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NOTE

BIPARTITIONAL POLYNOMIALS AND THEIR APPLICATIONS IN COMBINATORICS AND STATISTICS

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Bipartitional polynomials are multivariable polynomials

$$Y_{mn} = Y_{mn}(cy_{01}, cy_{10}, cy_{11}, \ldots, cy_{mn}), \qquad c^k \equiv c_k,$$

defined by a sum over all partitions of the bipartite number (mn). Recurrence relations, generating functions and some basic properties of these polynomials are given. Applications in Combinatorics and Statistics are briefly indicated.

1. Introduction

Partition polynomials have been introduced and studied by Bell [1, 2]; they are multivariable polynomials defined by a sum over all partitions of their indexes. Riordan [6, 7] called Bell polynomials the partition polynomials associated with the derivatives of a composite function; he discussed several applications of these polynomials in Combinatorics, Statistics and Number theory. Other properties and statistical applications have been discussed by the author [3]. Touchard [8] in his work on the cycles of permutations generalized the Bell polynomials in order to be able to study some problems of enumeration of the permutations when the cycles possess certain properties.

A natural generalization of the Bell partition polynomials are the bipartitional polynomials which are the subject of this paper. The idea of using bipartitional functions in Statistics goes back to R.A. Fisher [4].

2. Bipartitional polynomials and their applications

The bipartitional polynomials, denoted by

$$Y_{mn} = Y_{mn}(cy_{01}, cy_{10}, cy_{11}, \ldots, cy_{mn}), \qquad c^k \equiv c_k,$$

are defined by the sum

$$Y_{mn}=\sum\frac{m!\,n!\,c_k}{k_{01}!\,k_{10}!\,k_{11}!\cdots k_{mn}!}\left(\frac{y_{01}}{0!\,1!}\right)^{k_{01}}\cdot\left(\frac{y_{10}}{1!\,0!}\right)^{k_{10}}\cdot\left(\frac{y_{11}}{1!\,1!}\right)^{k_{11}}\cdots\left(\frac{y_{mn}}{m!\,n!}\right)^{k_{mn}},\tag{2.1}$$

over all partitions of the bipartite number (mn), that is over all solutions in non-negative integers of the equations:

$$\sum_{i=1}^{m} i\sum_{j=0}^{n} k_{ij}=m,\qquad \sum_{j=1}^{n} j\sum_{i=0}^{m} k_{ij}=n;$$

k is the number of parts in the partition. Note that by substituting y_{rs} by y_{sr} in the expression of Y_{mn} we get Y_{nm}.

Using the umbral calculus we may obtain from (2.1) the exponential generating function of these polynomials

$$Y(u,v)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}Y_{mn}\frac{u^m}{m!}\frac{v^n}{n!}=\exp[c\{y(u,v)-y(0,0)\}],$$
$$c^k\equiv c_k,\qquad y(u,v)=\sum_{s=0}^{\infty}\sum_{r=0}^{\infty}y_{rs}\frac{u^r}{r!}\frac{v^s}{s!},\tag{2.2}$$

from which we may easily deduce the recurrence relation

$$Y_{m,n+1}=\sum_{s=0}^{n}\sum_{r=0}^{m}\binom{m}{r}\binom{n}{s}y_{r,s+1}cY_{m-r,n-s},\qquad Y_{00}=1,\tag{2.3}$$

and the following properties

$$Y_{mn}(cby_{01},cay_{10},caby_{11},\dots,ca^mb^ny_{mn})=a^mb^nY_{mn}(cy_{01},cy_{10},cy_{11},\dots,cy_{mn}),\tag{2.4}$$

$$Y_{mn}(x_{01}+y_{01},x_{10}+y_{10},x_{11}+y_{11},\dots,x_{mn}+y_{mn})=\sum_{r=0}^{m}\sum_{s=0}^{n}\binom{m}{r}\binom{n}{s}Y_{rs}(x_{01},x_{10},x_{11},\dots,x_{rs})\times Y_{m-r,n-s}(y_{01},y_{10},y_{11},\dots,y_{m-r,n-s}),\tag{2.5}$$

$$Y_m(v)=\sum_{n=0}^{\infty}Y_{mn}\frac{v^n}{n!}=Y_m(\alpha y_1(v),\alpha y_s(v),\dots,\alpha y_m(v)),\qquad \alpha^k\equiv\alpha_k\tag{2.6}$$

where

$$\alpha_k=c^k\exp[c\{y_0(v)-y_0(0)\}],\qquad y_r(v)=\sum_{s=0}^{\infty}y_{rs}\frac{v^s}{s!}$$

and $Y_m=Y_m(\alpha x_1,\alpha x_2,\dots,\alpha x_m)$, $\alpha^k\equiv\alpha_k$, the Bell (unipartitional) polynomials.

The derivatives of the composite function $h(s,t)=f(g(s,t))$ expressed in terms of the derivatives of the component functions form a set of bipartitional polynomials, that is, with

$$h_{mn}=\frac{\partial^n}{\partial t^n}\frac{\partial^m}{\partial s^m}h(s,t), \qquad g_{mn}=\frac{\partial^n}{\partial t^n}\frac{\partial^m}{\partial s^m}g(s,t),$$
$$f_k=\frac{\mathrm{d}^k}{\mathrm{d}u^k}f(u)\big|_{u=g(s,t)}, \tag{2.7}$$

we have

$$h_{mn}=Y_{mn}(fg_{01}, fg_{10}, fg_{11}, \ldots, fg_{mn}), \qquad f^k \equiv f_k. \tag{2.8}$$

The proof of (2.8) may be carried out by following Riordan's technique [6, pp. 34–37] with the necessary modifications.

This result has a direct use in Statistics in expressing the probabilities and moments of a generalized random variable (r.v) in terms of the probabilities and moments of the generalizing r.v.'s; indeed, if Z and (X, Y) are independent discrete r.v.'s with probability generating functions (p.g.f.'s) $f(u)$ and $g(s,t)$ respectively, then the r.v. (V, W) with p.g.f. $h(s,t)=f(g(s,t))$ is called a generalized r.v. The probability function of (V, W) by differentiating h and using (2.8) may be obtained in the form

$$P_{mn}=P(V=m, W=n)=\frac{1}{m!\,n!}Y_{mn}(fp_{01}, fp_{10}, fp_{11}, \ldots, fm!\,n!\,p_{mn}),$$
$$f^k\equiv f_k=\frac{\mathrm{d}^k}{\mathrm{d}u^k}f(u)\big|_{u=p_{00}}, \qquad p_{kr}=P(X=k, Y=r), \qquad P_{00}=f(p_{00}). \tag{2.9}$$

The factorial moments $\mu_{(m,n)}=E[(V)_m\cdot(W)_n]$ may be obtained as

$$\mu_{(m,n)}=Y_{mn}(\alpha\beta_{(0,1)}, \alpha\beta_{(1,0)}, \alpha\beta_{(1,1)}, \ldots, \alpha\beta_{(m,n)}), \qquad \alpha^k\equiv\alpha_{(k)}, \tag{2.10}$$

with $\alpha_{(k)}$ and $\beta_{(k,r)}$ the factorial moments of Z and (X, Y) respectively.

Two classes of bipartitional polynomials, which are an important feature of many combinatorial and statistical problems, will be briefly discussed in the sequel. They are obtained from (2.1) by specifying the sequence c_k, $k=0,1,2,\ldots$. Letting $c_k=1$, $k=0,1,2,\ldots$, the generating function (2.2) reduces to the exponential function $Y(u,v)=\exp\{y(u,v)-y(0,0)\}$ and the polynomials $Y_{mn}(y_{01}, y_{10}, y_{11}, \ldots, y_{mn})$ may be called exponential (bipartitional) polynomials. For $c_0=0$, $c_k=(-1)^{k-1}(k-1)!$, $k=1,2,\ldots$ the generating function (2.2) becomes a logarithmic function $L(u,v)=\log[1+\{y(u,v)-y(0,0)\}]$ and the corresponding polynomials $L_{mn}(y_{01}, y_{10}, y_{11}, \ldots, y_{mn})$ may be called logarithmic (bipartitional) polynomials. Note that Y_{mn} and L_{mn} are inverse polynomials.

After these remarks it will not be difficult to verify the following relations connecting the moments μ'_{mn} and cumulants κ_{mn} of a bivariate probability distribution (for the univariate case see Riordan [6, p 37])

$$\begin{aligned} \mu'_{mn} &= Y_{mn}(\kappa_{01}, \kappa_{10}, \kappa_{11}, \ldots, \kappa_{mn}), \\ \kappa_{mn} &= L_{mn}(\mu'_{01}, \mu'_{10}, \mu'_{11}, \ldots, \mu'_{mn}). \end{aligned} \tag{2.11}$$

The polynomials Y_{mn} and L_{mn} have also a direct use in combinatorics in expressing the elementary symmetric functions a_{mn} and the homogeneous product sum symmetric functions h_{mn} in terms of the power sum symmetric functions s_{mn} and vice versa (c.f. MacMahon [5, p. 282–284] and Riordan [6, p. 47])

$$\begin{aligned} (-1)^{m+n} m!\, n!\, a_{mn} &= Y_{mn}(-s_{01}, -s_{10}, -s_{11}, \ldots, -(m+n-1)!\, s_{mn}), \\ (m+n-1)!\, s_{mn} &= (-1)^{m+n-1} L_{mn}(a_{01}, a_{10}, a_{11}, \ldots, m!\, n!\, a_{mn}), \end{aligned} \tag{2.12}$$

$$\begin{aligned} m!\, n!\, h_{mn} &= Y_{mn}(s_{01}, s_{10}, s_{11}, \ldots, (m+n-1)!\, s_{mn}), \\ (m+n-1)!\, s_{mn} &= L_{mn}(h_{01}, h_{10}, h_{11}, \ldots, m!\, n!\, h_{mn}), \end{aligned} \tag{2.13}$$

Moreover, with

$$\begin{aligned} & Z_{mn}(y_{01}, y_{10}, y_{11}, \ldots, y_{mn}) = Y_{mn}(cy_{01}, cy_{10}, cy_{11}, \ldots, cy_{mn}), \\ & c^k \equiv c_k = (-1)^k k!, \quad k = 0, 1, \ldots \end{aligned} \tag{2.14}$$

we may easily show that

$$\begin{aligned} (-1)^{m+n} m!\, n!\, a_{mn} &= Z_{mn}(h_{01}, h_{10}, h_{11}, \ldots, m!\, n!\, h_{mn}), \\ (-1)^{m+n} m!\, n!\, h_{mn} &= Z_{mn}(a_{01}, a_{10}, a_{11}, \ldots, m!\, n!\, a_{mn}). \end{aligned} \tag{2.15}$$

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