Available online at www.sciencedirect.com
SCIENCE DIRECT
DISCRETE APPLIED MATHEMATICS
Discrete Applied Mathematics 154 (2006) 1633 – 1639
www.elsevier.com/locate/dam

Note

The elimination procedure for the competition number is not optimal ☆

Stephen G. Hartke

Department of Mathematics, University of Illinois, Urbana, IL 61801, USA

Received 20 January 2004; received in revised form 24 May 2005; accepted 8 November 2005
Available online 22 March 2006

Abstract

Given an acyclic digraph D, the competition graph $C(D)$ is defined to be the undirected graph with $V(D)$ as its vertex set and where vertices x and y are adjacent if there exists another vertex z such that the arcs (x, z) and (y, z) are both present in D. The competition number $k(G)$ for an undirected graph G is the least number r such that there exists an acyclic digraph F on $|V(G)| + r$ vertices where $C(F)$ is G along with r isolated vertices. Kim and Roberts [The Elimination Procedure for the Competition Number, Ars Combin. 50 (1998) 97–113] introduced an elimination procedure for the competition number, and asked whether the procedure calculated the competition number for all graphs. We answer this question in the negative by demonstrating a graph where the elimination procedure does not calculate the competition number. This graph also provides a negative answer to a similar question about the related elimination procedure for the phylogeny number introduced by the current author in [S.G. Hartke, The Elimination Procedure for the Phylogeny Number, Ars Combin. 75 (2005) 297–311].

MSC: 05C20; 68R10

Keywords: Competition graph; Competition number; Elimination procedure; Phylogeny graph; Phylogeny number

1. Introduction

Given an acyclic digraph D, the competition graph $C(D)$ is defined to be the undirected graph with $V(D)$ as its vertex set and where vertices x and y are adjacent if there exists another vertex z such that the arcs (x, z) and (y, z) are both present in D. Competition graphs were introduced by Cohen [1] to study ecosystems. The vertices of an acyclic digraph D, known as a food web, represent species, and the arc (x, z) indicates that z is a prey of x. An edge exists between two vertices x and y in $C(D)$ if and only if x and y have a common prey. In addition to ecology, competition graphs have also found application in studying communication over noisy channels, interfering radio transmissions, and models of complex economic and energy problems—see the discussions in Raychaudhuri and Roberts [10] and Roberts [12]. Lundgren [7], Roberts [13], and Kim [4] survey the extensive literature of competition graphs.

In [11], Roberts noted that for any graph G, G along with r isolated vertices is the competition graph of some acyclic digraph if r is sufficiently large. The competition number $k(G)$ is defined to be the least such r. In general, determining

☆ Supported in part by a National Defense Science and Engineering Graduate Fellowship and National Science Foundation Grants EIA 0205116, DBI 9982983, and SBR 9709134.

E-mail address: hartke@math.uiuc.edu.

0166-218X/$ - see front matter © 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.dam.2005.11.009

the competition number of a graph is difficult: Opsut [8] showed that this problem is NP-complete. Kim and Roberts in [11,5] have determined the competition number of graphs with 0, 1, and 2 triangles, but for few other graph classes is the competition number known. As another approach, Roberts considered using an elimination procedure to calculate $k(G)$. An elimination procedure takes as input G and an ordering $\mathcal{O} = v_1, \dots, v_n$ of the vertices of G and produces an acyclic digraph D such that $C(D) = G \cup I_r$; that is, the competition graph of D is G along with r isolated vertices. The procedure "eliminates" each vertex in order by ensuring that all of the edges incident on the vertex will appear in $C(D)$. The goal is to create an elimination procedure that for some ordering, $\mathcal{O}$ outputs an acyclic digraph D where $|V(D) \setminus V(G)| = k(G)$.

Elimination procedures which seek to determine a graph-theoretical parameter through step-wise elimination of vertices have various applications in graph theory. A common example are algorithms for determining if a graph is chordal by finding perfect elimination orders such that each vertex is "simplicial" in the graph of remaining vertices. Roberts [11] was led to consider an elimination procedure for the competition number through variants of perfect elimination used by Parter [9], Rose [15], and Golumbic [2] in connection with numerical analysis. Here, elimination procedures are used to find a good order for eliminating variables during Gaussian elimination of a matrix.

Opsut [8] found an example of a graph G where Roberts' original elimination procedure does not calculate the competition number $k(G)$, thus giving a counterexample to Roberts' conjecture that the procedure always calculates $k(G)$. Kim and Roberts [6] then modified the elimination procedure and asked whether their modified procedure works for all graphs. They were able to show that the modified version calculates the competition number for a large class of graphs, the so-called "kite-free" graphs.

In this work, we present a graph L where Kim and Roberts' elimination procedure does not always calculate the competition number, in the following sense: for each order $\mathcal{O}$ of vertices of L, the elimination procedure can produce an acyclic digraph with more than $k(L)$ additional vertices.

This graph also is a counterexample to a similar question about the related elimination procedure for the phylogeny number. Given an acyclic digraph D, the phylogeny graph $P(D)$ is defined to be the undirected graph with $V(D)$ as its vertex set and with adjacencies as follows: two vertices x and y are adjacent if one of the arcs (x, y) or (y, x) is present in D, or if there exists another vertex z such that the arcs (x, z) and (y, z) are both present in D. Phylogeny graphs were introduced by Roberts and Sheng [14] from an idealized model for reconstructing phylogenetic trees in molecular biology. For a simple graph G, the phylogeny number $p(G)$ is the least number r such that there exists an acyclic digraph D on $|V(G)| + r$ vertices where G is an induced subgraph of $P(D)$. The current author introduced in [3] an elimination procedure for the phylogeny number based on Kim and Roberts' modified elimination procedure for the competition number. It was also asked whether this procedure calculated the phylogeny number for all graphs. The graph L shown in this paper also shows that the elimination procedure for the phylogeny number does not calculate the phylogeny number for all graphs.

Note that the focus of creating elimination procedures is not on efficiency, since calculating the competition number or the phylogeny number with an elimination procedure requires $n!$ runs (one for each ordering of the vertices). In fact, calculating both the competition number [8] and the phylogeny number [14] have been shown to be NP-complete. Instead, the focus is on whether an elimination procedure *could* be created that exactly calculates the relevant number. This is interesting both for historical reasons and because many of the practical examples are relatively small, exactness is sometimes more important than efficiency. Our result shows that this might be much more difficult than originally thought.

In this work, the graph G for which we wish to calculate the competition or phylogeny number need not be connected. For convenience, we will sometimes also describe a subgraph H of a graph G only as "consisting of" certain edges of G. It is understood that H has no isolated vertices: the vertices of H are only the endpoints of edges in H.

2. The elimination procedure for the competition number

We will first formalize our definitions and describe Kim and Roberts' elimination procedure using our terminology; however, its workings are the same as the elimination procedure described in [6].

Definition 1. Let $D = (V, A)$ be an acyclic digraph. The *competition graph* $C(D)$ is a simple graph with vertex set V where two vertices x and y are adjacent in $C(D)$ if there exists a vertex z such that both (x, z) and (y, z) are arcs in D. From the ecological origins of competition graphs, z is known as a *prey* of x if (x, z) is an arc of D.

Definition 2. For a simple graph G, the *competition number* $k(G)$ is the least number r such that there exists an acyclic digraph D on $|V(G)|+r$ vertices where $C(D)$ is G along with r isolated vertices.

We give an informal description of the elimination procedure for the competition number before presenting it formally. Given a graph G and an ordering $\mathcal{O}=v_1,\ldots,v_n$ of the vertices of G, we eliminate each vertex iteratively, in the process building up an acyclic digraph $D=D_n$ with the desired properties. When eliminating vertex v_i, we "cover" every edge incident to v_i that has not been covered in a previous iteration. By "covering" an edge e, we mean that the appropriate arcs and possibly vertices have been added to D_i so that e is an edge in $C(D_i)$. Here, D_i is the acyclic digraph built up through the ith iteration. The subgraph G_i is a spanning subgraph of G that contains the edges of G that have not been covered in an iteration prior to the ith iteration. The subgraph G_i' consists of the edges of G_i that are incident on v_i, and so the edges of G_i' must be covered in the ith iteration. Cliques are used to maximize the coverage of G_i' using the least number of added vertices. If C is a clique, then by adding arcs in D_i from the vertices of C to a common vertex x, all of the edges in C appear in $C(D_i)$. Thus, all of the vertices in C are "preying" on the same "species" x, and hence competing with each other. The set S_i contains the vertices available as prey at the beginning of the ith iteration, and a vertex x is chosen from S_i for each clique C in the clique cover of G_i'. If not enough prey vertices are available in S_i, then additional new vertices are added to D_i so that there are enough prey vertices among S_i and these new vertices.

Note that implementing the elimination procedure is not straightforward: in each iteration a minimum edge cover by maximal cliques must be obtained, and this problem is already NP-complete. However, as discussed below, for each order of the vertices there is a choice of edge clique covers where the elimination procedure attains the competition number.

The improvement of Kim and Roberts' modified elimination procedure over Roberts' original procedure was in recognizing that the edges in G_i' are the *only* edges that must be covered in the ith iteration. Roberts' original procedure required that *all* edges in the subgraph of G_i induced by $N_{G_i}[v_i]$ be covered in the ith iteration. For choosing the cliques, Kim and Roberts utilize the subgraph H_i consisting of the edges from v_i to vertices of higher index. The cliques covering G_i' are chosen from H_i, even though some of the edges in H_i might already be covered. By using maximal cliques of H_i, the size of the clique cover may be smaller and more uncovered edges that are not in G_i' may be covered.

Definition 3. Let $E_G(v)$ denote the subgraph of G with vertex set $N_G[v]$ and containing only those edges of G incident to the vertex v.

The Kim–Roberts elimination procedure for the competition number.

Input: A graph G and an ordering $\mathcal{O}=v_1,v_2,\ldots,v_n$ of the vertices of G.

Output: An acyclic digraph $D=D_n$ such that $C(D)$ is G along with M additional isolated vertices.

Initialization: Set D_0 to the digraph with vertex set $V(G)$ and no arcs. D_i is an acyclic digraph constructed during the ith iteration.

Set $G_1=G$. G_i is a spanning subgraph of G that contains the edges of G that do not appear in $C(D_{i-1})$.

Set $S_1=\emptyset$. S_i is a set of vertices available as prey at the beginning of the ith iteration.

ith Iteration, $i=1,\ldots,n$: Set G_i' to $E_{G_i}(v_i)$, and set H_i to the subgraph of G induced by $\{v_i\}\cup\{v_j : j>i \text{ and } v_j\in N_G(v_i)\}$. Let $\mathcal{E}_i=\{C_1,\ldots,C_{k_i}\}$ be a minimum size edge covering of G_i' by maximal cliques of H_i, ordered arbitrarily. Note that if v_i is isolated in G_i', then $\mathcal{E}_i$ is empty and $k_i=0$. Form G_{i+1} from G_i by removing the edges of C_j from G_i for $j=1,\ldots,k_i$.

Form the digraph D_i by adding vertices and arcs to D_{i-1} as follows: let $\ell_i=\min(k_i,|S_i|)$. Pick ℓ_i distinct vertices $u_1,\ldots,u_{\ell_i}$ from S_i. If $|S_i|<k_i$, then add $k_i-|S_i|$ additional vertices $u_{\ell_i+1},\ldots,u_{k_i}$ to D_i. For each clique $C_j\in\mathcal{E}_i$, add the arcs (w,u_j) to D_i for each $w\in C_j$.

Form S_{i+1} by $S_{i+1}=(S_i\backslash\{u_1,\ldots,u_{k_i}\})\cup\{v_i\}$.

Definition 4. Given a graph G and an ordering $\mathcal{O}$, let $\mathcal{E}=\mathcal{E}(G,\mathcal{O})=\{\mathcal{E}_1,\mathcal{E}_2,\ldots,\mathcal{E}_n\}$ be edge clique coverings obtained during the Kim–Roberts elimination procedure. Of course, the notation is ambiguous since the way to choose the clique covers $\mathcal{E}_i$ is not completely specified in the procedure. The *elimination number* $M(G)$ is the minimum of $M(G,\mathcal{O},\mathcal{E})$ over all orders $\mathcal{O}$ and some $\mathcal{E}$ obtained when using $\mathcal{O}$.

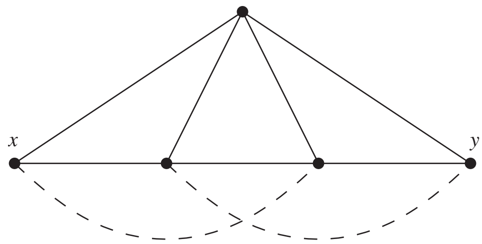

Fig. 1. A kite. The solid edges must be present, the dotted edges cannot be present, and the edge xy may or may not be present.

Kim and Roberts showed that for certain classes of graphs, if $M(G)$ is this minimum and is attained for $\mathcal{O}$ and some $\mathcal{E}$, then it is attained for the same $\mathcal{O}$ and any $\mathcal{E}$ corresponding to $\mathcal{O}$. If this is the case, $M(G)$ is unambiguously defined.

The determination of necessary and sufficient conditions for $M(G)$ to be unambiguously defined is an interesting open problem. Lemma 24 of [3] shows that there is always a "right" clique cover for each order such that the minimum over vertex orders attains the competition number $k(G)$. Kim and Roberts showed that there is a class of graphs, known as the kite-free graphs, for which $M(G)$ is unambiguously defined and equals $k(G)$.

Definition 5. A *kite* is the configuration shown in Fig. 1. In a kite, the solid edges must be present, and the dotted edges cannot be present. The edge between vertices x and y may or may not be present. A *kite-free* graph does not have a kite as a configuration, meaning that neither of the two graphs on five vertices that are kites are present as induced subgraphs.

Theorem 6 (*Kim and Roberts [6]*). *For a kite-free graph G, the elimination number $M(G)$ is unambiguously defined and equals the competition number $k(G)$.*

The current author presented an alternate proof of Theorem 6 in [3].

Kim and Roberts asked if $M(G)$ is unambiguously defined and equals $k(G)$ for all graphs. However, in the next section, we exhibit a graph L such that for each order $\mathcal{O}$ there is a choice of clique cover $\mathcal{E}_i$ in the Kim–Roberts elimination procedure such that $M(G, \mathcal{O}, \mathcal{E}) > k(G)$. This answers the Kim–Roberts question negatively.

3. The counterexample

By Theorem 6, any graph where the elimination procedure is not optimal must contain a kite. The graph L in Fig. 2 contains several kites, but we focus our attention on the kite with vertices $\{1, 2, 3, 10, 11\}$. When eliminating vertices 1 or 2 first, two different clique covers of two triangles each can be used to eliminate the incident edges. One of these choices is a *good* choice for the edge clique cover, but one is a *bad* choice. Our effort in constructing the counterexample is to force 1 or 2 to be eliminated first, so that a bad choice is made. When 1 or 2 is not eliminated first, then we will show that no choices allow the elimination procedure to attain the competition number.

Proposition 7. *For each ordering $\mathcal{O}$ of the vertices of the graph L in Fig. 2, there is a choice of edge clique coverings $\mathcal{E}$ within the elimination procedure such that $M(L, \mathcal{O}, \mathcal{E}) > 2$.*

Proof. Let $\mathcal{O} = v_1, v_2, \ldots, v_{14}$ be an ordering of the vertices of L. We consider several cases:

Case 1: $v_1 = 1$. We make the bad choice of the cliques $\{1, 2, 3\}$ and $\{1, 10, 11\}$. Any choice for v_2 other than vertex 2 cannot be eliminated without increasing the number of extra vertices added to D since its remaining incident edge cannot be covered by a single clique. Thus, v_2 must be vertex 2. But after vertex 2 is eliminated, no vertex has its remaining incident edges coverable by a single clique. Thus, $M(L, \mathcal{O}, \mathcal{E}) > 2$ if $v_1 = 1$.

Case 2: $v_1 = 2$. We make the bad choice of the cliques $\{2, 1, 3\}$ and $\{2, 6, 10\}$. Analogously to Case 1, vertex 1 is the only vertex that can then be eliminated as v_2 without increasing the number of added vertices, but after that no vertex has its remaining incident edges coverable by a single clique.

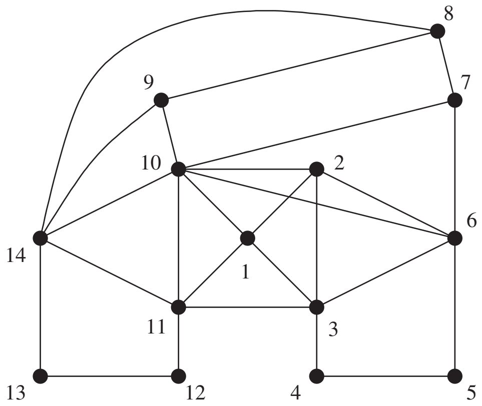

Fig. 2. The graph L.

Case 3: $v_1 = 3$, 6, 10, 11, or 14. Each of these vertices requires at least three cliques to cover its incident edges.

Case 4: $v_1 = 4$ or 5. One of these vertices can be eliminated using two cliques, and the other is then the only vertex that can be eliminated without increasing the number of added vertices. But then no vertex has its remaining incident edges coverable by a single clique.

Case 5: $v_1 = 7$. Vertex 7 can be eliminated with two cliques. Then vertices 8 and 9 in that order are the only vertices that can then be eliminated without increasing the number of added vertices. But after that no vertex has its remaining incident edges coverable by a single clique.

Case 6: $v_1 = 8$ or 9. One of these vertices can be eliminated using two cliques, and then the other vertex and vertex 7 are the only vertices that can then be eliminated without increasing the number of added vertices. Vertex 7 must be eliminated after vertex 8 for this to be the case. But after that no vertex has its remaining incident edges coverable by a single clique.

Case 7: $v_1 = 12$ or 13. One of these vertices can be eliminated using two cliques, and the other is the only vertex that can then be eliminated without increasing the number of added vertices. But after that no vertex has its remaining incident edges coverable by a single clique.

Thus, there exists a choice $\mathscr{E}$ of clique cover such that $M(L, \mathscr{O}, \mathscr{E}) > 2$ for any order $\mathscr{O}$. $\square$

Proposition 8. *The competition number of the graph L in Fig. 2 is* 2.

Proof. First note that there is no vertex in L whose incident edges can be covered with one clique. Thus, $k(L) \geqslant 2$. But the elimination procedure using the order $1, 2, \ldots, 14$ and the good choice of cliques $\{1, 2, 10\}$ and $\{1, 3, 11\}$ for vertex 1 produces an elimination number $M(L, \mathscr{O}, \mathscr{E})$ of 2. Thus, $k(L) = 2$. $\square$

4. The elimination procedure for the phylogeny number is not optimal

Because of the similarities in the elimination procedures for the competition and phylogeny numbers, the same graph L is also a counterexample to the optimality of the elimination procedure for the phylogeny number.

Definition 9. Let $D = (V, A)$ be an acyclic digraph. The *phylogeny graph* $P(D)$ is a simple undirected graph with vertex set V and with adjacencies as follows: two vertices x and y are adjacent if one of the arcs (x, y) or (y, x) is present in D, or if there exists another vertex z such that the arcs (x, z) and (y, z) are both present in D.

Definition 10. For a simple graph G, the phylogeny number $p(G)$ is the least number r such that there exists an acyclic digraph D on $|V(G)| + r$ vertices where G is an induced subgraph of $P(D)$.

The competition number problem is essentially a problem about minimum edge clique covers, where the "value" of a cover is computed in a weighted manner. The phylogeny number problem is similar in this regard. Thus, we can formulate an elimination procedure for the phylogeny number similar to that of the competition number and obtain analogous results. The elimination procedure for the phylogeny number was introduced in [3]. Note that the only difference from the elimination procedure for the competition number is how edges of G are "accounted for" in D.

The elimination procedure for the phylogeny number.

Input: A graph G and an ordering $\mathcal{O} = v_1, v_2, \ldots, v_n$ of the vertices of G.

Output: An acyclic digraph $D = D_n$ such that G is an induced subgraph of $P(D)$.

Initialization: Set D_0 to the digraph with vertex set $V(G)$ and no arcs. D_i is an acyclic digraph constructed at the ith iteration.

Set $G_1 = G$. G_i is a spanning subgraph of G that contains the edges of G that do not appear in $P(D_{i-1})$.

ith Iteration, $i = 1, \ldots, n$: Set G_i' to $E_{G_i}(v_i)$, and set H_i to the subgraph of G induced by $\{v_i\} \cup \{v_j : j > i$ and $v_j \in N_G(v_i)\}$. Let $\mathcal{E}_i = \{C_1, \ldots, C_{k_i}\}$ be a minimum size edge covering of G_i' by maximal cliques of H_i, ordered arbitrarily. Note that if v_i is isolated in G_i', then $\mathcal{E}_i$ is empty and $k_i = 0$. Form G_{i+1} from G_i by removing the edges of C_j from G_i for all j.

Form the digraph D_i by adding vertices and arcs to D_{i-1} as follows: add the arcs (w, v_i) to D_i for all vertices $w \in C_1 \backslash \{v_i\}$. For each clique $C_j \in \mathcal{E}_i \backslash \{C_1\}$, add a vertex b_j to $V(D_i)$ and add the arcs (w, b_j) to D_i for each $w \in C_j$.

Definition 11. Given a graph G and an ordering $\mathcal{O}$, let $\mathcal{E} = \mathcal{E}(G, \mathcal{O}) = \{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n\}$ be edge clique coverings obtained during the elimination procedure for the phylogeny number. Again, the notation is ambiguous since the way to choose the clique covers $\mathcal{E}_i$ is not completely specified in the procedure. The *phylogeny elimination number* $e_{\mathrm{p}}(G)$ is the minimum of $e_{\mathrm{p}}(G, \mathcal{O}, \mathcal{E})$ over all orders $\mathcal{O}$ and some $\mathcal{E}$ obtained when using $\mathcal{O}$.

As with $M(G)$, the determination of necessary and sufficient conditions for $e_{\mathrm{p}}(G)$ to be unambiguously defined is an interesting open problem. Lemma 9 of [3] shows that there is always a "right" clique cover for each order such that the minimum over orders attains the phylogeny number $p(G)$. Analogous to Kim and Roberts' result, the current author showed that for the kite-free graphs $e_{\mathrm{p}}(G)$ is unambiguously defined and equals $p(G)$.

Theorem 12 (*Hartke [3]*). *For a kite-free graph G, the phylogeny elimination number $e_{\mathrm{p}}(G)$ is unambiguously defined and equals the phylogeny number $p(G)$.*

Because of the similarities in the elimination procedures for the competition and phylogeny numbers, the same graph L in Fig. 2 also shows that the elimination procedure for the phylogeny number does not always attain $p(G)$. Both of the following propositions are proved in a fashion similar to Propositions 7 and 8 above.

Proposition 13. *For each ordering $\mathcal{O}$ of the vertices of the graph L in Fig. 2, there is a choice of edge clique coverings $\mathcal{E}_i$ such that the number of added vertices by the elimination procedure for the phylogeny number is greater than* 1.

Proposition 14. *The phylogeny number of the graph L in Fig. 2 is* 1.

5. Conclusion

Many questions still exist about the existence and efficacy of elimination procedures that calculate the competition number or the phylogeny number of a graph. Despite the existence of the counterexample graph L, the Kim–Roberts elimination procedure is still of interest, particularly in determining for which graphs the procedure calculates $k(G)$. For instance, is L the smallest graph where the elimination procedure fails, or is there a smaller example? Is there an example with only one kite? In some graphs that contain kites there is no choice for the clique cover $\mathcal{E}(G, \mathcal{O})$. Is the elimination procedure exact for these graphs? Can we characterize the graphs that contain kites but admit no choice of clique cover? A complete characterization for all graphs of when the procedure is exact and when it is not is still open.

The existence of the graph L where both the Kim–Roberts elimination procedure and the elimination procedure for the phylogeny number fail suggests a natural question: can a different elimination procedure be created that succeeds for all graphs? To answer this question, a more strict definition of what constitutes an elimination procedure is needed. One reasonable condition might be to restrict what portion of the graph the procedure may consider when eliminating a vertex v. For instance, the procedure might only be able to consider vertices that are a fixed distance from v. In such instances where an elimination procedure can only consider local information, it seems unlikely that the procedure will calculate $k(G)$ or $p(G)$ for all graphs, even with the power of taking a minimum over all vertex orders. One indication supporting this view would be if it can be shown that the Kim–Roberts elimination procedure needs to solve an NP-complete problem about cliques to guarantee producing the competition number, despite the extra power of the minimum over orders. It might be possible to prove the NP-completeness using the "widgetlike" construction of the graph L. Another reasonable condition is requiring that all computation for eliminating a vertex is done in polynomial time. It seems in this case that examining factorial number of different vertex orders should give sufficient power to exactly solve either problem. However, an explicit procedure that accomplishes this is still needed.

A more general study of elimination procedures might also give insight into what graph parameters could be effectively calculated using elimination properties. Other parameters related to clique coverings are natural candidates, but perhaps other parameters such as chromatic number could also be considered.

Acknowledgments

The author would like to thank Fred Roberts and Suh-Ryung Kim for helpful discussions and encouragement and an anonymous referee for suggestions that improved the readability of the paper.

References

[1] J.E. Cohen, Interval graphs and food webs: a finding and a problem, RAND Corporation Document 17696-PR, Santa Monica, CA, 1968.
[2] M.C. Golumbic, A note on perfect Gaussian elimination, J. Math. Anal. Appl. 64 (1978) 455–457.
[3] S.G. Hartke, The elimination procedure for the phylogeny number, Ars Combin. 75 (2005) 297–311.
[4] S.-R. Kim, The competition number and its variants, in: J. Gimbel, J.W. Kennedy, L.V. Quintas (Eds.), Quo Vadis Graph Theory? Ann. Discrete Math. 55 (1993) 313–325.
[5] S.-R. Kim, F.S. Roberts, Competition numbers of graphs with a small number of triangles, Discrete Appl. Math. 78 (1997) 153–162.
[6] S.-R. Kim, F.S. Roberts, The elimination procedure for the competition number, Ars Combin. 50 (1998) 97–113.
[7] J.R. Lundgren, Food webs, competition graphs, competition-common enemy graphs, and Niche graphs, in: F.S. Roberts (Ed.), Applications of Combinatorics and Graph Theory in the Biological and Social Sciences, IMA Volumes in Mathematics and its Applications, vol. 17, Springer, New York, 1989, pp. 221–243.
[8] R.J. Opsut, On the computation of the competition number of a graph, SIAM J. Algebraic Discrete Methods 3 (1982) 420–428.
[9] S. Parter, The use of linear graphs in Gauss elimination, SIAM Rev. 3 (1961) 119–130.
[10] A. Raychaudhuri, F.S. Roberts, Generalized competition graphs and their applications, in: P. Brucker, R. Pauly (Eds.), Methods of Operations Research, Academic Press, New York, London, 1969, pp. 295–311.
[11] F.S. Roberts, Food webs, competition graphs, and the boxicity of ecological phase space, in: Y. Alavi, D. Lick (Eds.), Theory and Applications of Graphs, Springer, New York, 1978, pp. 477–490.
[12] F.S. Roberts, Graph theory and its applications to problems of society, CBMS-NSF Monograph No. 29, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1978.
[13] F.S. Roberts, Competition graphs and phylogeny graphs, in: L. Lovasz (Ed.), Graph Theory and Combinatorial Biology, Bolyai Society of Mathematical Studies, vol. 7, Bolyai Mathematical Society, Budapest, Hungary, 1999, pp. 333–362.
[14] F.S. Roberts, L. Sheng, Phylogeny numbers, Discrete Appl. Math. 87 (1998) 213–228.
[15] D.J. Rose, Triangulated graphs and the elimination process, J. Math. Anal. Appl. 32 (1970) 597–607.