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LINEAR ALGEBRA AND ITS APPLICATIONS

Matrix sandwich problems

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Abstract

The Π Matrix Sandwich Problem (Π-MSP) is introduced here as follows: Given a $\{0, 1, *\}$ valued matrix A, where $*$ is interpreted as "do not care", does there exist a fill-in of the asterisks $*$ with 0s and 1s such that the completed $\{0, 1\}$ valued matrix M satisfies property Π? We study the computational complexity of this problem for several matrix properties including the Ferrers property, block decompositions and certain forbidden submatrices. Matrix sandwich problems are an important special case of matrix completion problems, the latter being generally defined over the real numbers rather than simply $\{0,1\}$.

Keywords: Matrix sandwich problems; Matrix completion problems; Ferrers diagrams; Block decomposition of matrices; Forbidden submatrix problems

1. Introduction

Let M_1 and M_2 be $\{0, 1\}$-valued $m \times n$ matrices. We say that M_1 is *dominated by* M_2 (M_2 *dominates* M_1), denoted by $M_1 \subseteq M_2$ if for all i, j we have

$$M_1(i,j) = 1 \quad \Rightarrow \quad M_2(i,j) = 1.$$

Consider the following problem: given $M_1 \subseteq M_2$ and a matrix property Π, is there an $m \times n$ $\{0,1\}$-valued matrix M satisfying Π such that $M_1 \subseteq M \subseteq M_2$? This problem is called the Π *matrix sandwich problem* and M is called a Π*-completion of* M_1 *in* M_2.

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An alternate way of regarding a sandwich problem is to consider an $m \times n$ matrix A with entries $\{0, 1, *\}$, and asking the question whether each $*$ entry (interpreted as "do not care") can be filled in by a 0 or 1 such that the filled-in matrix M satisfies property Π. We call M a Π-completion of A.

Graph sandwich problems [20] are precisely those for which the matrices are the adjacency matrices of a graph. *Hypergraph sandwich problems* [22] are precisely those for which the matrices are the (hyperedges-versus-vertices) incidence matrices of a hypergraph. Sandwich problems that have been studied recently for graph properties include NP-complete sandwich results for interval graphs [19,21], chordal graphs [6,30], unit interval graphs [19], permutation and comparability graphs [20] and k-trees for general k [22], and polynomial sandwich algorithms for split graphs, threshold graphs, cographs [20], unit interval graphs with bounded clique size [25], k-trees for fixed k [22] and graphs containing a homogeneous set [9].

Matrix sandwich problems are an important special type of matrix completion problems. The latter are generally defined over the real numbers, rather than simply $\{0,1\}$, and the unspecified entries are to be filled in so as to achieve a matrix with a desired numerical property. For example, the positive definite and semi-definite matrix completion problem [2–4,23], the Euclidean distance matrix completion problem [1], band matrix completions [11], Jordan and Hessenbery matrix completions [27], have been studied. See also [24].

1.1. Previous matrix sandwich results

A $\{0, 1\}$ matrix has the *consecutive ones* property, if its columns can be permuted so that the ones in each row are consecutive. Matrices with the consecutive ones property correspond to interval hypergraphs and to the cliques of interval graphs [5,12,13,15,17,18,31]. The consecutive ones property also plays a central role in many applications such as databases [14], genetics [16,19], and through its close connection with interval graphs arises in many other practical problems including temporal reasoning [21], medical diagnosis [32], scheduling, circuit design, psychology and others [17,29].

A $\{0, 1\}$ matrix has the *circular ones* property if its columns can be permuted so that the ones in each row are circularly consecutive (as in wrapping the matrix around a cylinder). Matrices with the circular ones property correspond to circular-arc hypergraphs. A matrix can be tested for the consecutives ones or the circular ones property in linear time [7].

Golumbic and Wassermann [22] have shown the following intractibility result.

Theorem 1. *The consecutive ones matrix sandwich problem and the circular ones matrix sandwich problem are NP-complete.*

In this paper we study the sandwich complexity for several other matrix properties. In Section 2 we give a linear time algorithm to solve the Ferrers matrix sandwich problem. In Section 3 we prove that the square block decomposition sandwich (SBDS) problem is NP-complete, but the rectangular block decomposition sandwich problem can be solved in linear time. Section 4 deals with some forbidden submatrix sandwich problems.

2. The Ferrers matrix sandwich problem

A $\{0,1\}$ matrix has the *Ferrers property* if its rows and columns can be reordered so that that 1s in each row and column appear consecutively with the rows left justified and the columns top justified.

Matrices in this stepwise form are known as *Ferrers diagrams*, and are of interest in representation theory of finite groups, partially ordered sets [10] and graph theory [28].

Lemma 2. *A* $\{0,1\}$ *matrix* M *satisfying the Ferrers property must either have a row with all* 1*s or a column with all* 0*s.*

Proof. Let r_i be the row of M which is reordered to the top row in the resulting Ferrers diagram, and let c_j be the column which is reordered to the last (rightmost) column. Clearly, either r_i is all 1s or c_j is all 0s. □

Lemma 2 provides us a simple method for checking whether a $\{0,1\}$ matrix has the Ferrers property, namely, *repeatedly delete any row having all* 1*s or any column having all* 0*s*. It is easy to show that this process will end with the empty matrix if and only if the original matrix has the Ferrers property. Moreover, the Ferrers diagram D can be constructed at the same time, starting with an empty $m \times n$ matrix, by filling in the remainder of the next row with all 1s when a row is deleted or the remainder of the next rightmost column with all 0s when a column is deleted. In fact, all of this can be done "virtually" by keeping a count of the row-sums as this elimination procedure is carried out.

We now present the main result of this section.

Theorem 3. *The Ferrers matrix sandwich problem can be solved in* $O(mn)$ *time.*

Proof. Our algorithm for solving the sandwich problem is the following elimination procedure:

Let A be a $\{0,1,*\}$-valued matrix. Repeatedly apply rules (1) and (2) in any order until neither applies.

(1) Delete any row containing only 1s and $*$s.

(2) Delete any column containing only 0s and $*$s.

Claim 4. *This process ends with the empty matrix if and only if A has a Ferrers completion. Moreover, if in* (1) *each asterisk in the row is filled in by* 1, *and in* (2) *each asterisk in the column is filled in by* 0, *then the resulting matrix M will have the Ferrers property.*

Suppose the process ends with the empty matrix, and let M be the completion of A as defined in Claim 4. Let D be the result of reordering the rows and columns of M according to the order in which they were filled in (i.e., deleted from A), rows from top to bottom and columns from right to left. Clearly, D is a Ferrers diagram.

Conversely, suppose the process stops with the non-empty submatrix A' (i.e., neither rule (1) nor (2) applies). It is well known, and easy to show that a $\{0, 1\}$matrix has the Ferrers property iff each submatrix has the Ferrers property. Thus, if A' had a Ferrers completion M', then either M' would have an all 1s row r_i in which case row i of A' would satisfy rule (1) or M' would have an all 0s column c_j and rule (2) would apply to column j of A', a contradiction. □

3. Block decomposition sandwich problems

A $\{0, 1\}$ matrix M has a *rectangular block decomposition* if its rows and columns can be reordered so that the 1s form rectangular blocks along a diagonal pattern as illustrated in Fig. 1. This property has an easy interpretation on the bipartite graph $B(M)$ whose vertices are $\{x_1, \ldots, x_m\} \cup \{y_1, \ldots, y_n\}$ corresponding to the m rows and n columns of M with x_i joined to y_j by an edge if an only if $M(i, j) = 1$. Clearly,

Remark 5. M has a rectangular block decomposition if and only if each connected component of $B(M)$ is a complete bipartite graph.

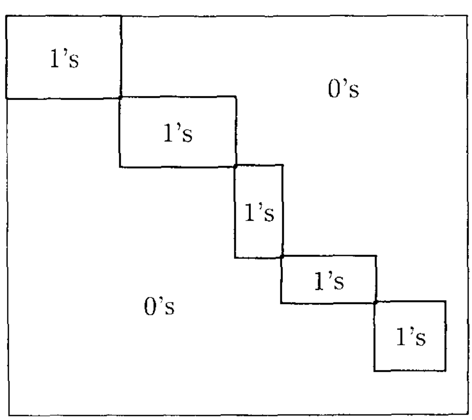

Fig. 1. A rectangular block pattern.

(A bipartite graph $G = (X, Y, E)$ is *complete* if $(x, y) \in E$ for every $x \in X$ and $y \in Y$. The complete bipartite graph with $| X |= m$ and $| Y |= n$ is commonly denoted by $K_{m,n}$. By convention, we also allow $K_{m,0}$ and $K_{0,n}$ as sets of independent vertices.)

In this way, we will often refer to a rectangular block decomposition of a matrix M in terms of the disjoint union

$$B(M) \cong K_{m_1,n_1} \cup K_{m_2,n_2} \cup \cdots \cup K_{m_k,n_k}, \tag{3.1}$$

where $m = \sum m_i$ and $n = \sum n_i$.

A $\{0, 1\}$ matrix M is said to have a *square block decomposition* if in (3.1) $m_i = n_i$ for all i. In such a case, the blocks of 1s in Fig. 1 will be squares.

Using elementary algorithmic graph theory, testing whether a $\{0,1\}$-matrix has a rectangular or square block decomposition can be done in linear time in the size of the input, i.e., $O(mn)$, simply by checking that each component of $B(M)$ is complete for the rectangular case or complete and balanced for the square case. We will show in the remainder of this section that the Rectangular block sandwich problem is also linearly solvable but the Square Block sandwich problem is NP-complete.

3.1. Rectangular block sandwich problem

Since rectangular block decompositions are equivalent to partitioning a bipartite graph into disjoint complete bipartite subgraphs, we may solve the sandwich problem using the bipartite graph model.

Let $G_1 = (X, Y, E_1)$ and $G_2 = (X, Y, E_2)$ be bipartite graphs on the same set of vertices satisfying $E_1 \subseteq E_2$. Let $E_0 = E_2 - E_1$ and $E_3 = (X \times Y) - E_2$. The Rectangular Block decomposition sandwich problem is equivalent to determining if there exists a bipartite graph $G = (X, Y, E)$ with $E_1 \subseteq E \subseteq E_2$ such that $G \cong K_{m_1,n_1} \cup \cdots \cup K_{m_k,n_k}$, i.e., G is the disjoint union of complete bipartite graphs. We follow the usual interpretation where E_1 corresponds to the required edges (the 1s in A), E_0 to the optional edges (the $*$s in A) and E_3 to the forbidden edges (the 0s in A).

We solve this sandwich problem as follows: Choose a vertex and generate its connected component C in G_1, denoting the vertices spanned by C as $X_C \cup Y_C$. If $X_C \cup Y_C$ has a forbidden edge (i.e., $(X_C \times Y_C) \cap E_3 \neq \phi$), then there is no sandwich; exit with failure. Otherwise, $X_C \cup Y_C$ is complete in G_2. Delete $X_C \cup Y_C$ from the graphs and repeat the same process. If this procedure succeeds in eliminating all vertices, exit with success.

Given this algorithm, we obtain the following result, which is straightforward.

Theorem 6. *The rectangular block decomposition sandwich problem can be solved in* $O(mn)$ *time.*

Proof. Clearly, each connected component in G_1 must be filled in by optional edges (from $E_0 = E_2 - E_1$) if there is to be a sandwich solution. Moreover, if this is done for each component, then the result will be a disjoint union of complete bipartite graphs and hence a sandwich solution. The complexity follows immediately. □

3.2. Square block sandwich problem

Theorem 7. *The SBDS problem is NP-complete.*

Proof. For square blocks, we consider the following equivalent bipartitite graph version of the SBDS problem.

Input: Bipartite graphs $G_1 = (X, Y, E_1)$ and $G_2 = (X, Y, E_2)$ with $E_1 \subseteq E_2$.

Question: Does there exist a bipartite graph $G = (X, Y, E)$ with $E_1 \subseteq E \subseteq E_2$ such that $G \cong K_{m_1,m_1} \cup \cdots \cup K_{m_k,m_k}$?

The problem is clearly in NP since any potential sandwich can be generated and tested in polynomial time. We prove that the problem is NP-hard by a reduction from the SET PARTITION problem.

SET PARTITION

Input: A collection $\mathscr{S} = \{S_i\}_{i\in I}$ of disjoint sets.

Question: Can $\mathscr{S}$ be partitioned into two equal parts, i.e., $I = A \cup B, A \cap B = \phi$ such that $\sum_{i\in A} \mid S_i \mid = \sum_{i\in B} \mid S_i \mid$?

Let $\mathscr{S} = \{S_i\}_{i\in I}$ be an instance of SET PARTITION, and let

$$\sum_{i\in I} \mid S_i \mid = 2p, \quad \mid I \mid = k, \quad n_i = \mid S_i \mid \quad \text{and} \quad S_i = \{s_i^1, s_i^2, \ldots, s_i^{n_i}\}. \tag{3.2}$$

We construct an instance of the bipartite graph version of the SBDS as follows: Let

$$X = \{a, b\} \cup \{x_1, \ldots, x_k\} \cup \bigcup_{i\in I} S_i,$$

$$Y = \{a', b'\} \cup \{y_1, \ldots, y_k\} \cup \{w_1, \ldots, w_p\} \cup \{z_1, \ldots, z_p\},$$

where all these vertices are distinct, and further define

$$E_1 = E_a \cup E_b \cup \bigcup_{i\in I} T_i,$$

where

$$E_a = \{(a, a')\} \cup \{(a, w_i) \mid i = 1, \ldots, p\},$$

$$E_b = \{(b, b')\} \cup \{(b, z_i) \mid i = 1, \ldots, p\},$$

$$T_i = \{(x_i, y_i)\} \cup \{(s_i^j, y_i) \mid j = 1, \ldots, n_i\}$$

and

$$E_2 = (X \times Y) - \{(a, b'), (a', b)\}.$$

The construction is illustrated in Fig. 2.

We observe the following.

(i) $|X| = |Y| = 2 + k + 2p$

(ii) Since $(a, a'), (b, b') \in E_1$ but $(a', b), (a, b') \notin E_2$, any (potential) sandwich solution would have to put a and b into disjoint components.

Suppose the SET PARTITION instance has a solution $I = A \cup B, A \cap B = \phi$ with $p = \sum_{i \in A} |S_i| = \sum_{i \in B} |S_i|$. Then $G = G_A \cup G_B$ is a sandwich solution, where

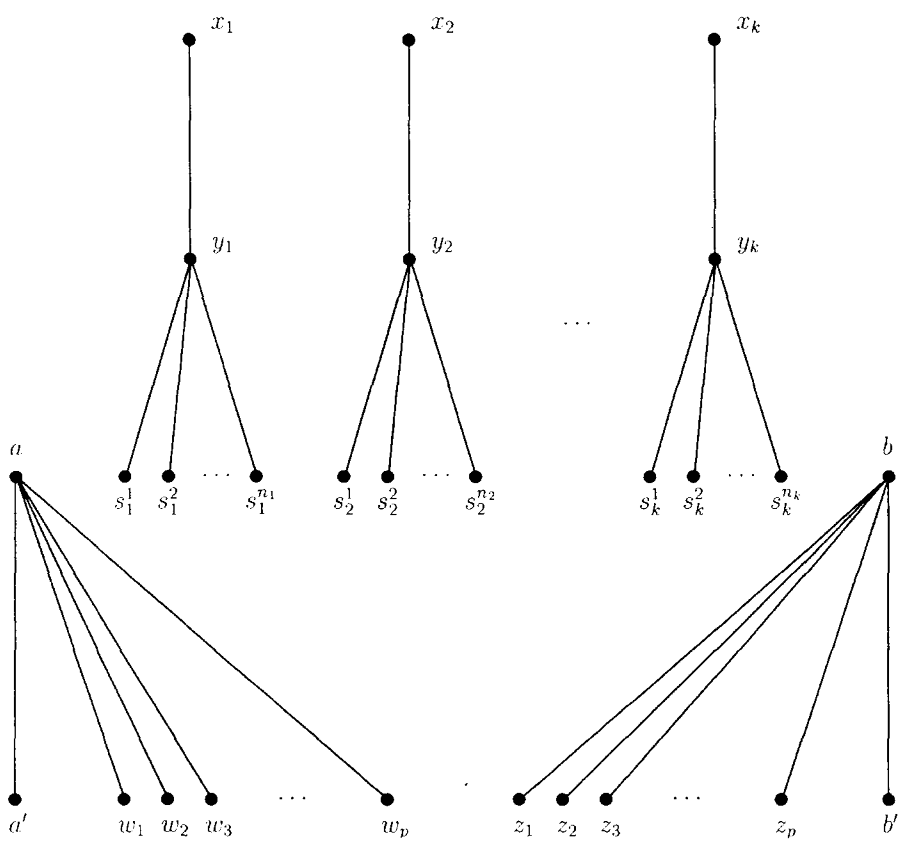

Fig. 2. The edges of E_1, required in any sandwich solution.

$$G_A = \left(\{a\} \cup \{x_i \mid i \in A\} \cup \bigcup_{i \in A} S_i \right)$$
$$\times (\{a'\} \cup \{y_i \mid i \in A\} \cup \{w_1, \ldots, w_p\})$$
$$G_B = \left(\{b\} \cup \{x_i \mid i \in B\} \cup \bigcup_{i \in B} S_i \right)$$
$$\times (\{b'\} \cup \{y_i \mid i \in B\} \cup \{z_1, \ldots, z_p\})$$

since all edges of E_1 are included, neither forbidden edge (a, b') nor (a', b) was added, and G_A and G_B are disjoint balanced complete bipartite graphs since the cardinality of both sides of the Cartesian products are equal for G_A and G_B.

Conversely, suppose the SBDS instance has a sandwich solution E. Then it must consist of exactly two disjoint components according to the following reasoning. Let H_A and H_B denote the connected components of E which contain a and b, respectively. Define $A = \{i \mid y_i \in H_A\}$ and $B = \{i \mid y_i \in H_B\}$. The number of vertices in H_A is $2 + 2 \mid A \mid + \sum_{i \in A} \mid S_i \mid + p$ corresponding to $\{a, a'\} \cup \{x_i, y_i \mid i \in A\} \cup \{S_i^j \mid i \in A, j = 1, \ldots, n_i\} \cup \{w_1, \ldots, w_p\}$, and since H_A is balanced, we have $p = \sum_{i \in A} \mid S_i \mid$. Similarly, the number of vertices in H_B is $2 + 2 \mid B \mid + \sum_{i \in B} \mid S_i \mid + p$, so $p = \sum_{i \in B} \mid S_i \mid$. Therefore, since all vertices are accounted for by (3.2), there are no other connected components, and A and B give a solution to the SET PARTITION problem. This concludes the proof of the theorem. □

4. Forbidden submatrix problems

Klinz et al. [26] have studied a large number of matrix properties of the form, (4.1): permute the rows and columns so that the result does not contain as a submatrix any element of $\mathscr{F}$, where $\mathscr{F}$ is a specified set of "forbidden" matrices, and by the term "submatrix" we mean "obtained by cancelling rows and columns", i.e., not necessarily a contiguous submatrix.

Denote by $\mathscr{M}(\mathscr{F})$ the family of $\{0, 1\}$-valued matrices satisfying (4.1), and denote by $\mathscr{S}(\mathscr{F})$ the family of $\{0, 1, *\}$ -valued matrices which have a sandwich solution (a $\{0, 1\}$-valued completion) satisfying (4.1). For example,

$$\mathscr{M}\left((0, 1), \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

are precisely those matrices having the Ferrers property, and

$$\mathscr{S}\left((0, 1), \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

are those having a Ferrers matrix completion.

In [26] the computational complexity of the membership problem for many instances of $\mathscr{M}(\mathscr{F})$ is investigated, and for similar versions for only column permutations $\mathscr{M}^{\mathrm{C}}(\mathscr{F})$ and for simultaneous row and column permutations on square matrices $\mathscr{M}^{\mathrm{S}}(\mathscr{F})$. Our interest here is the membership problem for $\mathscr{S}(\mathscr{F})$ which is, in fact, exactly the sandwich problem for permuted forbidden submatrices. For example, membership in $\mathscr{S}((1,0,1))$ is equivalent to the consecutive one's matrix sandwich problem since permuting rows is irrelevant. This problem we know to be NP-complete as opposed to the Ferrers sandwich problem which is polynomial. We now present some other polynomial time cases.

Proposition 8. *The complexity of membership in*

$$\mathscr{S}(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix})$$

is the same as the complexity of membership in

$$\mathscr{M}(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}),$$

hence, polynomial time.

Proof. We note that for this case, permutations of rows and columns are irrelevant. Given a $\{0,1,*\}$-valued matrix A, let B denote the completion of A with each asterisk $*$ replaced by a zero. We claim that

$$B \in \mathscr{M}(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix})$$

if and only if

$$A \in \mathscr{S}(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}),$$

that is, A has a $\{0,1\}$ completion in

$$\mathscr{M}(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}).$$

The "only if" implication is immediate. Suppose M is a completion of A with no

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

submatrix. Then, changing any filled-in one to a zero instead, cannot create a new

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

submatrix, and will result in obtaining B. □

It is shown in [26] that the complexity of membership in

$$\mathscr{M}\left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right)$$

for an $m \times n$ matrix with f entries equal to one is $\mathrm{O}(mn + \min\{m^2, n^2, f^{3/2}\})$.

Proposition 9. *Membership in $\mathscr{S}((0,1,1))$ has complexity* $\mathrm{O}(mn)$.

Proof. Row permutations are irrelevant again in this case. Let A be a $\{0,1,*\}$-valued matrix. The desired column permutations and fill-in of asterisks can be done in a greedy manner (similar to our solution for the Ferrers sandwich problem), applying the following three rules in any order until no rule applies:

(1) Delete any column containing only 1s and $*$s (permute it to the extreme left in the solution matrix and fill in each $*$ with 1).

(2) Delete any column containing only 0s and $*$s (permute it to the extreme right in the solution matrix and fill in each $*$ with 0).

(3) Delete any row containing at most one 1 (and fill in each $*$ with 0 in the solution matrix).

Claim 10. *This process ends with the empty matrix if and only if $A \in \mathscr{S}((0,1,1))$.*

Let A' be the submatrix when the process stops (i.e., none of rules (1)–(3) applies). If A' is empty, then the solution matrix we have filled in contains no submatrix (0,1,1). Conversely, the manner in which we fill in columns and rows in rules (1)–(3) insures that they cannot participate in any submatrix (0,1,1). Thus, it is sufficient to look at A', i.e., suppose $A' \in \mathscr{S}((0,1,1))$ and let M' be a permuted completion of A' which has no submatrix (0,1,1). By rule (1), the leftmost column of M' contains an original 0 (i.e., not a filled-in 0), say in row i. By rule (3), row i contains at least two original 1s, both to the right of the 0 entry. This is a contradiction and proves Claim 10.

The complexity result follows immediately. □

5. Concluding remarks

The matrix sandwich problems studied in this paper may be regarded as *existential* completion problems over the domain {0,1} since we ask if there *exists* an assignment for each asterisk from the possible values {0,1} such that a desired property holds. In this way we interpret $*$ as "do not care". A different approach to working with missing values, not studied here, is to consider *universal* completion problems where one asks whether *for all* assignments of values to the asterisks will the desired property hold. Yet another variation is to treat each asterisk as a variable with its own domain of possible values, and ask both existential and universal completion questions. This last approach is known in the artificial intelligence literature as constraint satisfaction, (see [21,32] and their references.) The special case of a missing value (variable) being designated simply "non-zero" is a completion problem commonly found in the matrix algebra literature.

Joel Brawley (personal communication, Clemson University) has pointed out that the matrix sandwich problem of fixed (i.e., specified) row and column sums can be solved by linear programming. Let A be a $\{0,1,*\}$-valued $n \times m$ matrix, and let $\{r_i\}$ and $\{c_j\}$ $(1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m)$ be integers. Let $0 \leqslant x_{i,j} \leqslant 1$ be a constraint on the variable $x_{i,j}$ if position $a_{i,j}$ corresponds to an asterisk, and $x_{i,j} = a_{i,j}$ otherwise. Consider the following linear program:

$$\sum_j x_{i,j} = r_i, \quad \sum_i x_{i,j} = c_j, \quad 0 \leqslant x_{i,j} \leqslant 1. \tag{5.1}$$

By a classical theorem, if (5.1) has a solution, then it has an integer solution, hence, a $\{0,1\}$-valued solution, and this will be a sandwich solution for the matrix.

Another related topic is that of finding extensions of partially defined Boolean functions with missing data. Like other sandwich problems, these can occur in classification and knowledge acquisition where positive and negative examples provide partial data to be generalized or clues to a hypothesis to be verified. Boros et al. [8] investigate the complexity of finding extensions of a desired type (positive, Horn, self-dual, threshold, etc.) providing polynomial algorithms in some cases and NP-hardness results in other cases.

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