]^{*}$ such that $H(i w j) \neq \varnothing$.

It can be shown $[14,16]$ that if every irreducible vector $p$ has a unique fixpoint solution, then every $p$ has a unique fixpoint solution.

Before discussing the two cases mentioned in Theorem 3.4, we need a few notations and preliminary lemmas.

Definition 4.1. For each $i \in[n]$, let

$$
\begin{gathered}
D_{i}^{\prime}=\operatorname{dom} q_{i}^{\prime}, \quad D_{i}^{\prime \prime}=\operatorname{dom} q_{i}^{\prime \prime}, \quad D_{i}=D_{i}^{\prime} \cap D_{i}^{\prime \prime} \\
V_{i}^{\prime}=q_{i}^{\prime-1}\left(X_{m_{i}^{\prime}}\right), \quad V_{i}^{\prime \prime}=q_{i}^{\prime \prime-1}\left(X_{m_{i}^{\prime \prime}}\right) \\
V_{i}=V_{i}^{\prime} \cup V_{i}^{\prime \prime}, \\
B_{i}^{\prime}=q_{i}^{\prime-1}(\perp), \quad B_{i}^{\prime \prime}=q_{i}^{\prime \prime-1}(\perp), \\
E_{i}=\left\{z \in D_{i}: q_{i}^{\prime}(z)=q_{i}^{\prime \prime}(z)\right\} .
\end{gathered}
$$

Lemma 4.2. For every $i, j \in[n]$
(a) $H(i j)$ and $H(i)$ are totally unordered,
(b) $V_{i}^{\prime} \subseteq D_{i}^{\prime}, V_{i}^{\prime \prime} \subseteq D_{i}^{\prime \prime}$ and both $V_{i}^{\prime}$ and $V_{i}^{\prime \prime}$ are totally unordered
(c) for every $u \in H(i j)$,

$$
\begin{gathered}
q_{i}^{\prime}\left[s_{i}^{\prime}\right] \upharpoonright u=q_{j}^{\prime}\left[r_{j}^{\prime}\right], \\
q_{i}^{\prime \prime}\left[s_{i}^{\prime \prime}\right] \upharpoonright u=q_{j}^{\prime \prime}\left[r_{j}^{\prime \prime}\right], \\
u \operatorname{dom} q_{j}^{\prime}\left[r_{j}^{\prime}\right] \subseteq \operatorname{dom} q_{i}^{\prime}\left[s_{i}^{\prime}\right], \\
u \operatorname{dom} q_{j}^{\prime \prime}\left[r_{j}^{\prime \prime}\right] \subseteq \operatorname{dom} q_{i}^{\prime \prime}\left[s_{i}^{\prime \prime}\right],
\end{gathered}
$$

(d) for every $u \in\left(H(i) \omega^{*}\right)^{\text {c }}$

$$
q_{i}^{\prime}\left[s_{i}^{\prime}\right](u)=q_{i}^{\prime \prime}\left[s_{i}^{\prime \prime}\right](u)
$$

Lemma 4.3. For every $i, j \in[n]$
(a) $B_{i}^{\prime}=q_{i}^{\prime-1}\left[r_{i}^{\prime}\right](\perp)=q_{i}^{\prime-1}\left[s_{i}^{\prime}\right](\perp)$,
(b) $H(i j) B_{j}^{\prime} \subseteq B_{i}^{\prime}$,
(c) for every $m>0$ and $w \in[n]^{m}, H(i w j) B_{j}^{\prime} \subseteq B_{i}^{\prime}$,
(d) $H(i j){ }^{1} B_{i}^{\prime} \subseteq B_{j}^{\prime}$,
(e) for every $m>0$ and $w \in[n]^{m}, H(i w j)^{-1} B_{i}^{\prime} \subseteq B_{j}^{\prime}$,
(f) $H(i j)\left(\downarrow B_{j}^{\prime}\right) \subseteq \downarrow B_{i}^{\prime}$,
(g) $H(i j)^{-1}\left(\downarrow B_{i}^{\prime}\right) \subseteq \downarrow B_{j}^{\prime}$.

Similar statements hold with $q_{l}^{\prime \prime}, r_{i}^{\prime \prime}, s_{i}^{\prime \prime}, B_{i}^{\prime \prime}$, and $B_{j}^{\prime \prime}$ replacing $q_{i}^{\prime}, r_{i}^{\prime}, s_{i}^{\prime}, B_{i}^{\prime}$, and $B_{j}^{\prime}$, respectively.

Proof. (a) Let $z \in B_{i}^{\prime}$. Then $q_{i}^{\prime}(z)=1$. By Definition 2.6, $z \in \operatorname{dom} q_{i}^{\prime}\left[r_{i}^{\prime}\right]$ and $q_{i}^{\prime}\left[r_{i}^{\prime} \mid(z)=\perp\right.$. Hence $B_{i}^{\prime} \subseteq q_{i}^{\prime}\left[r_{i}^{\prime}\right]^{-1}(\perp)$. To prove the opposite inclusion, let $q_{i}^{\prime}\left[r_{i}^{\prime}\right](z)=\perp$. By 2.6, either $z \in D_{i}^{\prime}$ and $q_{i}^{\prime}(z)=\perp$, or there exist $j \in\left[m_{i}^{\prime}\right]$ and $u, v \in \omega^{*}$ such that $z=u v, q_{i}^{\prime}(u)=x_{j}$ and $q_{i}^{\prime}\left[r_{i}^{\prime}\right](z)=r_{l j}^{\prime}(v)$. The latter alternative is impossible since $r_{i j}^{\prime} \in \bar{R}_{i}(Y)$. Hence $q_{i}^{\prime}(z)=\perp$ and $z \in B_{i}^{\prime}$.

The proof that $B_{i}^{\prime}=q_{i}^{\prime}\left[s_{i}^{\prime}\right]^{-1}(\perp)$ is similar.
(b) Let $z \in B_{j}^{\prime}$ and $u \in H(i j)$. By (a), $q_{j}^{\prime}\left[r_{j}^{\prime}\right](z)=\perp$ and by Lemma 4.2, $q_{i}^{\prime}\left[s_{i}^{\prime}\right](u z)=1$. By (a) again, $u z \in B_{i}^{\prime}$.

The proof of the other statements is similar.
Case I. $a \in \bar{R}_{\Sigma}(Y)^{k}$.
Lemma 4.4 shows that the solution of the fixed-point equation $x=p_{R_{\Sigma}(r)}(x, a)$ is a total tree if and only if the parameter $a$ is total.

Lemma 4.4. The following are equivalent:
(1) $a \notin \bar{R}_{\Sigma}(Y)^{k}$,
(2) $B_{i}^{\prime} \neq \varnothing$ for some $i \in[n]$,
(3) $B_{i}^{\prime} \neq \varnothing$ for every $i \in[n]$,
(4) $B_{i}^{\prime \prime} \neq \varnothing$ for some $i \in[n]$,
(5) $B_{i}^{\prime \prime} \neq \varnothing$ for every $i \in[n]$.

Proof. It is sufficient to prove the equivalence of (1), (2), and (3).
(1) $\Rightarrow$ (2) If $a \in R_{\Sigma}(Y)^{k}-\bar{R}_{\Sigma}(Y)^{k}$, then there exist $j \in[k]$ and $z \in \omega^{*}$ such that $a_{j}(z)=1$. By 4.0, there exist $i \in[n]$ such that $H(i(n+j)) \neq \varnothing$. Let $y \in H(i(n+j))$. Then $q_{i}^{\prime}\left[s_{i}^{\prime}\right](y z)=p_{i}\left[q_{1}^{\prime}\left[r_{i}^{\prime}\right], \ldots, q_{n}^{\prime}\left[r_{n}^{\prime}\right], a\right](y z)=a_{j}(z)=\perp$. Hence $y z \in B_{i}^{\prime}$.
(2) $\Rightarrow$ (3) By Lemma 4.3 and the fact that $p$ is irreducible, $B_{i}^{\prime} \neq \varnothing$ for every $i \in[n]$.
(3) $\Rightarrow$ (1) Now assume that $B_{i}^{\prime} \neq \varnothing$ for every $i \in[n]$ and let $z$ be a word of minimal length in $\bigcup\left\{B_{i}^{\prime}: i \in[n]\right\}$. We can assume, without loss of generality, that $z \in B_{1}^{\prime}$. By Lemma $4.3 p_{1}\left[q_{1}^{\prime}\left[r_{1}^{\prime}\right], \ldots, q_{n}^{\prime}\left[r_{n}^{\prime}\right], a\right](z)=\perp$. By Definition 2.6, there are three mutually exclusive cases:

Case 1. $z \in \operatorname{dom} p_{1}$ and $p_{1}(z)=\perp$. This cannot happen, since $p \in T_{\Sigma}(n+k)^{n}$.
Case 2. There exist $u, v \in \omega^{*}$ and $j \in[n]$ such that $z=u v, u \in H(1 j)$ and $q_{j}^{\prime}\left[r_{j}^{\prime}\right](v)=\perp$. Then $v \in B_{j}^{\prime}$ and $u \neq \lambda$ because $p$ is ideal. This contradicts the minimality of the length of $z$.

Case 3. There exist $j \in[k]$ and $u, v \in \omega^{*}$ such that $z=u v, u \in H(1(n+j))$ and $a_{j}(v)=\perp$. Then $a_{j} \notin \bar{R}_{\Sigma}(Y)$ and this implies that $a \notin \bar{R}_{\Sigma}(Y)^{k}$.

Proposition 4.5. If $a \in \bar{R}_{\Sigma}(Y)^{k}$, then $\left(q_{i}^{\prime}\left[s_{i}^{\prime}\right], q_{i}^{\prime \prime}\left[s_{i}^{\prime \prime}\right]\right) \in K^{\prime}$ for every $i \in[n]$.

Proof. Immediate from Lemma 4.4, Definition 3.2, and Proposition 3.3.
Case II. $\quad a \notin \bar{R}_{\Sigma}(Y)^{k}$.
We first show that no variable can occur on the " $H$-paths."

Lemma 4.6. For every $i \in[n]$
(1) $V_{i}^{\prime} \cap(\downarrow H(i))=\varnothing$,
(2) $V_{i}^{\prime \prime} \cap(\downarrow H(i))=\varnothing$,
(3) for every $m>0$ and $w \in[n]^{m}, V_{i}^{\prime} \cap H(i w)(\downarrow H(w(m)))=\varnothing$,
(4) for every $m>0$ and $w \in[n]^{m}, V_{i}^{\prime \prime} \cap H(i w)(\downarrow H(w(m)))=\varnothing$,
(5) $V_{i} \cap \downarrow\left(\cup\left\{H(i w): m>0, w \in[n]^{m}\right\}\right)=\varnothing$.

Proof. The proof of (1) and (2) is somewhat simpler than the proof of (3) and can be derived from it.
(3) Suppose, looking for a contradiction, that there exist $m>0, j \in\left[m_{i}^{\prime}\right]$, $w \in[n]^{m}, y \in H(i w)$, and $v \in \downarrow H(w(m))$ such that $q_{i}^{\prime}(y v)=x_{j}$. Since $v \in \downarrow H(w(m))$, there exists $u \in \omega^{*}$ such that $v u \in H(w(m))$ and hence $y v u \in H(i w) H(w(m))=H(i w)$ $\left(\bigcup\left\{H\left(w(m) j^{\prime}\right): j^{\prime} \in[n]\right\}\right)$. Let $j^{\prime}$ be such that $y v u \in H(i w) H\left(w(m) j^{\prime}\right)$ and let $z \in B_{j^{\prime}}^{\prime}$. Then $y v u z \in B_{i}^{\prime}$ by Lemma 4.3. This contradicts the assumption that $r_{i}^{\prime} \in \bar{R}_{\Sigma}(Y)^{m_{i}}$.

The proof of (4) is direct by symmetry and (5) follows from (3) and (4).

Lemma 4.7. For every $i \in[n], m>0$ and $w \in[n]^{m}$
(a) $\operatorname{dom} q_{i}^{\prime}\left[r_{i}^{\prime}\right] \cap H(j)^{*} \subseteq\left(D_{i}^{\prime}-V_{i}^{\prime}\right) \cup\left(V_{i}^{\prime} \cap H(j)^{*}\right) \omega^{*}$,
(b) $\operatorname{dom} q_{i}^{\prime}\left[r_{i}^{\prime}\right] \cap H(i w) H(w(m))^{*} \subseteq\left(D_{i}^{\prime}-V_{i}^{\prime}\right) \cup\left(V_{i}^{\prime} \cap H(i w) H(w(m))^{*}\right) \omega^{*}$.

Similar statements hold with $s_{i}^{\prime}$ replacing $r_{i}^{\prime}$ and when $q_{i}^{\prime \prime}, r_{i}^{\prime \prime}, s_{i}^{\prime \prime}, D_{i}^{\prime \prime}$, and $V_{i}^{\prime \prime}$ replace $q_{i}^{\prime}, r_{i}^{\prime}, s_{i}^{\prime}, D_{i}^{\prime}$, and $V_{i}^{\prime}$, respectively.

Lemma 4.8. For every $i \in[n], m>0$ and $w \in[n]^{m}, \quad \downarrow H(i) \subseteq E_{i}$ and $H(i w)(\downarrow H(w(m))) \subseteq E_{i}$.

Proof. The proof is by induction on $m$, with the first inclusion providing the base ( $m=0$ ) of the induction. Let $v \in \downarrow H(i)$. Then by Lemma 4.2, $q_{i}^{\prime}\left[s_{i}^{\prime}\right](v)=q_{i}^{\prime \prime}\left[s_{i}^{\prime \prime}\right](v)$ and, by Lemma 4.6, $v \in D_{i}-V_{i}$. Therefore $q_{i}^{\prime}(v)=q_{i}^{\prime}\left[s_{i}^{\prime}\right](v)=q_{i}^{\prime \prime}\left[s_{i}^{\prime \prime}\right](v)=q_{i}^{\prime \prime}(v)$ and thus $v \in E_{i}$.

Now assume the lemma true for $m$ and let $z \in H(i w)(\downarrow H(w(m+1)))$ for some $w \in[n]^{m+1}$. Let $j=w(1)$ and define $y \in[n]^{m}$ by $y(k)=w(k+1)$. Then there exist $u \in H(i j), u^{\prime} \in H(j y)$ and $v \in \downarrow H(w(m+1))=\downarrow H(y(m))$ such that $z=u u^{\prime} v$. By the inductive hypothesis, $u^{\prime} v \in E_{j}$ and furthermore $q_{i}^{\prime}\left[s_{i}^{\prime}\right](z)=q_{j}^{\prime}\left[r_{j}^{\prime}\right]\left(u^{\prime} v\right)=q_{j}^{\prime}\left(u^{\prime} v\right)=$ $q_{j}^{\prime \prime}\left(u^{\prime} v\right)=q_{j}^{\prime \prime}\left[r_{j}^{\prime \prime}\right]\left(u^{\prime} v\right)=q_{i}^{\prime \prime}\left[s_{i}^{\prime \prime}\right](z)$. Since $z \in \operatorname{dom} q_{i}^{\prime}\left[s_{i}^{\prime}\right]$, either $z \in D_{i}^{\prime}-V_{i}^{\prime}$ or there exists $v^{\prime} \leqslant z$ such that $v^{\prime} \in V_{i}^{\prime}$. The latter cannot occur since $v^{\prime} \leqslant z$ and
$z \in H(i w)(\downarrow H(w(m)))$ imply that $v^{\prime} \in \downarrow H(i, m+1)$, which contradicts Lemma 4.6. Therefore $z \in D_{i}^{\prime}-V_{i}^{\prime}$. Similarly $z \in D_{i}^{\prime \prime}-V_{i}^{\prime \prime}$. Hence $q_{i}^{\prime}(z)=q_{i}^{\prime}\left[s_{i}^{\prime}\right](z)=q_{i}^{\prime \prime}\left[s_{i}^{\prime \prime}\right](z)=$ $q_{i}^{\prime \prime}(z)$.

Corollary 4.9. For every $i \in[n], \downarrow\left(\bigcup\{H(i, r): r>0\} \subseteq E_{i}\right.$.
The previous corollary shows that along all possible " $H$-paths" starting with $H(i)$, $q_{i}^{\prime}$ and $q_{i}^{\prime \prime}$ agree. These $H$-paths will be part of the domain of $q_{i}$ and, for words on these $H$-paths, we will define $q_{i}$ to have the common values of $q_{i}^{\prime}$ and $q_{i}^{\prime \prime}$.

Definition 4.10. For every $i \in[n]$, let $L_{i, 0}=D_{i} \cap H(i)^{*}$ and for every $m>0$, $L_{i, m}=D_{i} \cap\left(\bigcup\left\{H(i w) H(w(m))^{*}: w \in[n]^{m}\right\}\right)$ and $V_{i, m}=L_{i, m} \cap V_{i}$.

Lemma 4.11. For every $i \in[n]$ and $m \geqslant 0$
(1) $L_{i, m} \subseteq\left(V_{i, m} \omega^{+}\right)^{\mathrm{c}}$,
(2) $V_{i, m}$ is totally unordered.

Definition 4.12. For every $i \in[n]$ and $m>0$, let

$$
\begin{gathered}
V_{i, 0}^{\prime}=V_{i}^{\prime} \cap H(i)^{*}, \quad V_{i, 0}^{\prime \prime}=V_{i}^{\prime \prime} \cap H(i)^{*}, \\
V_{i, m}^{\prime}=V_{i}^{\prime} \cap\left(\bigcup\left\{H(i w) H(w(m))^{\#}: w \in[n]^{m}\right\}\right), \\
V_{i, m}^{\prime \prime}=V_{i}^{\prime \prime} \cap\left(\bigcup\left\{H(i w) H(w(m))^{*}: w \in[n]^{m}\right\}\right), \\
G_{i, m}=\operatorname{MIN}\left(H(i, m)^{-1}\left(V_{i, m}^{\prime} \cup V_{i, m}^{\prime \prime}\right)\right), \\
W_{i, 0}=G_{i, 0} \quad \text { and } \quad W_{i, m+1}=\operatorname{MIN}\left(G_{i, m+1} \cup\left(\bigcup\left\{G_{j, k}: j \in[n], k \in[m]\right\}\right)\right), \\
T_{i, 0}=W_{i, 0} \cap H(i)^{\#} \cap D_{i}, \\
T_{i, m}=\left(\bigcup\left\{H(i w)\left(W_{i, m} \cap H(w(m))^{\#}\right): w \in[n]^{m}\right\}\right) \cap D_{i} .
\end{gathered}
$$

It is easy to see that for every $i \in[n]$ and $m \geqslant 0, G_{i, m}$ and $W_{i, m}$ are totally unordered and $T_{i, m} \subseteq L_{i, m}$ and that for every $j \in[n], W_{l, m} \subseteq W_{j, m+1} \omega^{*}$.

In defining $q_{i}$, we want to take the "common part" of $q_{i}^{\prime}$ and $q_{i}^{\prime \prime}$. In cutting at the $m$ th level $\left(T_{i, m}\right)$, we want the cutting points to be not only in $D_{i}^{\prime}$ and $D_{i}^{\prime \prime}$, but also no further from the $H$-paths corresponding to $i$ than all the cutting points at the ( $n-1$ )th level of all $H$-paths.

Lemma 4.13. For every $i \in[n]$ and $m>0$
(1) $V_{i, 0} \subseteq\left(G_{i, 0} \omega^{*}\right) \cap D_{i}$,
(2) $V_{i, m} \subseteq\left(H(i, m) G_{i, m} \omega^{*}\right) \cap D_{i}$,
(3) $\left(V_{i} \cap D_{i}\right) \omega^{+} \cap H(i)^{*}=V_{i, 0} \omega^{+}$,
(4) $\left(V_{i} \cap D_{i}\right) \omega^{+} \cap\left\{H(i w) H(w(m))^{*}: w \in[n]^{m}\right\}=V_{i, m} \omega^{+}$.

Proof. Again we prove only (2) and (4).
(2) $V_{i, m}^{\prime} \subseteq H(i, m)\left(H(i, m)^{-1} V_{i, m}^{\prime}\right)$ and $V_{i, m}^{\prime \prime} \subseteq H(i, m)\left(H(i, m)^{-1} V_{i, m}^{\prime \prime}\right)$ imply, by Definition 1.8, that $H(i, m)^{-1}\left(V_{i, m}^{\prime} \cup V_{i, m}^{\prime \prime}\right) \subseteq G_{i, m} \omega^{*}$. Therefore $V_{i, m}=D_{i} \cap V_{i} \cap$ $\left(\bigcup\left\{H(i w) H(w(m))^{*}: w \in[n]^{m}\right\}\right) \subseteq D_{i} \cap H(i, m) G_{i, m} \omega^{*}$.
(4) By Definition 4.10, $V_{i, m} \omega^{+} \subseteq\left(V_{i} \cap D_{i}\right) \omega^{+} \cap\left(\bigcup\left\{H(i w) H(w(m))^{*} \omega^{+}\right.\right.$: $\left.w \in[n]^{m}\right\}$ ). By Lemma 1.7, $H(i w) H(w(m))^{*} \omega^{+} \subseteq H(i w) H(w(m))^{*}$. Therefore, $V_{i, m} \omega^{+} \subseteq\left(V_{i} \cap D_{i}\right) \omega^{+} \cap\left(\bigcup\left\{H(i w) H(w(m))^{*}: w \in[n]^{m}\right\}\right)$.

To prove the opposite inclusion, let $z$ be an element on the left side of (4). Then there exist $w \in[n]^{m}, u \in H(i w), v \in H(w(m))^{*}$, and $y \in V_{i} \cap D_{i}$ such that $z=u v$ and $y<z$. By Lemma 4.6, there exists $v^{\prime}<v, v^{\prime} \in H(w(m))^{*}$ such that $y=u v^{\prime}$. Thus $u v^{\prime}<z$ and $u v^{\prime} \in V_{i} \cap D_{i} \cap H(i w) H(w(m))^{*} \subseteq V_{i, m}$. Therefore $z \in V_{i, m} \omega^{+}$.

Lemma 4.14. For every $i \in|n|$ and $m>0$
(a) $T_{i, m}$ is totally unordered,
(b) $T_{i, 0} \subseteq D_{i} \cap H(i)^{*} \cap\left(T_{i, 0} \omega^{+}\right)^{\mathrm{c}}$ and $T_{i, m} \subseteq D_{i} \cap\left(\bigcup\left\langle H(i w) H(w(m))^{*}:\right.\right.$ $\left.\left.w \in[n]^{m}\right\}\right) \cap\left(T_{i, m} \omega^{+}\right)^{c}$,
(c) for every $j \in[n], H(i j) T_{j, m} \subseteq T_{i, m+1} \omega^{*}$.

Proof. (a) Since $W_{i, m}$ and $H(w(m))^{*}$ are totally unordered, $W_{i, m} \cap H(w(m))^{*}$ is totally unordered. Then $H(i w)\left(W_{i, m} \cap H(w(m))^{*}\right)$ is totally unordered, by Lemma 1.6, since every $H(i j)$ is. Finally, $H(i, m)$ is totally unordered and therefore $T_{i, m} \subseteq \bigcup\left\{H(i w) H(w(m))^{*}: w \in[n]^{m}\right\}$ is totally unordered.
(b) Follows directly from Definition 4.12, (a), and Lemma 1.5.
(c) $H(i j) T_{j, m}=\bigcup\left\{H(i j) H(j w)\left(W_{j, m} \cap H(w(m))^{*}\right): w \in[n]^{m}\right\} \cap H(i j) D_{j} \subseteq$ $\bigcup\left\{H(i y)\left(W_{i, m+1} \omega^{*} \cap H(y(m+1))^{*} \omega^{*}\right): y \in[n]^{m+1}\right\} \cap D_{i} \omega^{*}=T_{i, m+1} \omega^{*}$.

Lemma 4.15. For every $i \in[n]$ and $m \geqslant 0$
(1) $V_{i, m} \subseteq T_{i, m} \omega^{*}$,
(2) $\downarrow T_{i, m+1} \cap H(i j) V_{J, m}=\varnothing$,
(3) $H(i j)^{-1}\left(\left(T_{i, m+1} \omega^{+}\right)^{\mathrm{c}} \subseteq\left(T_{j, m} \omega^{+}\right)^{\mathrm{c}}\right.$.

We now show that the "cuts" ( $T_{i, m}$ ) have been carried out properly and that $q_{i}^{\prime}$ and $q_{i}^{\prime \prime}$ agree for all the words which are incomparable with the cutting points.

Lemma 4.16. For every $i \in[n], m>0$ and $w \in[n]^{m}$
(a) $\left(D_{i}^{\prime} \cup D_{i}^{\prime \prime}\right) \cap H(i)^{*} \cap\left(T_{i, 0} \omega^{+}\right)^{\mathrm{c}} \subseteq E_{i}$,
(b) $\quad\left(D_{i}^{\prime} \cup D_{i}^{\prime \prime}\right) \cap H(i w) H(w(m))^{*} \cap\left(T_{l, m} \omega^{+}\right)^{\mathrm{c}} \subseteq E_{i}$.

Proof. (a) Let $z \in D_{i}^{\prime} \cap H(i)^{\#} \cap\left(T_{i, 0} \omega^{+}\right)^{\mathrm{c}}$. Then, by Lemma 4.2, $q_{i}^{\prime}(z)=$
$q_{i}^{\prime}\left[s_{i}^{\prime}\right](z)=q_{i}^{\prime \prime}\left[s_{i}^{\prime \prime}\right](z)$. By Lemma 4.7, either $z \in D_{i}^{\prime \prime}-V_{i}^{\prime \prime}$, in which case $q_{i}^{\prime \prime}(z)=$ $q_{i}^{\prime \prime}\left[s_{i}^{\prime \prime}\right](z)=q_{i}^{\prime}(z)$, or there exists $v<z$ such that $v \in V_{i}^{\prime \prime} \cap H(i)^{*}$. But the latter case would give us a contradiction, since it implies that $v \in D_{i}^{\prime} \cap D_{i}^{\prime \prime} \cap H(i)^{*} \cap V_{i}^{\prime \prime} \subseteq T_{i, 0}$ contrary to the choice of $z$. Hence $z \in E_{i}$. By symmetry of the roles of $D_{i}^{\prime}$ and $D_{i}^{\prime \prime}$, the statement is proved.
(b) The proof is by induction on $m$, with (a) providing the base step. Assume the statement true for $m$, any $j \in[n]$ and every $y \in[n]^{m}$ and let $z \in D_{i}^{\prime} \cap H(i w)$ $H(w(m+1))^{*} \cap\left(T_{i, m+1} \omega^{+}\right)^{c}$ for some $w \in[n]^{m+1}$. Let $j=w(1)$ and define $y \in[n]^{m}$ by $y(k)=w(k+1)$ for every $k \in[m]$. Then $z=u u^{\prime} v$ for some $u \in H(i j), u^{\prime} \in H(j y)$ and $v \in H(y(m))^{*}$. By Lemma 4.7, $u^{\prime} v \in\left(D_{j}^{\prime}-V_{j}^{\prime}\right) \cup\left(V_{j}^{\prime} \cap H(j y) H(y(m))^{*}\right) \omega^{+}$ and thus either $u^{\prime} v \in D_{j}^{\prime}-V_{j}^{\prime}$ or there exists $v^{\prime}<v$ such that $u^{\prime} v^{\prime} \in V_{j}^{\prime} \cap H(j y)$ $H(y(m))^{*}$. The latter cannot occur because $u^{\prime} v^{\prime} \in H(i j)^{-1}\left(T_{i, m+1} \omega^{+}\right)^{c} \subseteq\left(T_{j, m} \omega^{*}\right)^{c}$, which implies, (b) the inductive hypothesis, that $u^{\prime} v^{\prime} \in D_{j}^{\prime} \cap H(j y) H(y(m))^{*} \cap$ $\left(T_{j, m} \omega^{+}\right)^{c} \subseteq E_{j} \subseteq D_{j}^{\prime \prime}$. But then $u^{\prime} v^{\prime} \in D_{j}^{\prime} \cap D_{j}^{\prime \prime} \cap H(j y) H(y(m))^{*} \cap V_{j}^{\prime} \subseteq V_{j, m} \subseteq$ $T_{j, m} \omega^{*}$ and therefore $u u^{\prime} v \in T_{i, m+1} \omega^{*}$, contrary to the choice of $z$.
(1) Therefore $u^{\prime} v \in D_{j}^{\prime}-V_{j}^{\prime}$.

Now $z \in\left(T_{i, m+1} \omega^{+}\right)^{\mathfrak{c}}$ implies $u^{\prime} v \in\left(T_{j, m} \omega^{*}\right)^{\text {c }}$ by Lemma 4.15 and hence $u^{\prime} v \in$ $D_{j}^{\prime} \cap H(j y) H(y(m))^{*} \cap\left(T_{j, m} \omega^{+}\right)^{c} \subseteq E_{j}$ by the inductive hypothesis. In particular $u^{\prime} v \in D_{j}^{\prime \prime}$. Then, by Lemma 4.7, either $u u^{\prime} v \in D_{i}^{\prime \prime}-V_{i}^{\prime \prime}$ or there exists $v^{\prime}<v$ such that $u u^{\prime} v^{\prime} \in V_{i}^{\prime \prime} \cap H(i w) H(w(m+1))^{*}$. If the latter holds, then $u u^{\prime} v^{\prime} \in D_{i} \cap V_{i}^{\prime \prime} \cap$ $H(i w) H(w(m+1))^{*} \subseteq V_{i, m+1}$ and thus $u u^{\prime} v \in T_{i, m+1} \omega^{*}$, which again contradicts the choice of $z$.
(2) Therefore $u u^{\prime} v \in D_{i}^{\prime \prime}-V_{i}^{\prime \prime}$. Finally

$$
\begin{aligned}
q_{i}^{\prime}(z) & =q_{i}^{\prime}\left(u u^{\prime} v\right)=q_{i}^{\prime}\left[s_{t}^{\prime}\right]\left(u u^{\prime} v\right)=q_{j}^{\prime}\left[r_{j}^{\prime}\right]\left(u^{\prime} v\right)=q_{j}^{\prime}\left(u^{\prime} v\right)=q_{j}^{\prime \prime}\left(u^{\prime} v\right) \\
& =q_{j}^{\prime \prime}\left[r_{j}^{\prime \prime}\right]\left(u^{\prime} v\right)=q_{i}^{\prime \prime}\left[s_{i}^{\prime \prime}\right]\left(u u^{\prime} v\right)=q_{i}^{\prime \prime}\left(u u^{\prime} v\right)=q_{i}^{\prime \prime}(z) .
\end{aligned}
$$

Therefore $z \in E_{i}$. By symmetry of the roles of $D_{i}^{\prime}$ and $D_{i}^{\prime \prime}$, the proof of the lemma is complete.

Next, observe that, since $r_{i}^{\prime}, s_{i}^{\prime} \in \bar{R}_{\Sigma}(Y)^{m_{i}^{\prime}}$, we have, for every $i \in[n]$,

$$
\left.\left\{u \in \operatorname{dom} q_{i}^{\prime}\left[r_{i}^{\prime}\right]: q_{i}^{\prime}\left[r_{i}^{\prime}\right] \upharpoonright u \in \bar{R}_{\Sigma}(Y)\right\}=\left\{u \in \operatorname{dom} q_{i}^{\prime} \mid s_{i}^{\prime}\right]: q_{i}^{\prime}\left[s_{i}^{\prime}\right] \upharpoonright u \in \bar{R}_{\Sigma}(Y)\right\} .
$$

Similarly with " replacing '.
We are now ready to show that none of the cutting points has separated bottom elements from the $H$-paths.

Lemma 4.17. For every $i \in[n]$, every $m \geqslant 0$ and every $z \in T_{i, m}$

$$
q_{i}^{\prime}\left[s_{i}^{\prime}\right]\left\lceilz \in \overline { R } _ { \Sigma } ( Y ) \quad \text { and } \quad q _ { i } ^ { \prime \prime } [ s _ { i } ^ { \prime \prime } ] \left\lceil z \in \bar{R}_{\Sigma}(Y) .\right.\right.
$$

Proof. The proof for $m=0$ is straightforward using Lemma 4.2 and Definition 4.12. We can then assume that $m>0$. Let $z \in T_{i, m}$. Then, by $4.12, z \in D_{i}$ and there exist $w \in[n]^{m}, u \in H(i w)$ and $v \in W_{i, m} \cap H(w(m))^{\#}$ such that $z=u v$. Then either $v \in H(j, k)^{-1}\left(V_{j, k}^{\prime} \cup V_{j, k}^{\prime \prime}\right)$ for some $j \in[n]$ and $k \in[m-1]$, or $v \in H(i, k)^{-1}\left(V_{i, k}^{\prime} \cup V_{i, k}^{\prime \prime}\right)$ for some $k \in[m]$.

We can assume that $v \in H(i, k)^{-1} V_{i, k}^{\prime}$; the other three cases are dealt with in a similar way. By choice of $v$, there exist $w^{\prime} \in[n]^{k-1}$ and $u^{\prime} \in H\left(i w^{\prime}\right)$ such that $w^{\prime}(k-1)=w(m)$ and $u^{\prime} v \in V_{i, k}^{\prime}$. Thus there is a $j_{0} \in\left[m_{i}^{\prime}\right]$ such that $q_{i}^{\prime}\left(u^{\prime} v\right)=x_{j_{0}}$ and

$$
\begin{equation*}
q_{i}^{\prime}\left[s_{i}^{\prime}\right] \upharpoonright u^{\prime} v=s_{i j_{0}}^{\prime} \in \bar{R}_{\Sigma}(Y) \tag{1}
\end{equation*}
$$

Now suppose, looking for a contradiction, that $q_{i}^{\prime}\left[s_{i}^{\prime}\right]\left\lceil z \notin \bar{R}_{\Sigma}(Y)\right.$. Then, for some $y \in \omega^{*}, q_{i}^{\prime}\left[s_{i}^{\prime}\right](z y)=1$, i.e., $z y \in B_{i}^{\prime}$. By Lemma 4.3, $v y \in H(i w)^{-1} B_{i}^{\prime} \subseteq B_{w(m)}^{\prime}$ and $u^{\prime} v y \in H\left(i w^{\prime}\right) B_{w(m)}^{\prime} \subseteq B_{i}^{\prime}$ contradicting (1). Similarly, if $q_{i}^{\prime \prime}\left[s_{i}^{\prime \prime}\right]\left\lceil z \notin \bar{R}_{\Sigma}(Y)\right.$, then let $y$ be such that $q_{i}^{\prime \prime}\left[s_{i}^{\prime \prime}\right](z y)=\perp$. Then $v y \in H(i w)^{-1} B_{i}^{\prime \prime} \subseteq B_{w(m)}^{\prime \prime}$ and $v y \in H(w(m))^{\#}$. But, by Lemma 4.2, $q_{i}^{\prime \prime}\left[s_{i}^{\prime \prime}\right](v y)=q_{i}^{\prime}\left[s_{i}^{\prime}\right](v y)$ and therefore $v y \in B_{w(m)}^{\prime}$. Again $u^{\prime} v y \in B_{i}^{\prime}$ contradicts (1).

Let $\left\{g_{i j}^{\prime}: j \in\left[k_{i}^{\prime}\right]\right\}$ and $\left\{g_{i j}^{\prime \prime}: j \in\left[k_{i}^{\prime \prime}\right]\right\}$ be the sets of all distinct subtrees of $\left.q_{i}^{\prime} \mid s_{i}^{\prime}\right]$ and $q_{i}^{\prime \prime}\left[s_{i}^{\prime \prime}\right]$, respectively. Define $M_{i}=k_{i}^{\prime} \cdot k_{i}^{\prime \prime}$ and identify $\left[M_{i}\right]$ with $\left[k_{i}^{\prime}\right] \times\left[k_{i}^{\prime \prime}\right]$.

We are now in a position to define, for every $i \in[n], q_{i}, t_{i}^{\prime}$ and $t_{i}^{\prime \prime}$ such that

$$
\begin{gather*}
q_{i} \in R_{\Sigma}\left(M_{i}\right)  \tag{4.18.1}\\
t_{i}^{\prime}, t_{i}^{\prime \prime} \in \bar{R}_{\Sigma}(Y)^{M_{i}},  \tag{4.18.2}\\
q_{i}\left[t_{i}^{\prime}\right]=q_{i}^{\prime}\left[s_{i}^{\prime}\right],  \tag{4.18.3}\\
q_{i}\left[t_{i}^{\prime \prime}\right]=q_{i}^{\prime \prime}\left[s_{i}^{\prime \prime}\right], \tag{4.18.4}
\end{gather*}
$$

and

$$
\begin{equation*}
\left|t_{i}^{\prime}\right|=\left|t_{i}^{\prime \prime}\right| . \tag{4.18.5}
\end{equation*}
$$

DEFINITION 4.19. For every $i \in[n]$, let $T_{i}=\bigcup\left\{T_{i, m}: m \geqslant 0\right\}$ and define dom $q_{i}=$ $\downarrow H(i) \cup \downarrow(\cup\{H(i, r): r>0\}) \cup T_{i} \cup \bigcup\left\{\left(D_{i}^{\prime} \cup D_{i}^{\prime \prime}\right) \cap H(i)^{*} \cap\left(T_{i, 0} \omega^{+}\right)^{c}\right\} \cup$ $\bigcup\left\{\left(D_{i}^{\prime} \cup D_{i}^{\prime \prime}\right) \cap H(i w) H(w(m))^{*} \cap\left(T_{i, m} \omega^{+}\right)^{c}: m>0, w \in[n]^{m}\right\}$.

Definition 4.20. To complete the definition of $q_{i}$, let $z \in \operatorname{dom} q_{i}$.
Case 1. $z=\downarrow H(i) \cup \downarrow(\cup\{H(i, r): r>0\})$. By Corollary 4.9, $z \in E_{i}$. Define $q_{i}(z)=q_{i}^{\prime}(z)$.

Case 2. $z \in\left(D_{i}^{\prime} \cup D_{i}^{\prime \prime}\right) \cap H(i)^{*} \cap\left(T_{i, 0} \omega^{+}\right)^{\mathrm{c}} \quad$ or $\quad z \in\left(D_{i}^{\prime} \cup D_{i}^{\prime \prime}\right) \cap H(i w) H(w(m))^{*}$
$\cap\left(T_{i, m} \omega^{+}\right)^{\mathrm{c}}$ for some $m>0$ and $w \in[n]^{m}$. By Lemma 4.16, $z \in E_{i}$. Again let $q_{i}(z)=q_{i}^{\prime}(z)$.

Case 3. $z \in T_{i, m}$ for some $m \geqslant 0$. By Lemma 4.17, there exist $j^{\prime} \in\left[k_{i}^{\prime}\right]$ and $j^{\prime \prime} \in\left[k_{i}^{\prime \prime}\right]$ such that $q_{i}^{\prime}\left[s_{i}^{\prime}\right] \upharpoonright z=g_{i j^{\prime}}^{\prime}$ and $q_{i}^{\prime \prime}\left[s_{i}^{\prime \prime}\right] \upharpoonright z=g_{i j^{\prime \prime}}^{\prime \prime}$. Define $q_{i}(z)=x_{\left(j^{\prime}, j^{\prime \prime}\right)}$.

It is not too hard to check that $q_{i}$ satisfies Definition 2.3.

Definition 4.21. We now define $t_{i}^{\prime}$ and $t_{i}^{\prime \prime}$ componentwise as follows: let $\left(j^{\prime}, j^{\prime \prime}\right) \in\left[M_{i}\right]$. If $q_{i}^{-1}\left(x_{\left(j^{\prime}, j^{\prime \prime}\right)}\right)=\varnothing$, then let $t_{i}^{\prime}\left(j^{\prime}, j^{\prime \prime}\right)=t_{i}^{\prime \prime}\left(j^{\prime}, j^{\prime \prime}\right)$ be arbitrary in $\bar{R}_{\Sigma}(Y)$. (See Notation 1.0.) If $q_{i}^{-1}\left(x_{\left(j^{\prime}, j^{\prime \prime}\right)}\right) \neq \varnothing$, then let $t_{i}^{\prime}\left(j^{\prime}, j^{\prime \prime}\right)=g_{i j^{\prime}}^{\prime}$ and $t_{i}^{\prime \prime}\left(j^{\prime}, j^{\prime \prime}\right)=$ $g_{i j^{\prime \prime}}^{\prime \prime}$. It is clear that (4.18.2) is satisfied.

We now prove (4.18.3); the proof for (4.18.4) is similar. Let $z \in \operatorname{dom} q_{i}\left[t_{i}^{\prime}\right]$. Then either $z \in \operatorname{dom} q_{i}-T_{i} \omega^{*}$, in which case $z \in E_{i}$ and $q_{i}\left[t_{i}^{\prime}\right](z)=q_{i}(z)=q_{i}^{\prime}(z)=$ $q_{i}^{\prime}\left[s_{i}^{\prime}\right](z)$, or $z \in \operatorname{dom} q_{i} \cap\left(T_{i, m} \omega^{*}\right)$ for some $m \geqslant 0$. In this case, let $u \in T_{i, m}$ be such that $u \leqslant z$; then, by definition of $t_{i}^{\prime}, q_{i}\left[t_{i}^{\prime}\right] \upharpoonright u=q_{i}^{\prime}\left[s_{i}^{\prime}\right] \upharpoonright u$ and in particular $q_{i}\left[t_{i}^{\prime}\right](z)=q_{i}^{\prime}\left[s_{i}^{\prime}\right](z)$.

Now let $z \in \operatorname{dom} q_{i}^{\prime}\left[s_{i}^{\prime}\right]$. By Lemma 4.7, there are three mutually exclusive cases:
Case 1. $z \in D_{i}^{\prime} \cap \perp(H(i) \cup H(i, r))$ for some $r>0$. In this case $q_{i}^{\prime}\left[s_{i}^{\prime}\right](z)=$ $q_{i}^{\prime}(z)=q_{i}(z)=q_{i}\left[t_{i}^{\prime}\right](z)$ by Definitions $4.20(1)$ and 2.6.

Case 2. Either $z \in\left(D_{i}^{\prime}-V_{i}^{\prime}\right) \cap H(i w) H(w(m))^{*}$ for some $m>0$ and $w \in[n]^{m}$ or $z \in\left(D_{i}^{\prime}-V_{i}^{\prime}\right) \cap H(i)^{*}$. We will consider the former possibility; the latter is similar.

If $z \in\left(T_{i, m} \omega^{*}\right)^{\mathrm{c}}$, then, by Definition 4.20(2), $q_{i}^{\prime}\left[s_{i}^{\prime}\right](z)=q_{i}^{\prime}(z)=q_{i}(z)=q_{i}\left[t_{i}^{\prime}\right](z)$. Otherwise, if $z \in T_{i, m} \omega^{*}$, let $u \in T_{i, m}$ be such that $u \leqslant z$. By Definition 4.21, $q_{i}^{\prime}\left[s_{i}^{\prime}\right] \upharpoonright u=q_{i}\left[t_{i}^{\prime}\right] \upharpoonright u$ and in particular $q_{i}^{\prime}\left[s_{i}^{\prime}\right](z)=q_{i}\left[t_{i}^{\prime}\right](z)$.

Case 3. Either $z \in\left(V_{i}^{\prime} \cap H(i w) H(w(m))^{*}\right) \omega^{*}$ for some $m>0$ and $w \in[n]^{m}$ or $z \in\left(V_{i}^{\prime} \cap H(i)^{*}\right) \omega^{*}$. Again we only consider the former case. Let $u \in V_{i}^{\prime} \cap H(i w)$ $H(w(m))^{*}$ be such that $u \leqslant z$. Since $u \in\left(T_{i, m} \omega^{*}\right)^{\mathrm{c}}$ would contradict Lemma 4.15, we must have $u \in T_{i, m} \omega^{*}$. But then $z \in T_{i, m} \omega^{*}$ and therefore $q_{i}\left[t_{i}^{\prime}\right](z)=q_{i}^{\prime}\left[s_{i}^{\prime}\right](z)$ as in Case 2.

To prove (4.18.5) componentwise, define for every $i \in[n], j^{\prime} \in\left[k_{i}^{\prime}\right]$ and $j^{\prime \prime} \in\left[k_{i}^{\prime \prime}\right]$ $m u_{i}\left(j^{\prime}, j^{\prime \prime}\right)=$ the smallest integer $m$ such that $q_{i}(z)=x_{\left(j^{\prime}, j^{\prime \prime}\right)}$ for some $z \in T_{i, m}$. If $m u_{i}\left(j^{\prime}, j^{\prime \prime}\right)$ is not defined, then $q_{i}^{-1}\left(x_{\left(j^{\prime}, j^{\prime \prime}\right)}\right)=\varnothing$, in which case $t_{i}^{\prime}\left(j^{\prime}, j^{\prime \prime}\right)=t_{i}^{\prime \prime}\left(j^{\prime}, j^{\prime \prime}\right)$ and there is nothing to prove. The proof of (4.18.5) is by induction on $m u_{i}\left(j^{\prime}, j^{\prime \prime}\right)$.

Base $m u_{i}\left(j^{\prime}, j^{\prime \prime}\right)=0$. Let $z \in T_{i, 0}$ be such that $q_{i}(z)=x_{\left(j^{\prime}, j^{\prime \prime}\right)}$. Since $T_{i, 0} \subseteq H(i)^{*}$, by Lemma 4.2 we have $t_{i}^{\prime}\left(j^{\prime}, j^{\prime \prime}\right)=g_{i j^{\prime}}^{\prime}=q_{i}^{\prime}\left[s_{i}^{\prime}\right] \upharpoonright z=q_{i}^{\prime \prime}\left[s_{i}^{\prime \prime}\right] \upharpoonright z=g_{i j^{\prime \prime}}^{\prime \prime}=t_{i}^{\prime \prime}\left(j^{\prime}, j^{\prime \prime}\right)$ and therefore $\left|t_{i}^{\prime}\left(j^{\prime}, j^{\prime \prime}\right)\right|=\left|t_{i}^{\prime \prime}\left(j^{\prime}, j^{\prime \prime}\right)\right|$.

Inductive step. Assume the claim true for all $i \in[n], j_{1} \in\left[k_{i}^{\prime}\right]$ and $j_{2} \in\left[k_{i}^{\prime \prime}\right]$ such that $m u_{i}\left(j_{1}, j_{2}\right)<m+1$ and let $m u_{l}\left(j^{\prime}, j^{\prime \prime}\right)=m+1$. Choose $z \in T_{i, m+1}$ so that $q_{i}(z)=x_{\left(j^{\prime}, j^{\prime \prime}\right)}$. Then $z \in H(i w) H(w(m+1))^{*}$ for some $w \in[n]^{m+1}$. Define $y \in\left[\left.n\right|^{m}\right.$ by $y(k)=w(k+1)$ and let $j=w(1)$. Then $z=u u^{\prime} v$ for some $u \in H(i j), u^{\prime} \in H(i y)$ and $v \in H(y(m))^{*}$. Now
(1) $t_{i}^{\prime}\left(j^{\prime}, j^{\prime \prime}\right)=q_{i}\left[t_{i}^{\prime}\right] \upharpoonright z=q_{i}^{\prime}\left[s_{i}^{\prime}\right] \upharpoonright u u^{\prime} v=q_{j}^{\prime}\left[r_{j}^{\prime}\right] \upharpoonright u^{\prime} v$, and
(2) $t_{i}^{\prime \prime}\left(j^{\prime}, j^{\prime \prime}\right)=q_{i}\left[t_{i}^{\prime \prime}\right] \upharpoonright z=q_{i}^{\prime \prime}\left[s_{i}^{\prime \prime}\right] \upharpoonright u u^{\prime} v=q_{j}^{\prime \prime}\left[r_{j}^{\prime \prime}\right] \upharpoonright u^{\prime} v$.

By Lemmas 4.13 and 4.15, $u^{\prime} v \in \operatorname{dom} q_{j}^{\prime}\left[r_{j}^{\prime}\right]$ implies $u^{\prime} v \in D_{j}^{\prime}$; similarly $u^{\prime} v \in D_{j}^{\prime \prime}$. So let $\bar{q}_{j}^{\prime}=q_{j}^{\prime} \upharpoonright u^{\prime} v$ and $\bar{q}_{j}^{\prime \prime}=q_{j}^{\prime \prime} \upharpoonright u^{\prime} v$. By Definition 4.19, $u^{\prime} v \in \operatorname{dom} q_{j}$ and so let $\bar{q}_{j}=q_{j} \upharpoonright u^{\prime} v$. Then by Definition 2.6
(3) $\bar{q}_{j}\left[t_{j}^{\prime}\right]=q_{j}\left[t_{j}^{\prime}\right] \upharpoonright u^{\prime} v=q_{j}^{\prime}\left[s_{j}^{\prime}\right] \upharpoonright u^{\prime} v=\bar{q}_{j}^{\prime}\left[s_{j}^{\prime}\right]$,
(4) $\bar{q}_{j}\left[t_{j}^{\prime \prime}\right]=q_{j}\left[t_{j}^{\prime \prime}\right] \upharpoonright u^{\prime} v=q_{j}^{\prime \prime}\left[s_{j}^{\prime \prime}\right] \upharpoonright u^{\prime} v=\bar{q}_{j}^{\prime \prime}\left[s_{j}^{\prime \prime}\right]$,
(5) $\bar{q}_{j}^{\prime}\left[r_{j}^{\prime}\right]=q_{j}^{\prime}\left[r_{j}^{\prime}\right] \upharpoonright u^{\prime} v=t_{i}^{\prime}\left(j^{\prime}, j^{\prime \prime}\right)$, by (1),
(6) $\bar{q}_{j}^{\prime \prime}\left[r_{j}^{\prime \prime}\right]=q_{j}^{\prime \prime}\left[r_{j}^{\prime \prime}\right] \upharpoonright u^{\prime} v=t_{i}^{\prime \prime}\left(j^{\prime}, j^{\prime \prime}\right)$, by (2).

Since $\left|r_{j}^{\prime}\right|=\left|s_{j}^{\prime}\right|$ and $\left|r_{j}^{\prime \prime}\right|=\left|s_{j}^{\prime \prime}\right|$, by Proposition 3.1 we have
(7) $\left|\bar{q}_{j}^{\prime}\left[r_{j}^{\prime}\right]\right|=\left|\bar{q}_{j}^{\prime}\left[s_{j}^{\prime}\right]\right|$, and
(8) $\left|\bar{q}_{j}^{\prime \prime}\left[r_{j}^{\prime \prime}\right]\right|=\left|\bar{q}_{j}^{\prime \prime}\left[s_{j}^{\prime \prime}\right]\right|$.

Now notice that dom $\bar{q}_{j} \subseteq H(j y) H(y(m))^{*}$ implies that if $\bar{q}_{j}^{-1}\left(x_{\left(j_{1}, j_{2}\right)}\right) \neq \varnothing$, then $m u_{j}\left(j_{1}, j_{2}\right)<m+1$. We can therefore apply the inductive hypothesis to all the components of $t_{j}^{\prime}$ and $t_{j}^{\prime \prime}$ appearing in $\bar{q}_{j}\left[t_{j}^{\prime}\right]$ and $\bar{q}_{j}\left[t_{j}^{\prime \prime}\right]$. Thus
(9) $\left.\left|\bar{q}_{j}\left[t_{j}^{\prime}\right]\right|=\left|\bar{q}_{j}\right| t_{j}^{\prime \prime}\right] \mid$, by Proposition 3.1.

Finally

$$
\begin{array}{rlrl}
\left|t t_{i}^{\prime}\left(j^{\prime}, j^{\prime \prime}\right)\right| & & \text { by (5) } \\
& =\left|\bar{q}_{j}^{\prime}\left[r_{j}^{\prime}\right]\right| & & \text { by (7) } \\
& =\left|\bar{q}_{j}^{\prime}\left[s_{j}^{\prime}\right]\right| & & \text { by (3) } \\
& =\left|\bar{q}_{j}\left[t_{j}^{\prime}\right]\right| & & \text { by (9) } \\
& =\left|\bar{q}_{j}\left[t_{j}^{\prime \prime}\right]\right| & & \text { by (4) } \\
& =\left|\bar{q}_{j}^{\prime \prime}\left[s_{j}^{\prime \prime}\right]\right| & & \text { by (8) } \\
& =\left|\bar{q}_{j}^{\prime \prime}\left[r_{j}^{\prime \prime}\right]\right| & & \text { by (6) } \\
& =\left|t_{i}^{\prime \prime}\left(j^{\prime}, j^{\prime \prime}\right)\right| . &
\end{array}
$$

In view of Definition 4.20 , to prove (4.18.1) it is sufficient to show that $q_{i}$ is a regular tree. We will use the characterization of Proposition 2.8.

By Lemma 4.2, $q_{i}^{-1}(y)$ is nonempty only if either $y \in X_{M_{i}}$ or $q_{i}^{\prime-1}(y)$ is nonempty. Hence $q_{i}^{-1}(y)$ is empty for all but a finite number of elements of $\Sigma \cup X_{M_{i}}$. We only need to show that $q_{i}$ satisfies Proposition 2.8(2)(ii).

Lemma 4.22. For every $i \in[n]$, let $N_{i}^{\prime} \in \omega$ be such that for every $r>0$ and every $u \in H(i, r)$, there exist $m<N_{i}^{\prime}$ and $v \in H(i, m)$ such that $q_{i}^{\prime} \upharpoonright u=q_{i}^{\prime} \upharpoonright v$. Define $N_{i}^{\prime \prime}$ similarly. Let $N_{i}=\sup \left\{N_{i}^{\prime}, N_{i}^{\prime \prime}\right\}$ and $N=\sup \left\{N_{i}: i \in[n]\right\}$. Then, for every $i \in[n]$ and $r>N, G_{i, r} \subseteq \bigcup\left\{G_{i, m} \omega^{*}: m=0, \ldots, N\right\}$.

Proof. Let $z \in G_{i, r}$. Then there exists $w \in H(i, r)$ such that $w z \in V_{i, r}^{\prime} \cup V_{i, r}^{\prime \prime}$. By definition of $N_{i}$, there exist $m<N_{i}$ and $v \in H(i, m)$ such that either $q_{i}^{\prime} \upharpoonright w=q_{i}^{\prime}\lceil v$ or $q_{i}^{\prime \prime} \upharpoonright w=q_{i}^{\prime \prime} \upharpoonright v . \quad$ In either case, $\quad v z \in V_{i, m}^{\prime} \cup V_{i, m}^{\prime \prime} \quad$ and therefore $z \in\left(H(i, m)^{-1}\left(V_{i}^{\prime} \cup V_{i}^{\prime \prime}\right)\right) \subseteq G_{i, m} \omega^{*}$.

Lemma 4.23. For every $i \in[n]$
(a) $W_{i, m}=W_{i, N}$ for every $m>N$;
(b) $W_{i, m}$ is a regular set for every $n$;
(c) $T_{i}=\bigcup\left\{T_{i, m}: m \geqslant 0\right\}$ is a regular set.

Proof. (a) Let $z \in W_{i, m}$. Then, by Lemma 4.22, $z \in\left(\cup\left\{G_{i, k} ; k=0, \ldots, N\right.\right.$, $i \in[n]\}) \omega^{*}$. If $z \in\left(\bigcup\left\{G_{i, k}: k=0, \ldots, N, i \in[n]\right\}\right) \omega^{+}$, then there exists $y<z$ such that $\quad y \in \bigcup\left\{G_{i, k}: k=0, \ldots, N, i \in[n]\right\} \subseteq G_{i, m} \cup \bigcup\left\{G_{i, k}: k=0, \ldots, m-1, i \in[n]\right\} \subseteq$ $W_{i, m} \omega^{*}$. Hence $z \in W_{l, m} \omega^{+}$, which contradicts Definition 4.12. Therefore $z \in\left(\bigcup\left\{G_{i, k}: k=0, \ldots, N, i \in[n]\right\}\right)$ and, by Remark $1.9, z \in W_{i, N}$.
(b) Since $q_{i}^{\prime}$ and $q_{i}^{\prime \prime}$ are regular trees, $V_{i, m}^{\prime}$ and $V_{i, m}^{\prime \prime}$ are regular sets. Then, by Definition 1.8, $G_{i, m}$ is regular and therefore $W_{i, m}$ is regular.
(c) By (b), $T_{i, 0}$ is regular since $D_{i}$ is regular and $H(i)$ is finite. Hence it is sufficient to consider $\bigcup\left\{T_{i, m}: m>0\right\}$. Then

$$
\begin{aligned}
& \bigcup\left\{T_{i, m}: m>0\right\} \\
&=D_{i} \cap \bigcup\left\{H(i w)\left(W_{i, m} \cap H(w(m))^{*}\right): m>0, w \in[n]^{m}\right\} \\
&=D_{i} \cap\left(\bigcup\left\{H(i w)\left(W_{i, m} \cap H(w(m))^{*}\right): m>N, w \in[n]^{m}\right\}\right. \\
&\left.\cup \bigcup\left\{H(i w)\left(W_{i, m} \cap H(w(m))^{*}\right): m=0, \ldots, N, w \in[n]^{m}\right\}\right) \\
&=D_{i} \cap\left(\left(\left(\bigcup\left\{H(i, m) W_{i, N^{\prime}}: m>N\right\}\right)\right.\right. \\
&\left.\cap\left(\bigcup\left\{H(i w) H(w(m))^{*}: m>N, w \in[n]^{m}\right\}\right)\right) \\
&\left.\cup \bigcup\left\{H(i w)\left(W_{i, m} \cap H(w(m))^{*}\right): m=0, \ldots, N, w \in[n]^{m}\right\}\right) .
\end{aligned}
$$

The result now follows since a finite union and a catenation product of regular sets is regular and from (b) and Lemma 1.14.

Finally, let $x_{\left(j^{\prime}, j^{\prime \prime}\right)} \in X_{M_{i}}$. By Definition 4.20, $q_{i}(z)=x_{\left(j^{\prime}, j^{\prime \prime}\right)}$ if and only if $z \in T_{i}$, $q_{i}^{\prime}\left[s_{i}^{\prime}\right] \upharpoonright z=g_{i j^{\prime}}^{\prime}$ and $q_{i}^{\prime \prime}\left[s_{i}^{\prime \prime}\right] \upharpoonright z=g_{i j^{\prime \prime}}^{\prime \prime}$. Hence $q_{i}^{-1}\left(x_{\left(j^{\prime}, j^{\prime \prime}\right.}\right)=T_{i} \cap\left\{u: q_{i}^{\prime}\left[s_{i}^{\prime}\right] \upharpoonright u=\right.$ $\left.g_{i j^{\prime}}^{\prime}\right\} \cap\left\{u: q_{i}^{\prime \prime}\left[s_{i}^{\prime \prime}\right] \upharpoonright u=g_{i j^{\prime \prime}}^{\prime \prime}\right\}$ and therefore, by Lemma 4.23 and the result stated in the Appendix, $q_{i}^{-1}\left(x_{\left(j^{\prime}, j^{\prime \prime}\right)}\right)$ is regular. For every $y \in \Sigma, q_{i}(z)=y$ if and only if $z \in \operatorname{dom} q_{i}\left[t_{i}^{\prime}\right]$ and $z \notin T_{i} \omega^{*}$. Hence $q_{i}^{-1}(y)=q_{i}\left[t_{i}^{\prime}\right]^{-1}(y) \cap\left(T_{i} \omega^{*}\right)^{c}$ and this set is regular since $q_{i}\left[t_{i}^{\prime}\right]=q_{i}^{\prime}\left[s_{i}^{\prime}\right]$ is a regular tree. This completes the proof that $q_{i}$ is a regular tree.

## 5. An Application

Iterative algebras deal with unique fixed points, while regular algebras deal with least fixed points. It is possible to extend the carrier of an iterative $\Sigma$-algebra $A$ to obtain a regular $\Sigma$-algebra $A_{R}$ in such a way that for every system of ideal fixed point equations, the unique solution in $A$ is also the least solution in $A_{R}$.

Definition 5.1. An iterative $\Sigma$-algebra $A$ admits a regular extension if there exist a regular $\Sigma$-algebra $A_{R}$ and $\varphi: A \rightarrow A_{R}$ such that:
(i) $\varphi \circ\left(p_{A}\right)^{+}=\left(p_{A_{R}}\right)^{\nabla} \circ \varphi$ for every $p \in T_{\Sigma}(n+k)^{n}$,
(ii) for every regular algebra $B$ and homomorphism $f: A \rightarrow B$ satisfying (i) with $f$ replacing $\varphi$, there is a unique regular homomorphism $f^{*}: A_{R} \rightarrow B$ such that $f^{*} \circ \varphi=f$.

The pair $\left(A_{R}, \varphi\right)$ is called a regular extension of $A$. An algebra $A$ admits a faithful regular extension if $A$ admits a regular extension $\left(A_{R}, \varphi\right)$ with $\varphi$ injective.

Comments. Condition (i) is the correspondence between unique fixed-point and least fixed-point. Condition (ii) is the requirement that the construction from $A$ to $A_{R}$ be "universal." It also guarantees the uniqueness, up to regular isomorphism, of regular extensions.

We will now use Theorem 3.4 to answer a question asked by Tiuryn. In [12], he proves that every iterative algebra $A$ admits a regular extension $\left(A_{R}, \varphi\right)$ and gives an example of an iterative algebra which does not admit a faithful regular extension. He then raises the question of whether $A_{R}$ is again iterative. A negative answer, in the nonfaithful case, is given in [13]. We will show that if $\varphi$ is one-to-one, then $A_{R}$ is always iterative. We first need the following result.

Proposition 5.2 [18]. Let $A$ and $B$ be iterative $\Sigma$-algebras and let $h: A \rightarrow B$ be a $\Sigma$-homomorphism. Define $K=\operatorname{ker} h$. Then $A / K$ is an iterative $\Sigma$-algebra.

Theorem 5.3. If an iterative $\Sigma$-algebra A admits a faithful regular extension $\left(A_{R}, \varphi\right)$, then $A_{R}$ is again iterative.

Proof. It can be shown [12] that if $\varphi$ is one-to-one, then $A_{R}$ is isomorphic to $R_{\Sigma}(A) /=_{A}$, where $={ }_{A}$ is a congruence relation defined as follows:

Let $h: \bar{R}_{\Sigma}(A) \rightarrow A$ be the extension of id: $A \rightarrow A$ to a $\Sigma$-homomorphism (the evaluation map for derived operations in $A$ ) and let $t, t^{\prime} \in R_{\Sigma}(A)$. Define $t={ }_{A} t^{\prime}$ if and only if there exist $n \in \omega, q \in R_{\Sigma}(n)$ and $r, r^{\prime} \in \bar{R}_{\Sigma}(A)^{n}$ such that $t=q[r]$, $t^{\prime}=q\left[r^{\prime}\right]$ and $h(r)=h\left(r^{\prime}\right)$.

Notice that $=_{A}$ is exactly the extension of ker $h$ described in Definition 3.2.
The conclusion now follows from Proposition 5.2 and Theorem 3.4.

## Appendix

Lemma. Let $t$ be a regular $\Sigma$-tree and $t_{1}$ a subtree of $t$. Then $Q=\{u \in \operatorname{dom} t$ : $\left.t \upharpoonright u=t_{1}\right\}$ is a regular set.

Proof. Let $\left\{t_{1}, \ldots, t_{m}\right\}$ be the set of all distinct subtrees of $t$ and, for every $i, j=1, \ldots, m$, let $z_{i j} \in \omega^{*}$ be such that $t_{i}\left(z_{i j}\right) \neq t_{j}\left(z_{i j}\right)$. Define $c_{j}=t_{1}\left(z_{1 j}\right)$ and let $S_{j}=t^{-1}\left(c_{j}\right)$.

Claim. $Q=\bigcap\left\{S_{j}\left(z_{1 j}\right)^{-1}: j=1, \ldots, n\right\}$.
To prove the claim, let $u \in Q$. Then, for every $j, t\left(u z_{1 j}\right)=t_{1}\left(z_{1 j}\right)=c_{j}$ and $u z_{1 j} \in S_{j}$. Hence $u \in S_{j}\left(z_{1 j}\right)^{-1}$ for every $j$. On the other hand, if $u \notin Q$, then $t \upharpoonright u=t_{j}$ for some $j \neq 1$. Hence $t\left(u z_{1 j}\right) \neq c_{j}$ which implies that $u \notin S_{j}\left(z_{1 j}\right)^{-1}$. Therefore $u \notin \bigcap\left\{S_{j}\left(z_{1 j}\right)^{-1}\right.$ : $j \in[m]\}$ and the proof of the claim is complete.

Now, since $t$ is regular, each $S_{j}$ is regular and only a finite number of them are nonempty. Hence $\bigcap\left\{S_{f}\left(z_{1 j}\right)^{-1}: j=1, \ldots, m\right\}$ is a finite intersection of regular sets and therefore regular.

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