



Note

Extremal connected graphs for independent domination number

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Abstract

A general characterization of connected graphs on n vertices having the maximum possible independent domination number of $\lfloor n + 2 - 2\sqrt{n} \rfloor$ is given. This result leads to a structural characterization of such graphs in all but a small finite number of cases. For certain situations, one of which occurs when n is a perfect square, the extremal graphs have a particularly simple structure.

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1. Introduction

A subset S of vertices of a graph G is a *dominating set* if each vertex in the graph is either in S or is adjacent to some vertex of S . The *independent domination number*, $i(G)$, of graph G is the smallest size of a dominating set which also is an independent set. An extensive discussion of this parameter can be found in [6].

Bollobas and Cockayne [1] proved that $i(G) \leq n - \gamma(g) + 1 - \lceil (n - \gamma(g))/\gamma(g) \rceil$ for any connected graph. Favaron [2] maximized the right-hand side of this inequality to show that for any connected graph $i(G) \leq n - 2\sqrt{n} + 2$, which clearly can be sharpened to $\lfloor n - 2\sqrt{n} + 2 \rfloor$. In this paper we restrict our attention to connected graphs and define a graph to be *extremal* if it obtains the Favaron bound.

Favaron [2] conjectured that $i(G) \leq n - 2\sqrt{n\delta} + 2\delta$ where δ is the minimum degree of G . This conjecture has been proven in the case of $\delta = 2$ by Glebov and Kostochka [4] and for any δ by Sun and Wang [7].

A related parameter, *total matching*, is a collection of edges and vertices of G such that no two elements of the matching are adjacent or incident. The minimum size of

a maximal total matching is denoted $\beta'_2(G)$. Gimbel and Vestergaard [3] showed that, except for a few small graphs involving an odd number of at most seven vertices, the minimum size of a maximal total matching in a connected graph obeys the Favaron bound, i.e. $\beta'_2(G) \leq n - 2\sqrt{n} + 2$. They also showed that when $n = m^2$ the graph formed by joining $m - 1$ pendant vertices to each vertex of a K_m achieves this bound. We shall use GV to designate this class of graphs.

It is not hard to see that the graphs in GV also obtain the Favaron bound for independent domination number. In particular, if I is an independent set of vertices which dominate a graph G in GV then I contains at most one vertex in K_m . If I contains no vertex in the K_m then I must contain all of the pendant vertices and $|I| = m(m - 1)$. If I contains one vertex v in the K_m then I must contain all of the $(m - 1)(m - 1)$ pendant vertices not adjacent to v and $|I| = (m - 1)(m - 1) + 1$. If $m \geq 2$ then $(m - 1)(m - 1) + 1 \leq m(m - 1)$ and $i(G) = (m - 1)(m - 1) + 1 = m^2 - 2m + 2$. If $m = 1$ then $n = 1$ and, once again, $|I| = 1 = m^2 - 2m + 2$. For graphs in GV we have $m = \sqrt{n}$ so $m^2 - 2m + 2 = n - 2\sqrt{n} + 2$ and the graphs in GV are extremal.

Observation 4 in Section 2 provides a non-structural characterization of extremal graphs. Theorem 14 of Section 3 provides insight into the structure of extremal graphs while Theorems 17, 21, and 23 of Section 4 give a structural characterization for all but a finite number of cases.

2. Preliminaries and characterization

We begin with a reformulation of Favaron's bound. Let G be a graph on n vertices and $m = \lceil \sqrt{n} \rceil$. Define t by $t = 1$ if $(m - 1)^2 < n \leq m^2 - m$ and $t = 0$ if $m^2 - m + 1 \leq n \leq m^2$.

Lemma 1. For any n , $\lfloor n - 2\sqrt{n} + 2 \rfloor = n - 2m + 2 + t$.

Proof. If $t = 1$, then $(m - 1)^2 < n$ implies $n - 2\sqrt{n} + 2 < n - 2(m - 1) + 2 = n - 2m + 4$. Also, $n \leq m^2 - m$ implies $n < m^2 - m + \frac{1}{4} = (m - \frac{1}{2})^2$ or $2\sqrt{n} < 2m - 1$. Hence, $n - 2\sqrt{n} + 2 > n - (2m - 1) + 2 = n - 2m + 3$. Thus, $\lfloor n - 2\sqrt{n} + 2 \rfloor = n - 2m + 3 = n - 2m + 2 + t$. If, on the other hand, $t = 0$, then $m^2 - m + 1 \leq n$, so $4m^2 - 4m + 1 \leq 4n$ or $2m - 1 \leq 2\sqrt{n}$. Hence, $n - 2\sqrt{n} + 2 \leq n - (2m - 1) + 2 = n - 2m + 3$. Also, $n \leq m^2$ implies $n - 2\sqrt{n} + 2 \geq n - 2m + 2$. Thus, $\lfloor n - 2\sqrt{n} + 2 \rfloor = n - 2m + 2 = n - 2m + 2 + t$. \square

By Lemma 1, a graph G is extremal if and only if $i(G) = n - 2m + 2 + t$. We shall see that the extremal graphs in GV are a special case of a more general structure. Before proceeding we introduce some terminology.

Let G be a connected graph on n vertices, let I be a maximum independent set in $V(G)$, and let G_r be the subgraph of G induced by the $r = n - |I|$ vertices in $V(G) - I$. Vertices of I will be called *outvertices*. Observe that any independent dominating set of G has the form $X \cup (I - N(X))$ where X is an independent set of vertices in G_r and $N(X)$ denotes the set of vertices in G adjacent to at least one vertex of X . For a set $X \subseteq V(G_r)$, we define the *outneighborhood* of X , denoted $N'(X)$, to be $N(X) \cap I$.

If X is just a single vertex, v , we write $N'(v)$. The *outdegree* of a vertex v in $V(G_r)$ is the size of $N'(v)$. An independent set M of vertices in $V(G_r)$ is a *full independent set*, denoted FIS, if $M \cup (I - N'(M))$ dominates G . Finally, for any vertex v in $V(G_r)$, let $X_v = \{x \in V(G_r) : N'(x) \subseteq N'(v)\}$. The next lemma shows that there is a sense in which every vertex in $V(G_r)$ generates an FIS.

Lemma 2. *If M is a maximal independent subset of X_v which contains v , then M is an FIS.*

Proof. Observe that $N'(M) = N'(v)$. Suppose w is a vertex of G which is not dominated by any vertex in M , so $w \notin M$. If w is in I , then w must be in $I - N'(M)$. If w is in G_r , then, since M is maximal and hence a dominating set of X_v , w is not in X_v and some vertex in $N'(w)$ must not be in $N'(M)$. Therefore, either w is in $I - N'(M)$ or w is adjacent to a vertex in $I - N'(M)$. It follows that $M \cup (I - N'(M))$ is an independent dominating set of G . \square

For each vertex $v \in V(G_r)$, we fix a particular maximal independent subset of X_v which contains v and denote it by $M(v)$. Notice that every independent dominating set consists of some FIS, M , along with all vertices of $I - N'(M)$. In the case of graphs in GV, each vertex of the complete graph, (our G_r), is an FIS. The concept of full independent sets leads to the characterization of extremal graphs via the following lemma.

Lemma 3. *Let G be a graph on n vertices and let G_r be an induced subgraph of G such that $|V(G_r)| = r$ and $I = V(G) - V(G_r)$ is a maximum independent set of vertices in G . For an integer a , $i(G) \geq n - a$ if and only if for every FIS M of G $|N'(M)| \leq |M| + a - r$.*

Proof. Suppose $i(G) \geq n - a$. Let M be an FIS in G_r . By definition, $M \cup (I - N'(M))$ is an independent dominating set. Hence $|I| - |N'(M)| + |M| \geq i(G) \geq n - a$. Since $|I| = n - r$, we have $n - r - |N'(M)| + |M| \geq n - a$ or $|N'(M)| \leq |M| + a - r$.

Suppose every FIS, M , of $V(G_r)$ satisfies $|N'(M)| \leq |M| + a - r$. Let D be an independent dominating set in G and $Y = D \cap G_r$. By the independence of D , $Y \cup (I - N'(Y)) = D$; hence, Y is an FIS. By hypothesis, $|N'(Y)| \leq |Y| + a - r$. Hence $|D| = |Y| + n - r - |N'(Y)| \geq n - r + r - a = n - a$. Therefore, every independent dominating set has size at least $n - a$ which implies $i(G) \geq n - a$. \square

The following characterization is an easy consequence of Lemma 3.

Corollary 4. *Let G be a graph on n vertices and let G_r and I be as previously defined. G is extremal if and only if every full independent set M of $V(G_r)$ satisfies $|N'(M)| \leq |M| + 2m - 2 - t - r$.*

Proof. By Lemma 1, G is extremal if and only if $i(G) = n - 2m + 2 + t$. Take $a = 2m - 2 - t$ in Lemma 3 and the result follows. \square

3. Restrictions on the order of G_r for extremal graphs

Corollary 4 characterizes extremal graphs in terms of full independent sets. To obtain a structural characterization we need to establish bounds on r the order of G_r . The first such bound given in the following lemma is easy but useful.

Lemma 5. *If G is extremal, then $r \leq 2m - 2 - t$.*

Proof. For any graph G , a maximum independent set dominates the entire graph. Consequently, we have $i(G) \leq n - r$. Thus, if G is extremal, $n - 2m + 2 + t \leq n - r$ and the result follows. \square

We define r to be *feasible* if and only if $r + \lceil n/r \rceil = 2m - t$. In this section we will demonstrate that, if G is extremal then r must be feasible. The next lemma shows that $r + \lceil n/r \rceil$ is never less than $2m - t$.

Lemma 6. *For any positive integer r , $r + \lceil n/r \rceil \geq 2m - t$.*

Proof. Suppose $r + \lceil n/r \rceil < 2m - t$, so $r + \lceil n/r \rceil \leq 2m - t - 1$ in which case $n/r \leq 2m - t - r - 1$. This leads to the inequalities $0 \geq r^2 - (2m - 2)r + n$ if $t = 1$ and $0 \geq r^2 - (2m - 1)r + n$ if $t = 0$. Because $n > (m - 1)^2$ if $t = 1$ and $n \geq m^2 - m + 1$ if $t = 0$, both of the quadratics in r have negative discriminants. As the coefficient of r is positive, the inequalities cannot be satisfied by any value of r . \square

Not surprisingly, r is significantly less than n for extremal graphs, as is shown in the following lemma.

Lemma 7. *If G is extremal with $n = 9$ or $n \geq 11$, then $r < n/2$.*

Proof. We consider two cases depending on the value of t .

1. Assume $t = 0$ which implies $(m - 1)^2 + m \leq n \leq m^2$. If $n \geq 11$ then $m \geq 4$. Therefore, $n > (m - 1)^2 + m - 1 = (m - 1)m \geq 4(m - 1) = 2(2m - 2)$. Also, by Lemma 5, $r \leq 2m - 2 - t = 2m - 2$, hence $r < n/2$. If $n = 9$, $m = 3$ and $2m - 2 - t = 4$; and again by Lemma 5, $r \leq 4 < n/2$.
2. Assume $t = 1$. In this case, n cannot be 9. Thus, $n \geq 11$ and $n^2 - 12n + 20 = (n - 2)(n - 10) > 0$ implying $n^2 + 4n + 4 > 16n - 16$ or $n + 2 > 4\sqrt{n - 1}$. Using the inequality $(m - 1)^2 + 1 \leq n$, we have $n + 2 > 4(m - 1)$. Therefore, $n > 4m - 6 = 2(2m - 3)$. Once more by Lemma 5, $r \leq 2m - 2 - t = 2m - 3$ and $r < n/2$. \square

Before we can show that only feasible values for r yield extremal graphs, we need several lemmas which describe the outneighborhoods of vertices in G_r .

Lemma 8. *If S is an independent set of vertices in G_r , then $|S| \leq |N'(S)|$.*

Proof. The set $S \cup (I - N'(S))$ is independent and so has size no more than $|I| = n - r$. Thus, $|S| + n - r - |N'(S)| \leq n - r$. \square

For each vertex $v \in V(G_r)$ recall that $M(v)$ denotes a fixed FIS containing v as described following Lemma 2.

Lemma 9. Suppose G is extremal. If $v \in G_r$ and $|N'(v)| = \lceil (n - r)/r \rceil + q$ for $q \geq 0$, then $|M(v)| \geq q + 1$.

Proof. Since $N'(M(v)) = N'(v)$, we have $i(G) = n - 2m + 2 + t \leq |M(v) \cup [I - N'(M(v))]| = |M(v)| + n - r - (\lceil (n - r)/r \rceil + q)$. By Lemma 6, $r + \lceil n/r \rceil \geq 2m - t$ and the result follows. \square

Note that if $r + \lceil n/r \rceil > 2m - t$ then we obtain the strict inequality $|M(v)| > q + 1$. For any set of vertices P in G_r , it will be convenient to define the *exclusive outneighborhood* of P , denoted $EN'(P)$, as the subset of vertices in $N'(P)$ whose neighborhoods are contained in P .

Lemma 10. If G is extremal, then $|EN'(P)| \leq |P| \lceil (n - r)/r \rceil$ for any $P \subseteq V(G_r)$.

Proof. We proceed by induction on the size of P . Suppose $P = \{v\}$. If $|N'(v)| \leq \lceil (n - r)/r \rceil$, the result follows. Therefore, assume $|N'(v)| = \lceil (n - r)/r \rceil + q$, $q \geq 1$. By Lemmas 8 and 9, $|N'(M(v) - \{v\})| \geq |M(v) - \{v\}| \geq q$. All the vertices in $N'(M(v) - \{v\})$ are in $N'(v)$ and none of them are exclusive neighbors of v . Therefore, $|EN'(v)| \leq |N'(v)| - |N'(M(v) - \{v\})| \leq |N'(v)| - q = \lceil (n - r)/r \rceil$.

Let $k \geq 1$, assume the result for any set, $P \subseteq V(G_r)$, of size no larger than k , and consider adding an arbitrary vertex $x \in V(G_r) - P$. If $|EN'(P \cup \{x\})| \leq |EN'(P)| + \lceil (n - r)/r \rceil$, we are done by the inductive hypothesis, so suppose $|EN'(P \cup \{x\})| = |EN'(P)| + \lceil (n - r)/r \rceil + q$, $q \geq 1$. The vertices in $EN'(P \cup \{x\})$ which are not in $EN'(P)$ must be in $N'(x)$. Hence, $|N'(x)| = \lceil (n - r)/r \rceil + q + p$, where $p \geq 0$ of the vertices of $N'(x)$ have an edge to some vertex of $G_r - P$ as well as to x . By Lemma 9, $|M(x)| \geq q + p + 1$.

Let S be the set of vertices in $M(x) - \{x\}$ which are not in P and let Q be those which are in P . By the choice of S , any vertex in $N'(S)$ cannot be in $EN'(P \cup \{x\})$ but since $N'(S) \subseteq N'(x)$, $|N'(S)| \leq p$. Thus, by Lemma 8, $|S| \leq p$ which implies that $|Q| \geq q$.

Since $N'(Q) \subseteq N'(x)$ and $x \notin P$, the set $N'(Q)$ is disjoint from $EN'(P)$. Thus, $|EN'(P)| = |EN'(P - Q)|$ and $|EN'(P \cup \{x\})| = |EN'(P)| + \lceil (n - r)/r \rceil + q = |EN'(P - Q)| + \lceil (n - r)/r \rceil + q \leq (|P| - q) \lceil (n - r)/r \rceil + \lceil (n - r)/r \rceil + q \leq (|P| + 1) \lceil (n - r)/r \rceil$, where the first inequality follows from induction and the second from $\lceil (n - r)/r \rceil \geq 1$. \square

Lemma 11. Let G be extremal with $n = 9$ or $n \geq 11$. If $r + \lceil (n - r)/r \rceil > 2m - t - 1$ then, for any $P \subseteq G_r$, $|EN'(P)| \leq |P|(\lceil (n - r)/r \rceil - 1)$.

Proof. Observe that $r + \lceil (n-r)/r \rceil > 2m - t - 1$ is equivalent to $r + \lceil n/r \rceil > 2m - t$ so as was noted following Lemma 9 $|M(v)| > q + 1$. We parallel the proof of Lemma 10. If $P = \{v\}$ and $|N'(v)| = \lceil (n-r)/r \rceil + q$, $q \geq 0$, then $|N'(M(v) - \{v\})| \geq q + 1$ so $|EN'(v)| \leq |N'(v)| - q - 1 = \lceil (n-r)/r \rceil - 1$.

Assume that upon adding an arbitrary vertex x to set P , we get $|EN'(P \cup \{x\})| = |EN'(P)| + \lceil (n-r)/r \rceil + q$, $q \geq 0$. Then $|N'(x)| = \lceil (n-r)/r \rceil + q + p$ for some $p \geq 0$ and so $|M(x)| \geq q + p + 2$. If S is the set of vertices in $M(x) - \{x\}$ which are not in P and Q are those in P , then just as in the previous proof, $|S| \leq p$ which implies in this case that $|Q| \geq q + 1$. Thus, we arrive at the inequality $|EN'(P \cup \{x\})| \leq (|P| - q - 1)(\lceil (n-r)/r \rceil - 1) + \lceil (n-r)/r \rceil + q \leq (|P| + 1)(\lceil (n-r)/r \rceil - 1)$ where the first inequality follows from induction and the second by Lemma 7. \square

Corollary 12. *If G is extremal with $n \geq 11$ or $n = 9$, then r is feasible.*

Proof. By Lemma 6, $r + \lceil n/r \rceil \geq 2m - t$. If r is not feasible, then $r + \lceil n/r \rceil > 2m - t$. Therefore, by Lemma 11, the entire set $V(G_r)$ can dominate at most $r(\lceil (n-r)/r \rceil - 1) < r((n-r)/r + 1 - 1) = n - r$ vertices. However, G is connected so all of the $n - r$ independent vertices in $V(G) - V(G_r)$ must be dominated by a vertex in $V(G_r)$. \square

The remaining cases are handled individually.

Lemma 13. *If G is extremal with $n = 7, 8$, or 10 , then r is feasible.*

Proof. If $n = 7$ or $n = 8$, then $m = 3$, $t = 0$, and feasible values for r are 2, 3, and 4. If $r = 1$, G is $K_{1,n-1}$ and G clearly is not extremal. Furthermore, by Lemma 5, $r \leq 2m - 2 - t = 4$. If $n = 10$, then $m = 4$, $t = 1$, and feasible values are 2, 3, 4, and 5. Again, it is clear that r can not be 1 for an extremal graph and, by Lemma 5, $r \leq 2m - 2 - t = 5$. \square

Theorem 14. *If G is extremal then r is feasible.*

Proof. By examining all graphs with $n \leq 6$, as shown in [5], we have found those which are extremal and observed that r is feasible for each of them. The case $n \geq 7$ follows by Corollary 12 and Lemma 13. \square

If we let $\beta(G)$ be the independence number of a graph then, for extremal graphs, $\beta(G) = n - r$. Also, by Lemma 1, $i(G) = n - 2m + 2 + t$. By definition r is feasible if and only if $r + \lceil n/r \rceil = 2m - t$. These equalities lead to the following corollary to Theorem 14 which relates $i(G)$ to $\beta(G)$.

Corollary 15. *If G is extremal then $i(G) = \beta(G) - \lceil n/(n - \beta(G)) \rceil + 2$.*

4. Special cases and structural characterizations

When r divides a sufficiently large n , we can obtain a strong bound on the outdegree of any vertex in G_r .

Lemma 16. *Suppose G is extremal with $n \geq 11$ or $n=9$. If r divides n , then $|N'(v)| \leq (n-r)/r$ for all $v \in V(G_r)$.*

Proof. Suppose there is a vertex $v \in V(G_r)$ with $|N'(v)| = (n-r)/r + q$ and $q \geq 1$. By Lemma 9, $|M(v)| \geq q+1$. By the connectivity of G , there are at most $r - |M(v)| \leq r - (q+1)$ vertices which can have the vertices in $I - N'(v)$ as their exclusive neighborhood. Therefore, by Lemma 10, $(r - (q+1))(n-r)/r \geq n - r - |N'(v)| = n - r - (n-r)/r - q$. Simplifying results in the inequality $n \leq 2r$ which contradicts Lemma 7. \square

When $n = 9$ or $n \geq 11$, extremal graphs for which r divides n have a unique structure.

Theorem 17. *Assume $n = 9$ or $n \geq 11$. If r divides n , then G is extremal if and only if G_r is complete and each $v \in G_r$ has $|N'(v)| = |EN'(v)| = (n-r)/r$.*

Proof. Suppose G is extremal and let $v \in V(G_r)$. We have $|N'(V(G_r) - \{v\})| \leq \sum_{x \in V(G_r) - \{v\}} |N'(x)| \leq (r-1)(n-r)/r$, where the second inequality follows from Lemma 16. Hence, $|I| - |N'(V(G_r) - \{v\})| \geq n - r - (r-1)(n-r)/r = (n-r)/r$. By the definition of exclusive neighborhood, $|EN'(v)| = |I| - |N'(V(G_r) - \{v\})|$. Therefore, again using Lemma 16 for the first of the following inequalities, we obtain $(n-r)/r \geq |N'(v)| \geq |EN'(v)| \geq (n-r)/r$. Thus, the inequalities may be replaced with equalities.

If G_r is not complete then there exist vertices v and w in $V(G_r)$ which are not adjacent. Taking the union of v , w , and the outvertices not dominated by v or w we obtain an independent dominating set of size $2 + n - r - 2(n-r)/r = 2 + n - (n-r)/r - (r + (n-r)/r) = 2 + n - (n-r)/r - (2m - t - 1)$. Therefore, $i(G) \leq n - 2m + 2 + t + (1 - (n-r)/r)$. This contradicts G being extremal since $n > 2r$ and r divides $n - r$ imply that $(n-r)/r \geq 2$.

For the converse, observe that a minimum independent dominating set is formed from a vertex v of G_r and the $(r-1)((n-r)/r)$ outvertices not adjacent to v . There are $1 + (r-1)(n-r)/r = n - (r + n/r) + 2 = n - 2m + t + 2$ vertices in this dominating set so G is extremal. \square

According to Theorem 17, when r divides n then up to isomorphism there is a unique extremal graph which is obtained by joining $(n-r)/r$ pendant vertices to each vertex of a K_r . This is precisely the method used by Gimble and Vestergaard to construct the graphs in GV. As a consequence of the following corollary we see that these graphs are the only extremal graphs when $n = m^2$.

Corollary 18. Suppose $n=m^2$ with $m \geq 3$ or $n=m^2-m$ with $m \geq 4$. Then G is extremal if and only if G_r is complete and each $v \in G_r$ has $|N'(v)| = |EN'(v)| = (n-r)/r$.

Proof. It is easy to see that the only feasible values for r are $r = m$ if $n = m^2$, and $r = m$ or $r = m - 1$ if $n = m^2 - m$. The result follows from Theorem 17. \square

The following two lemmas show that the structure in the $n=m^2$ case is very similar to a more general situation and allow us to determine a structural characterization for most extremal graphs. Indeed, most of the time G_r is complete and each vertex has outdegree of no more than $\lceil (n-r)/r \rceil$.

Lemma 19. If G is extremal and $m \geq 5$, ($n \geq 17$) no vertex in G_r has outdegree greater than $\lceil (n-r)/r \rceil + 1$. Furthermore, if $m \geq 5$ and some vertex in G_r has outdegree equal to $\lceil (n-r)/r \rceil + 1$, one of the following holds:

- (a) $n = m^2 - m + 1$ and $m \leq r \leq m + 1$,
- (b) $n = m^2 - m + 2$ and $m \leq r \leq m + 1$,
- (c) $n = m^2 - 2m + 2$ and $m - 1 \leq r \leq m + 1$,
- (d) $n = m^2 - 2m + 3$ and $r = m$.

Proof. Suppose some vertex, v , in G_r has outdegree $\lceil (n-r)/r \rceil + q$ with $q \geq 1$. By Lemma 9, $|M(v)| \geq q + 1$. Let P be the at most $r - q - 1$ vertices of $G_r - M(v)$. The outneighborhood of P must include the $n - r - \lceil (n-r)/r \rceil - q$ vertices of $I - N'(v)$. Thus, by Lemma 10, it must be true that $(r - q - 1)\lceil (n-r)/r \rceil \geq n - r - \lceil (n-r)/r \rceil - q$. Since r is feasible and $\lceil (n-r)/r \rceil = \lceil n/r \rceil - 1$, we obtain $(r - q - 1)(2m - r - t - 1) \geq n - 2m + t + 1 - q$ which can be transformed into the following quadratic in r : $-r^2 + (2m - t + q)r - 2mq + qt + 2q - n \geq 0$. We note that $t = t^2$ and compute the discriminant as $q^2 + (2t + 8 - 4m)q + 4m^2 - 4mt + t - 4n$.

If $t = 0$ (which implies $n \geq m^2 - m + 1$), this discriminant is non-negative when $q \geq 2m - 4 + 2\sqrt{n + 4 - 4m}$ or $q \leq 2m - 4 - 2\sqrt{n + 4 - 4m}$. If the first inequality holds then $q \geq 2m - 4 + 2\sqrt{n + 4 - 4m} \geq 2m - 4 + 2\sqrt{m^2 - 5m + 5} > 2m - 4 + 2(m - 3) = 4m - 10$. By Lemma 5, $q < r \leq 2m - 2$ which implies $2m - 2 > 4m - 10$ contrary to the assumption $m \geq 5$. Thus, $q \leq 2m - 4 - 2\sqrt{n + 4 - 4m} < 2m - 4 - 2(m - 3) = 2$ which implies $q = 1$.

If $t = 1$ (which implies $n \geq m^2 - 2m + 2$), the discriminant is non-negative when $q \geq 2m - 5 + 2\sqrt{n + 6 - 4m}$ or $q \leq 2m - 5 - 2\sqrt{n + 6 - 4m} < 2m - 5 - 2(m - 4) = 3$. Once again, the first inequality leads to a contradiction, so $q \leq 2$. However, substituting $t = 1$ and $q = 2$ along with the condition $n \geq m^2 - 2m + 2$ into our original inequality of $-r^2 + (2m - t + q)r - 2mq + qt + 2q - n \geq 0$, we obtain $-r^2 + (2m + 1)r - m^2 - 2m + 4 \geq 0$ which has a negative discriminant when $m \geq 5$. Hence, once again $q = 1$.

Finally, we substitute $q = 1$ into our inequality to obtain $-r^2 + (2m - t + 1)r - 2m + t + 2 - n \geq 0$. It is easy to see that the discriminant of this quadratic is negative if $t = 0$ and $n \geq m^2 - m + 3$ or if $t = 1$ and $n \geq m^2 - 2m + 4$ and hence, in these cases, there can be no solution. Thus the value of n is restricted to the ones given in conditions *a* through *d* and the corresponding integer solutions of the inequality provide the specified ranges for r . \square

Lemma 20. *If G is extremal, $m \geq 5$ ($n \geq 17$), and M is a maximum independent set in G_r , then $|M| \leq 2$. Furthermore, if none of conditions (a)–(d) of Lemma 19 hold then $|M| = 1$.*

Proof. Choose a maximum independent set, M , in G_r . If $|M| = 1$, we are done so assume that $|M| = 1 + q$, $q \geq 1$. Then, because any maximal independent set in G_r is also an FIS, by Corollary 4, $|N'(M)| \leq (q + 1) + 2m - 2 - t - r$ which, because r is feasible, is equal to $\lceil (n - r)/r \rceil + q$. As in the proof to Lemma 19, let P be the $r - q - 1$ vertices of $G_r - M(v)$. Again, the outneighborhood of P must include the at least $n - r - \lceil (n - r)/r \rceil - q$ vertices of $I - N'(M)$, and we have the same quadratic inequality as in Lemma 19. \square

Most extremal graphs do not satisfy conditions (a)–(d) of Lemma 19. We immediately obtain a structural characterization of these graphs.

Theorem 21. *If $n \geq 17$ and none of conditions (a)–(d) of Lemma 19 hold, then G is extremal if and only if r is feasible, G_r is complete and each vertex, v , of G_r has $|N'(v)| \leq \lceil (n - r)/r \rceil$.*

Proof. Theorem 14 along with Lemmas 19 and 20 prove necessity. To show sufficiency, let M be an FIS in G_r . The completeness of G_r implies $M = \{v\}$ for some vertex v in $V(G_r)$. Therefore, $|N'(M)| = |N'(v)| \leq \lceil (n - r)/r \rceil = |M| + \lceil (n - r)/r \rceil - 1 = |M| + 2m - 2 - t - r$ where the last equality follows from the feasibility of r . Thus, G is extremal by Corollary 4. \square

From this theorem we see that when $n \geq 17$ and none of conditions (a)–(d) of Lemma 19 hold the general construction of an extremal graph is similar to the construction when r divides n . First join $n - r$ pendant vertices to the vertices of a K_r , being careful not to join more than $\lceil (n - r)/r \rceil$ vertices to any particular vertex. Then add additional edges between the $n - r$ vertices and the vertices of the K_r as long as $|N'(v)| \leq \lceil (n - r)/r \rceil$ for each v in the K_r . Note that the joined vertices in the resultant graph do not need to be monovalent. Also, these graphs are not unique up to isomorphism for a particular n and r . For example there are two non-isomorphic extremal graphs with $n = 19$ and $r = 5$.

We can also give a structural characterization of graphs which do satisfy conditions (a)–(d) of Lemma 19. Here, the structure of G_r is slightly more complicated as specified in the following lemma.

Lemma 22. *If G is extremal and $m \geq 5$ ($n \geq 17$), then either G_r is complete or one of conditions (a)–(d) of Lemma 19 hold and G_r contains K_{r-1} as a subgraph.*

Proof. Choose a maximum independent set, M , in G_r . If $|M| = 1$ then G_r is complete and we are done. Otherwise, by Lemma 20, $|M| = 2$ and one of conditions (a)–(d) of Lemma 19 must hold. Suppose G_r does not contain K_{r-1} as a subgraph in which case $V(G_r)$ contains two disjoint sets of independent vertices, $\{u, v\}$ and $\{x, y\}$. These

sets are maximum independent sets in $V(G_r)$ so each must be an FIS. It follows that $2 + n - r - |N'(\{u, v\})| \geq n - 2m + 2 + t$ which simplifies to $2m - t - r \geq |N'(\{u, v\})|$. Similarly, $2m - t - r \geq |N'(\{x, y\})|$. Let $P = V(G_r) - \{u, v, x, y\}$ so $|P| = r - 4$ and $|EN'(P)| = n - r - |N'\{u, v, x, y\}| \geq n - r - 2(2m - t - r)$. Thus by Lemma 10, $(r - 4)\lceil(n - r)/r\rceil \geq n - r - 2(2m - t - r)$ or $(r - 4)\lceil n/r \rceil + 4 \geq n - 4m + 2t + 2r$. If condition (a) of Lemma 19 holds, then $n = m^2 - m + 1$, $t = 0$, and $r = m$ or $r = m + 1$. If $r = m + 1$, we see that $\lceil n/r \rceil = \lceil (m^2 - m + 1)/(m + 1) \rceil = \lceil (m^2 + m - 2m - 2 + 3)/(m + 1) \rceil = m - 1$. In this case, our inequality becomes $(m - 3)(m - 1) \geq m^2 - m + 1 - 4m + 2m + 2$ which reduces to $4 \geq m$, contradicting $m \geq 5$. The other seven cases specified by conditions (a)–(d) of Lemma 19 imply contradictions by similar reasoning and the result follows. \square

The more complicated structure of G_r for graphs which satisfy conditions (a)–(d) of Lemma 19 leads to the following more complicated structure theorem for extremal graphs.

Theorem 23. *If $n \geq 17$ and one of the four conditions of Lemma 19 holds, then G is extremal if and only if r is feasible, G_r contains K_{r-1} as a subgraph, every pair of independent vertices $\{v, w\}$ in G_r satisfies $|N'(\{v, w\})| \leq \lceil(n - r)/r\rceil + 1$, and each vertex, v , of G_r has $|N'(v)| \leq \lceil(n - r)/r\rceil + 1$ with equality only if v is adjacent to at most $r - 2$ vertices of G_r .*

Proof. To prove necessity, we suppose G is extremal. Theorem 14 along with Lemmas 22 and 19 imply, respectively, r is feasible, G_r contains K_{r-1} as a subgraph, and each vertex, v , of G_r has $|N'(v)| \leq \lceil(n - r)/r\rceil + 1$. Suppose v is a vertex of G_r with $|N'(v)| = \lceil(n - r)/r\rceil + 1$ and v is adjacent to all of the other $r - 1$ vertices of G_r . Then v is an FIS with $|N'(v)| = \lceil(n - r)/r\rceil + 1 > \lceil(n - r)/r\rceil + |v| - 1 = |v| + 2m - 2 - t - r$ which contradicts G being extremal by Corollary 4. A similar contradiction is obtained from any pair of independent vertices, $\{v, w\}$ in G_r with $|N'(\{v, w\})| \geq \lceil(n - r)/r\rceil + 2$.

For sufficiency, let M be an FIS in G_r . The structure of G_r ensures that $|M| \leq 2$. Suppose $|M| = 1$, so $M = \{v\}$ for some vertex v in G_r . If $|N'(v)| = \lceil(n - r)/r\rceil + 1$ then v is adjacent to at most $r - 2$ vertices in G_r , so there exists a vertex u in G_r such that $\{u, v\}$ is independent. We observe that $N'(u)$ is not contained in $N'(v)$ because M is an FIS; hence, $|N'(u, v)| \geq \lceil(n - r)/r\rceil + 2$, a contradiction. Therefore, if $M = \{v\}$ we have $|N'(v)| \leq \lceil(n - r)/r\rceil$ and $|N'(M)| \leq |M| + 2m - 2 - t - r$ follows just as in the proof to Theorem 21. If, on the other hand, $M = \{u, v\}$ then the condition $|N'(\{u, v\})| \leq \lceil(n - r)/r\rceil + 1$ and r being feasible also ensures that $|N'(M)| \leq |M| + 2m - 2 - t - r$. We conclude by Corollary 4 that G is extremal. \square

When one of the four conditions of Lemma 19 holds we can construct the extremal graphs by first joining $n - r$ pendant vertices to a graph G_r on r vertices that contains a K_{r-1} and then adding additional edges between the $k - r$ vertices and the vertices of G_r , being careful that each additional edge abides by the bounds of the theorem on outdegrees and that the final graph is connected. These graphs are also not unique up to isomorphism for a particular n and r .

The only extremal graphs not covered by the results in this section or by Theorem 14 are those where r does not divide n and $n = 7, 8, 10, 11, 13, 14$, or 15 . Each of these graphs can be constructed on a case by case basis.

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