Talking About Singularities

Spiros Cotsakis

Research Group of Geometry, Dynamical Systems and Cosmology

Department of Information and Communication Systems Engineering

University of the Aegean

Karlovassi 83 200, Samos, Greece

E-mail: `skot@aegean.gr`

Abstract

We discuss some aspects of recent research as well as more general issues about motivation, useful methods and open problems in the field of cosmological singularities. In particular, we review some of the approaches to the general area and include discussions of the method of asymptotic splittings, singularity and completeness theorems and the use of the Bel-Robinson energy to prove completeness theorems and classify cosmological singularities.

'Knowledge should be shared', yes, of course it should, but it is extremely doubtful whether an organic chemist would put missionary zeal into an attempt to interest the pure mathematician in his latest researches, or vice versa. To be realistic, the sharing of knowledge throughout the whole field of science is a pious hope, not a passion, and it is only passion that could overcome the inescapable difficulties of communication. *J. L. Synge.*

1 Introduction

Singularities have always been important, and all successful theories of physics have them by necessity. From the Newtonian dynamics of particles and fields, to relativistic

1

problems, to quantum dynamics, the ever-presence of singularities can for the most part be adequately treated one way or another, even avoided or ignored, although in some cases their consideration has triggered fundamental changes.

But it is different in cosmology. Here, even though there are areas where the consideration of singularities can for the most part or wholly be neglected, no fundamental discussion of the basic cosmological issues can go on without facing them sooner or (preferably!) later. Up to the recent past, discussions of cosmological singularities have revolved around the same philosophical issues that we all know and grew up to respect, but which after a while any young, inquiring and productive cosmologist would wish to by-pass. But with certain exceptions. Not all discussions about singularities in cosmology are philosophical, they can be very mathematical indeed and at the same time touch upon very basic physical issues, probably at the cost of creating a situation that many people (even the creators themselves!) would wish to overpass.

We believe that there lies ahead of us a golden era for research in cosmological singularities. The basic pillars are now almost done, the proliferation of newer and more interesting cosmological models is a call to arms, the mathematical techniques are better understood and explained and are definitely there waiting to be used, and cosmological observations, although richer than ever, do not constrain the possible models to any considerable degree. It is unfortunate that all this richness is often stated as a source for despair, it is a paradise for those who don't expect ready-made answers.

In this review we are set to present the subject of cosmological singularities from a new angle, that of a logical and simplest form which however, may not be the most psychologically natural one. For motivation purposes, discussions of a new but already interesting subject are perhaps equally important to precise definitions and results, at least for the novice, and we have tried to present both here (resulting unfortunately in the usual unlucky compromise). In the next section after a short review of the basic definitions of completeness and singularity, we introduce the problem of cosmological singularities in two stages: Firstly, it is set at the level of special, exact cosmologies that is

when we have chosen a particular spacetime, a specific matter component and a geometric action to describe the gravitational field, for instance a fluid-filled Friedmann universe in general relativity. Secondly, we formulate the singularity issue at the different level of leaving the choice of spacetime unspecified. This leads us in turn into two ways of attack, one through spacetime differential geometry to formulate and prove general results about geodesic incompleteness (singularity theorems), the other being the analysis of the field partial differential equations describing the evolution of Einsteinian spacetimes.

It is important to realize that any information about the singularities of a system in the general case (that is without assuming any symmetry) cannot be usefully reduced to give us information about particular instances of the field in situations of symmetry (and vice versa of course!). This last case requires a different analysis. A concrete general technique, the method of asymptotic splittings, designed specifically to deal with the approach to a spacetime singularity of exact cosmologies is presented in section 3. The aim of this method is to decompose the vector field defining the model in a suitable way and follow the dominant features of it all the way down the finite time singularity. This leads to the construction of asymptotic series expansions of the unknowns in the neighborhood of the singularity, from which valuable information about the field can be extracted. Some examples of the application of the method of asymptotic splittings are discussed. This method is an adaptation to real dynamical systems (finite dimensions) of the philosophy of certain blow-up techniques in partial differential equations, and of methods that exist in the field of complex differential equations (singularity analysis).

Next we move on, in section 4, to discuss sufficient and necessary conditions for the occurrence of singularities. This section, besides the classic singularity theorems, includes more recent results about singularities associated primarily with the name of Y. Choquet-Bruhat. We also provide applications of these criteria for singularity formation to the isotropic category and show how these lead to a novel classification of spacetime singularity types therein through the use of the Bel-Robinson energy. We also use the Bel-Robinson energy to give a new characterization of the g-completeness and eternal

acceleration of Robertson-Walker (RW) spacetimes.

The absence of a discussion section at the end is a healthy one, and conclusions are out of place at this initial stage of development of this most beautiful subject.

2   Approaches to the singularity problem

In this section we give the basic definitions of what it means for a spacetime to be singular. We then discuss the general problem of spacetime singularities in the framework of exact cosmological solutions (physical approach). Finally, we extend the formulation of the problem to the case of spacetimes without symmetries and discuss various more mathematical approaches used to tackle this most basic cosmological issue.

2.1   Definitions

We give in this subsection the basic definitions of complete as well as singular spacetimes. Let $(\mathcal{M}, g)$ be a spacetime with $\mathcal{M} = (a, b) \times \Sigma$, where (a, b) is an interval in $\mathbb{R}$ (possibly the whole line), and the spatial slices $\Sigma_t \, (= \Sigma \times \{t\})$ are spacelike submanifolds endowed with the time-dependent spatial metric g_t. The standard geometric definition of spacetime completeness in general relativity is the following.

Definition 2.1 (Geometric) *We say that the spacetime* $(\mathcal{M}, g)$ *is non singular if it is causally geodesically complete, that is every causal (i.e., timelike or null) geodesic*

$\gamma : (a, b) \subset \mathbb{R} \to \mathcal{M}$, *defined in a finite subinterval, can be extended to the whole real line.*

However, this definition is not useful in evolution problems because in such studies one is typically interested not in the behaviour of geodesics but in that of the *dynamical* variables of the problem. These satisfy differential equations such as the Einstein or some other field equations. In such problems one is typically interested in proving local or global in time existence of the spacetime in question. This leads us to the following alternative.

Definition 2.2 (Dynamical) *We say that the spacetime $(\mathcal{M}, g)$ exists globally as a solution of the field equations if*

$$||(\mathbf{g}_t, \partial \mathbf{g}_t, \boldsymbol{\Theta})||_{L^p} < \infty, \quad \text{uniformly for all } p, t.$$

This definition, deliberatively vague, means intuitively that a spacetime such that its spatial metric variables $\mathbf{g}_t$, their derivatives $\partial \mathbf{g}_t$ and any matter fields $\boldsymbol{\Theta}$ present are finite for all time, must be considered as globally existent. In a specific problem of course one must choose the appropriate norms so that they make sense with the given conditions on the various data. However, even when, as it is usually the case, global existence comes together with non singularity (in the above sense of geodesic completeness), one must prove that this is indeed the case as the two notions are different.

The negations of the above definitions lead to the notions of singular and finitely existent spacetimes.

Definition 2.3 (Geometric) *We say that $(\mathcal{M}, g)$ is singular if it is geodesically incomplete, that is there is at least one incomplete causal geodesic $\gamma : (a, b) \subset \mathbb{R} \to \mathcal{M}$.*

For a spacetime to have only a finite duration, we require the following

Definition 2.4 (Dynamical) *We say that $(\mathcal{M}, g)$ exists as a solution of the field equations for only a finite time if the following holds:*

$$||(\mathbf{g}_t, \partial \mathbf{g}_t, \boldsymbol{\Theta})||_{L^p} \quad \text{becomes unbounded at some } p, t.$$

These definitions allow us to make several remarks.

Remark 2.1 (Equivalence) *The two definitions, geometric and dynamical, of completeness on the one hand, and singularity on the other, are considered as equivalent, but there is to our knowledge no* formal *proof of this in the literature. So one must check for instance that a spacetime that exists globally for all future time is also geodesically complete to prove that it contains no singularities. For an example of this, see [3, 1].*

Remark 2.2 (True singularity) *It is important to emphasize that one should be careful not to mix a singularity in the congruence of a bunch of geodesics with a true spacetime singularity. The singularity theorems (see below) give conditions when one ends up with a true spacetime singularity, the proof, however, of such statements is by contradiction and therefore one cannot use such arguments to really* construct *incomplete spacetimes.*

Remark 2.3 (Fixed and sudden singularities) *The actual finite duration of a singular spacetime (according to the dynamical definition), in other words the position on the time axis of the finite time singularity, is in fact its maximum interval of existence considered as a solution. This therefore may very well depend on the initial and final conditions set in the problem. One is therefore led to consider* fixed *singularities, those which have positions independent of the initial conditions as well as* movable, *or* sudden *singularities, whose position is actually very sensitive to the initial conditions. An interesting nontrivial problem is to characterize these two types of singularities using the geometric definitions of geodesic incompleteness. For more details see Section 4.2.*

2.2 The physical approach

Up to around the 1960's only three dynamical theories existed in physics: Newtonian dynamics (of particles, continua and fields), relativistic dynamics (including both the special and the general theory) and quantum dynamics (splitting naturally in its newtonian (mechanics) and relativistic (field theory) parts). Since then, however, there has been a proliferation of different dynamical theories some aiming at rectifying one or more

of the issues which arose in one of the three pillars, others pointing at the other direction, that of the *unification* (eventually of relativistic and quantum dynamics). This sparkling diversity is especially apparent in theories which include gravity, and the following table aims at offering a broad but humble picture of this most chaotic situation.

Cosmologies

Theories of gravity	Spacetimes	Matterfields
General Relativity	(Anti-) de Sitter	Vacuum
Higher Derivative	RW	Fluids
Scalar-Tensor and VC	Bianchi,	Scalar fields
Inflation and QC	Gödel	n-form fields
Braneworlds	T-NUT-M	Phantoms, tachyons

With the advent of string theory and its subsequent generalizations and extensions previous dynamical theories of gravity found their natural places under its umbrella in the form of *effective* actions and field equations and their overall position gave to the picture a notable matter-of-fact-ness.

But it is disturbing. The whole picture is blurred with the existence and apparent (mathematical at least) viability of all these theories, while their structures are connected to one another by complicated laws, duality and conformal transformations, making it impossible to choose some and to leave more out (the existence of the so-called string and Einstein frame formulations are good examples of this). We don't yet know what really lies behind this web of theories but the real question is: How is it that we can tolerate this apparently arbitrary multiplicity? The standard answer to this pressing question is that of course each and every one of these possible theories in the web represent different viable *mathematical models* of reality.

The least thing one needs in this game is to become inflated with long words. The singularity problem acquires the following simple form in this context: Analyze the possible models (like those that can be readily extracted from the Table above) and discover what they have to say about the existence and nature of their spacetime singularities. It

is to be expected that the obvious proliferation of models will lead to an equally obvious number of different kinds of singularities. This is a good thing, it is like discovering new kinds of life in zoology (for more on this, see [4, 5]).

The essential mathematical tools needed to construct such cosmological models (with the aim eventually to treat their singularities) come from tensor analysis. One ends up with tensor equations, invariant under arbitrary coordinate transformations, and these describe the dynamical content of the model. Then, techniques from the theory of dynamical systems enter the picture, and with geometry and dynamical systems one hopes to be able to tackle the many different subtleties which will be pertinent to the singularity structure of the spacetimes in question. Due to its richness, unraveling the singularities of different cosmological models is a game that must be played with adroitness, many surprises are still hidden from our ever expanding horizon. More on this approach will be given in Section 3.

2.3 The mathematical approach

There is another way of looking at the problem of cosmological singularities, but before we discuss this in this subsection, let us summarize in a sentence the content of the approach taken in the previous subsection. For the admitted variety of theories of gravity we have to pay a price, namely, the spacetimes to-be-used are mathematically simple, perhaps too simple, but there are at least two good reasons that make that a legitimate approach: Even if one does not believe that the actual models constructed this way are true in reality, one still has a reason to consider them for *only* in this way we have even the slightest chance to use the available observational data (cf. [6], p. 470). Or, we perhaps, like Einstein, must elevate the *homogeneity assumption* to the status of a physical principle, cf. [7], p. 16). Follow either of these two leads and you conveniently find yourself walking along with the other *physical cosmologists.*

Others, however, follow none. The point of view of the *mathematical cosmologist,* at least in its traditional sense, is that, because we know so little from observations, *it*

is legitimate to consider arbitrary manifolds and metrics (cf. [8], p.30). There is also another reason for this legitimacy. It is quite possible that homogeneities at small scales might change drastically the *global* mode of behaviour in any model when we follow its evolution in either early or late (proper) times. For the treatment of homogeneous cosmological models in any theory of gravity, this could affect the results obtained through the physical approach quite seriously, even deadly[1].

The mathematical cosmologist does not have to go far to find problems of his liking, general relativity is for the most part adequate. Even the simplest of theories when looked from such a perspective can have a richness of unprecedented beauty. However, sooner or later the mathematical approach faces a true obstacle: The transcendental difficulty of the equations. The Einstein equations are, in their most interesting form, a system of hyperbolic evolution equations together with elliptic constraints and this system will obviously have to be augmented by further equations describing the evolution of the various matter fields.

However, some progress has been made and here we have the Cauchy problem (see [8] and references therein). In this approach, spacetimes are not known in advance, they are in fact built up from initial data as the solutions that describe them evolve, or develop, through the field equations from that data. The issues of the existence and uniqueness of *maximal developments*, as well as that of the nature of singularities in these solutions are the central objects of study here, and it is of interest to apply these methods not only to general relativity but also to other interesting theories of gravity formulated in the string frame, something that has not yet been tried to any measurable degree of completeness.

Another area in which modern mathematical methods can be of great interest is

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[1]There are of course certain 'no-go' conjectures, the BKL conjecture being the primary example (see, the very recent work [9] for an account of the current status of the proof of this conjecture). For the singularity problem, this basically says that what happens in the most general *homogeneous* model on approach to the singularity (technically for the Bianchi IX spacetimes) describes the generic situation, that is the approach to the singularity when there are no symmetries.

in fact that which contains models of interest to the physical cosmologist. Here a more mathematical approach to models already formulated by physicists can help in improving the rigor of many presentations as well as unify and polish the essential elements of many models considered in the literature. We shall have to say more on this later.

Still another direction to the singularity problem open to the modern mathematical cosmologist is through Lorentzian geometry. This approach has led historically to the first singularity theorems, those of R. Penrose and S. W. Hawking (see [10] for a modern introductory account of these theorems).[2] Recently there has been a related body of results, the *completeness theorems* (see [1] and also [12] for an application of these results to cosmology). These theorems aim to give sufficient conditions for a spacetime to be geodesically complete, however, like the singularity theorems (but unlike the dynamical mathematical approaches described above), they require a spacetime of a *known* form in order to be applied. We shall have to say more about these theorems in Section 4.

3 Geometric asymptotics for exact models

A large part of the literature[3] on cosmological singularities is directly concerned with the problem of tracing out the exact, early or late time, asymptotic behaviour of various cosmological models. In all these studies, a cosmology (in the sense of the previous Section) is given and the problem is then to determine the nature of the spacetime and of the various fields in the neighborhood of a past or future singularity. This is basically an *asymptotic problem* since the form of the spacetime is known in advance but the precise asymptotic relations of the unknown quantities present in the cosmology

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[2]There are many results in this and related directions, see the book[11] where this whole approach is presented along with interesting theorems and open problems.

[3]To do full justice to the enormous number of published papers in the many different areas of this subject can lead to a project in itself, something that this humble review never intended to do (a comprehensive *Bibliography of Papers on Cosmological Singularities* is currently under preparation). We apologize in advance to many readers to whom the choice of references included here, and more importantly their omissions, seems arbitrary. The present author is aware of this issue only too well!

are to be determined. In this Section, we present an outline of a new (or more precisely cleaned-up) method, called hereafter *the method of asymptotic splittings* (cf. [13] for a more detailed presentation), which enables us to precisely determined such asymptotic relations. The utility of the method of asymptotic splittings to analyze cosmological singularities is then shown in a number of recently examined cases (with references to the relevant papers). We also add some comments and remarks about the issue of fixed vs. spontaneous (or sudden) singularities, the latter having recently gained some attention, in a slightly different context, through the work of J. D. Barrow and others (see [14, 15, 12] and refs. therein).

3.1 Method of asymptotic splittings

In this subsection, we follow [13] closely and this reference is to be consulted for notation, explanations, and proofs of any stated result. The general setting is a vector field $\mathbf{f}$: $\mathcal{M}^n \to \mathcal{T}\mathcal{M}^n$ and the associated dynamical system defined by $\mathbf{f}$ on the manifold $\mathcal{M}^n$,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{3.1}$$

with $(\cdot) \equiv d/dt$. Depending on the number of arbitrary constants, a solution of our dynamical system can be *general, particular* or *exact*. Any solution, however, can develop *finite-time singularities*, that is instances where a solution $\mathbf{x}(t; c_1, \cdots, c_k)$, $k \leq n$, misbehaves at a finite value t_* of the time t. This is made more precise as follows. We say that the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ (equivalently, the vector field $\mathbf{f}$) has a *finite-time singularity* if there exists a $t_* \in \mathbb{R}$ and a $\mathbf{x}_0 \in \mathcal{M}^n$ such that for all $M \in \mathbb{R}$ there exists an $\delta > 0$ such that

$$||\mathbf{x}(t; \mathbf{x}_0)||_{L^p} > M, \tag{3.2}$$

for $|t - t_*| < \delta$. Here $\mathbf{x} : (0, b) \to \mathcal{M}^n$, $\mathbf{x}_0 = \mathbf{x}(t_0)$ for some $t_0 \in (0, b)$, and $|| \cdot ||_{L^p}$ is any L^p norm. We say that the vector field has a *future* (resp. *past*) singularity if $t_* > t_0$ (resp. $t_* < t_0$). Note also, that t_0 is an arbitrary point in the domain $(0, b)$ and may be

taken to mean 'now'. We often write

$$\lim_{t \to t_*} ||\mathbf{x}(t; \mathbf{x}_0)||_{L^p} = \infty, \tag{3.3}$$

to denote a finite-time singularity at t_*. Our basic problem then is to find the structure of the set of points $\mathbf{x}_0$ in $\mathcal{M}^n$ such that, when evolved through the dynamical system defined by the vector field, the integral curve of $\mathbf{f}$ passing through a point in that set satisfies property (3.3).

There are two kinds of finite time singularities that a nonlinear dynamical system can possess, *fixed* and *movable* (sudden, or spontaneous in other terminology is often used for a movable singularity). A singularity is *fixed* if it is a singularity of $\mathbf{x}(t; \mathbf{C})$ for all $\mathbf{C}$; otherwise, we say it is a *movable singularity*. Our basic problem is then: What can a vector field do, or equivalently, how do the solutions of the associated dynamical system behave in the neighborhood of a finite time singularity? Assume that we are given a vector field and suppose that at some point, t_*, a system of integral curves, corresponding to a particular or a general solution, has a (future or past) finite-time singularity in the sense of definition (3.2). The vector field (or its integral curves, solutions of the dynamical system defined by $\mathbf{C}$) can basically do two things sufficiently close to the finite-time singularity, namely, it can either show some dominant feature or not. In the latter case, the integral curves can 'spiral' in some way around the singularity *ad infinitum* so that (3.2) is satisfied and the dynamics is totally controlled by subdominant terms, whereas in the former case solutions share a distinctly dominant behaviour on approach to the singularity at t_* determined by the most nonlinear terms.

Our approach to this problem is an asymptotic one. We decompose, or split, the vector field into simpler, component vector fields and examine whether the most nonlinear one of these shows a dominant behaviour while the rest become subdominant in some exact sense. We then built a system of integral curves corresponding, where feasible, to the general solution and sharing exactly its characteristics in a sufficiently small neighborhood of the finite-time singularity. In this way we are led to a general procedure to uncover the nature of singularities by constructing series expansion representations of

12

particular or general solutions of dynamical systems in suitable neighborhoods of their finite-time singularities.

This *method of asymptotic splittings* [13] consists of building splittings of vector fields that are valid asymptotically and trace the dominant behaviour of the vector field near the singularity. A resulting series expansion connected to a particular *dominant balance* helps to decide whether or not the arrived solution is a general one and to spot the exact positions of the arbitrary constants as well as their role in deciding about the nature of the time singularity. To apply the method of asymptotic splittings to a given dynamical system so as to discover the nature of its solutions near the time singularities, we must follow a number of steps:

1. Write the system of equations in the form of a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with $\mathbf{x} = (x_1, \cdots, x_n)$, and identify the vector field $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \cdots, f_n(\mathbf{x}))$.

2. Find all the different weight-homogeneous decompositions of the system, that is the splittings of the form

$$\mathbf{f} = \mathbf{f}^{(0)} + \mathbf{f}^{(1)} + \cdots + \mathbf{f}^{(k)},$$

   and choose one of these splittings to start the procedure.

3. Substitute the scale-invariant solution

$$\mathbf{x}^{(0)}(\tau) = \mathbf{a}\tau^{\mathbf{p}},$$

   into the equation $\dot{\mathbf{x}} = \mathbf{f}^{(0)}$. Study the resulting algebraic systems, and find all dominant balances $(\mathbf{a}, \mathbf{p})$ together with their orders.

4. Identify the non-dominant exponents, that is the positive numbers $q^{(j)}$, $j = 1, \cdots, k$, such that

$$\tau^{q^{(j)}} \sim \frac{\mathbf{f}^{\text{sub, }(j)}(\tau^{\mathbf{p}})}{\tau^{\mathbf{p}-1}} \to 0.$$

13

5. Construct the K-matrix $\mathcal{K}$:

$$\mathbf{f}^{(0)} \to D\mathbf{f}^{(0)} \to D\mathbf{f}^{(0)}(\mathbf{a}) \to D\mathbf{f}^{(0)}(\mathbf{a}) - \operatorname{diag}\mathbf{p}.$$

6. Compute the spectrum of $\mathcal{K}$,

$$\operatorname{spec}(\mathcal{K}) = (-1, \rho_2, \cdots, \rho_n).$$

Is $\mathcal{K}$ semi-simple? Are the balances hyperbolic?

7. Find the eigenvectors $\mathbf{v}^{(i)}$ of $\mathcal{K}$.

8. Identify s as the multiplicative inverse of the least common multiple of all the subdominant exponents and positive K-exponents.

9. Substitute the Puiseux series

$$x_i = \sum_{j=0}^{\infty} c_{ji}\, \tau^{p_i + \frac{j}{s}}$$

into the original system.

10. Identify the polynomials $\mathbf{P}_j$ and solve for the final recursion relations which give the unknown coefficients $\mathbf{c}_j$.

11. Check the compatibility conditions at the K-exponents,

$$\mathbf{v}_\rho^\top \cdot \mathbf{P}_\rho = 0, \quad \text{for each eigenvalue } \rho.$$

12. If the Puiseux series is valid, then the method is concluded for this particular splitting. Otherwise, if compatibility conditions are violated at the eigenvalue ρ^*, restart from step 9 by substituting the logarithmic series

$$\mathbf{x} = \tau^{\mathbf{p}}\left(\mathbf{a} + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mathbf{c}_{ij}\tau^{i/s}(\tau^\rho \log \tau)^{j/s}\right),$$

13. Get coefficient at order ρ^*. Write down the final expansion with terms up to order ρ^*.

14. Verify that compatibility at ρ^* is now satisfied.

15. Repeat whole procedure for each of the other possible decompositions.

Following the above steps even up to that of calculating a dominant balance in one particular decomposition (that is up to step 4), can be very useful since it offers one particular possible asymptotic behaviour of the system near the time singularity. In this respect, the whole method expounded here is truly generic since it helps to decide the generality of any behaviour found in an exact solution – that is, how many arbitrary constants there are in the final solution that shares that behaviour (particular or general solution). It is rare that a Puiseux series is inadequate to describe the dynamics (semi-simplicity of K), but in such cases one must resort to the more complex logarithmic solutions.

3.2 Example: Gauss-Bonnet cosmologies

There are many interesting systems that could be seen under the magnifying glass of the method of asymptotic splittings, cf. the systems in [13, 18, 19, 20]. Here we present a short summary of some of the results of our very recent work [21] in collaboration with A. Tsokaros, which is to be followed for motivation and further explanations. The starting point is the action

$$\mathcal{S} = \int_{\mathcal{M}^4} \mathcal{L}_{\text{total}} d\mu_g, \qquad d\mu_g = \sqrt{-g} d\Omega, \tag{3.4}$$

where $\mathcal{L}_{\text{total}}$ is the lagrangian density of the general quadratic gravity theory given in the form[4] $\mathcal{L}_{\text{total}} = \mathcal{L}(R) + \mathcal{L}_{\text{matter}}$, with

$$\mathcal{L}(R) = R + BR^2 + C\text{Ric}^2 + D\text{Riem}^2, \tag{3.5}$$

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[4]the conventions for the metric and the Riemann tensor are those of [17].

where B, C, D are constants. Since in four dimensions we have the Gauss-Bonnet identity,

$$\delta \int_{\mathcal{M}^4} R_{GB}^2 d\mu_g = 0, \quad R_{GB}^2 = R^2 - 4\text{Ric}^2 + \text{Riem}^2, \tag{3.6}$$

in the derivation of the field equations through variation of the action associated with (3.5), only terms up to Ric^2 will matter. Below we focus exclusively in spatially flat universes of the form

$$ds^2 = dt^2 - b(t)^2(dx^2 + dy^2 + dz^2), \tag{3.7}$$

which are radiation dominated ($P = \rho/3$), and use only the 00-component of the field equations, this being the following equation satisfied by the scale factor $b(t)$:

$$\frac{\dot{b}^2}{b^2} - \kappa \left[2\frac{\dddot{b}\,\dot{b}}{b^2} + 2\frac{\ddot{b}\dot{b}^2}{b^3} - \frac{\ddot{b}^2}{b^2} - 3\frac{\dot{b}^4}{b^4} \right] - \frac{b_1^2}{b^4} = 0, \tag{3.8}$$

where b_1 is a constant defined by

$$\frac{8\pi G\rho}{3c^4} = \frac{b_1^2}{b^4}, \quad (\text{from } \nabla_i T^{i0} = 0). \tag{3.9}$$

Note that the Friedmann solution $\sqrt{2b_1 t}$ of general relativity satisfies the above equation.

To apply the method of asymptotic splittings, we set $b = x$, $\dot{b} = y$ and $\ddot{b} = z$, and then Eq. (3.8) can be written as a dynamical system of the following form: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x} = (x, y, z)$:

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = \frac{y}{2\kappa} - \frac{b_1^2}{2\kappa yx^2} - \frac{yz}{x} + \frac{z^2}{2y} + \frac{3y^3}{2x^2}. \tag{3.10}$$

Then we look for the possible weight-homogeneous decompositions. The most interesting one has *dominant part*

$$\mathbf{f}^{(0)} = \left(y, z, \frac{z^2}{2y} - \frac{zy}{x} + \frac{3y^3}{2x^2} \right), \tag{3.11}$$

while the subdominant part reads

$$\mathbf{f}^{\text{sub}} = \left(0, 0, -\frac{b_1^2}{2\kappa yx^2} + \frac{y}{2\kappa} \right), \tag{3.12}$$

with $\mathbf{f} = \mathbf{f}^{(0)} + \mathbf{f}^{\text{sub}}$. The dominant balance for this decomposition turns out to be the unique (full order) form

$$(\mathbf{a}, \mathbf{p}) = \left(\left(\alpha, \frac{\alpha}{2}, -\frac{\alpha}{4} \right), \left(\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2} \right) \right), \tag{3.13}$$

16

where α is an arbitrary constant.

Next we turn to an examination of the form and properties of the K-matrix. The Kowalevskaya exponents for this particular decomposition, eigenvalues of the matrix $\mathcal{K} = D\mathbf{f}(\mathbf{a}) - \mathrm{diag}(\mathbf{p})$, are $\{-1, 0, 3/2\}$ with corresponding eigenvectors $\{(4, -2, 3), (4, 2, -1), (1, 2, 2)\}$. (The arbitrariness of the coefficient α in the dominant balance reflects the fact that one of the dominant exponents is zero with multiplicity one.) After following the other steps of the method we finally arrive at the following series expansion representation of the behaviour of the system in the neighborhood of the initial collapse singularity:

$$x(t) = \alpha \ (t - t_0)^{1/2} + c_{13} \ (t - t_0)^2 + \frac{\alpha^4 - 4b_1^2}{24\kappa\alpha^3} \ (t - t_0)^{5/2} + \cdots . \qquad (3.14)$$

The series expansions for $y(t)$ and $z(t)$ are given by the first and second time derivatives of the above expressions respectively.

There are several comments to be made for this solution. First, the series (3.14) has three arbitrary constants, α, c_{13}, t_0 (the last corresponding to the arbitrary position of the singularity) and is therefore a local expansion of the *general* solution around the movable singularity t_0. Secondly, our solutions are stable in the neighborhood of the singularity in the sense that all flat, radiation solutions of the quadratic gravity theory considered here are Friedmann-like regardless of the sign of the R^2 coefficient. This implies that there are no bounce solutions of the model analyzed here. It remains an open problem to find a physically reasonable modification of this model with stable, bouncing solutions valid at early times.

4 Differential geometric issues

There are many instances where one does not wish to put precise assumptions on the form of the cosmological spacetime but leave it unspecified instead. In such circumstances it is important to be able to find general conditions and criteria under which a 'universe' represented in such a generic way will develop a physical, past or future, singularity in a finite time. In fact, it turns out that one can formulate precise criteria for such purposes

and even begin to classify the possible mathematical and physical singularities of this generic situation.

In this section we review some attempts to tackle this problem, beginning with the (statement of the) time-honored singularity theorems, results predicting the general existence of spacetime singularities, first formulated in the sixties. These results utilized in their proof some of the most beautiful concepts and techniques from differential topology and Riemannian geometry, but since then the general theory has remained somewhat dormant and perhaps wanting. Recently, it has been possible to revive interest in this whole area and new results predicting the existence of complete as well as singular spacetimes have been made possible, mainly due to the efforts of Y. Choquet-Bruhat. An adaptation of these ideas to simple cosmological solutions reveals structures which are perhaps difficult to unearth by more traditional methods and analysis.

4.1   Completeness and singularities

The geometric criteria to be developed in this subsection for a spacetime to be either geodesically complete or incomplete will be based on certain properties of its spacelike submanifolds. Below we focus exclusively on spacelike *hypersurfaces*, although submanifolds of higher codimension also play a significant role for such purposes (the most well-known example of the latter situation is the notion of a closed trapped surface).

Consider a spacetime of the form $(\mathcal{M}, g)$ with $\mathcal{M} = \Sigma \times \mathcal{I}$, $\mathcal{I}$ being an interval in $\mathbb{R}$ (for simplicity we can take it to be the whole real line) and (Σ, g_t) a smooth, spacelike submanifold of dimension n, in which the smooth, $(n+1)$–dimensional, Lorentzian metric g splits as follows:

$$g \equiv -N^2(\theta^0)^2 + g_t, \quad g_t \equiv g_{ij}\,\theta^i\theta^j, \quad \theta^0 = dt, \quad \theta^i \equiv dx^i + \beta^i dt. \qquad (4.1)$$

Here $N = N(t, x^i)$ denotes the *lapse function*, $\beta^i(t, x^j)$ is the *shift vector field* and all spatial slices $\Sigma_t \,(= \Sigma \times \{t\})$ are spacelike hypersurfaces endowed with the time-dependent spatial metric $g_t \equiv g_{ij}dx^i dx^j$. We assume that our spacetime $(\mathcal{V}, g)$ is *regularly sliced,*

18

that is the lapse function is bounded, the shift vector field is uniformly bounded and the spatial metric is itself uniformly bounded below, cf. [1, 2].

There are two kinds of information about the slices Σ_t that play a role in what follows: topological and geometric. The topological part of the theory to-be-developed starts with a connection of the completeness of Σ_t with the global hyperbolicity of $(\mathcal{M}, g)$, as in the following result from [1, 2] which may be consulted for the proof.

Theorem 4.1 *Let $(\mathcal{M}, g)$ be a regularly sliced spacetime. Then the following are equivalent:*

1. *(Σ_0, g_0) is a g_0-complete Riemannian manifold*

2. *The spacetime $(\mathcal{M}, g)$ is globally hyperbolic*

We thus see that our ability to successfully exercise classical determinism in a general spacetime is intimately connected to the topological property of geodesic completeness, not of the whole of spacetime, but of each one of its spacelike slices (considered as Riemannian manifolds, so that, in particular the Hopf-Rinow theorem is valid). Unfortunately to say something about the geodesic completeness of the *whole* of our general spacetime $(\mathcal{M}, g)$ is not possible by using solely topological pieces of information as the one given above (unless $(\mathcal{M}, g)$ is trivial in some sense, see[2]).

We need to know in addition more about the geometry of $(\mathcal{M}, g)$. What we specifically need to know in order to state the following singularity as well as completeness theorems, is some *submanifold geometry* (cf., for example, [10] chap. 4 and parts of chap. 10). For both kinds of theorem, the notion the *shape tensor* (or second fundamental form) of the slice Σ_t becomes vital. Consider the mean curvature vector field H of Σ_t, denote by $\bar{\nabla}$ the induced connection on Σ_t, and let $K = \mathrm{nor}\,\bar{\nabla}$ be its shape tensor which is defined on pairs of vector fields of Σ_t.

The trace and norm of K play a fundamental role. For a unit future pointing vector field U normal to Σ_t, we define the *convergence* of Σ_t to be the real-valued function θ on

the normal bundle $N\Sigma_t$

$$\theta = \langle U, H \rangle = \frac{1}{n-1} \operatorname{trace} K. \tag{4.2}$$

A natural interpretation of θ is to say that when θ is positive the spacelike slice is bent inward so that its outward pointing normals converge, whereas when it is negative we have the reverse situation. One may easily then imagine a spatial slice such that θ is positive *everywhere* on it: we will picture it as a convex spacelike arc of infinite extent.

The only other piece of geometric information we shall need is this: the g_t norms of K as well as of the spatial gradient of the lapse function, namely, $|K|_{\mathbf{g}_t}$ and $|\nabla N|_{\mathbf{g}_t}$. These are the natural objects to consider and estimate in an effort to see how large or small they can become and so to develop a feeling on how the slice itself evolves with proper time.

We are now ready to state a basic singularity theorem and a completeness theorem to get a taste of this kind of result (for more on this see the references above). The first result is Hawking's singularity theorem.

Theorem 4.2 *If:*

- *$Ric(X, X) \geq 0$ for all causal vector fields X of the spacetime $(\mathcal{M}, g)$*

- *$\theta \geq C > 0$, everywhere on the Cauchy slice Σ,*

then no future-directed causal curve from Σ can have length greater than $1/C$.

This theorem continues to hold if in the place of the words 'Cauchy slice' one substitutes the weaker 'compact slice', although in this case the proof is somewhat more involved (cf. [10] pp. 431-3). The proof is by contradiction: One cannot have both a complete spacetime *and* a Cauchy (or compact) slice with a positive convergence *everywhere.* That is unlike the case where the convergence were positive only locally, then nothing bad would happen. (This is the case, for instance, when one considers a convex spacelike arc *of finite length* with positive convergence in two-dimensional Minkowski space: its normals are converging, they meet at some point to the future of the arc but there is

no singularity, spacetime is geodesically complete.) Therefore this theorem predicts that under this situation spacetime must be geodesically incomplete, that is singular, but in a very special way: the whole of spacetime will eventually fold around into itself!

On the brighter side, we have the following completeness result [1].

Theorem 4.3 *If*

- *$(\mathcal{M}, g)$ is a globally hyperbolic, regularly sliced spacetime*

- *for each finite t_1, $|\nabla N|_{\mathbf{g}_t}$ and $|K|_{\mathbf{g}_t}$ are integrable functions on $[t_1, +\infty)$,*

then $(\mathcal{M}, g)$ is future causally geodesically complete.

The proof of this result is achieved by showing that all future-directed geodesics have an infinite length, and it is done by analyzing properties of the geodesic equation. For various applications of this theorem to cosmological spacetimes, see [12, 16].

4.2   Fixed and sudden singularities

We pause in this subsection to discuss, in passing, some qualitative differences between the two possible *dynamical* kinds of finite time singularities inherent in the method of asymptotic splittings, namely, fixed and movable (or sudden) singularities. Both types of singularities are of course in accordance with the assumptions of the singularity theorems, although in discussions of the latter the distinction is never made explicit. (For concreteness, the reader if he so wishes, may take a simple cosmological metric, for instance one from the Robertson-Walker family, which has only one unknown function, the scale factor a.) We assume that we have a situation where the assumptions of the singularity theorems are valid and the model has a true, finite time singularity. Our problem currently is to obtain conditions under which this singularity is fixed or movable.

Suppose we have a singly infinite family of geodesics in our spacetime. These geodesics form a 2-dimensional geodesic surface parametrized by

$$x^r = x^r(u, v), \tag{4.3}$$

and one understands that the geodesics defined in this way are the parametric lines of u in the geodesic surface. We denote by $p^r = \partial x^r / \partial u$ the unit tangent vector field to the geodesics, and by $\eta^r = (\partial x^r / \partial v) dv$ the connecting, infinitesimal vector field between two adjacent geodesics of the family, also called a *Jacobi field*. These two vectors p^r, η^r satisfy two fundamental equations, namely the geodesic equation and the Jacobi equation (or equation of geodesic deviation) respectively. Both equations are very familiar and play a basic role in any discussion of geodesic incompleteness. They read:

$$\dot{p}^r + \Gamma^r_{mn} p^m p^n = 0, \quad \text{Geodesic Equation,} \tag{4.4}$$

and

$$\ddot{\eta}^r + R^r_{smn} p^s \eta^m p^n = 0, \quad \text{Jacobi Equation.} \tag{4.5}$$

Here, differentiation is meant with respect to arc length (wherever it makes sense). Both equations are nonlinear but their main difference is this: Whereas the equation of geodesics is not a tensor equation but a frame-dependent one and so any conclusions using it may change under coordinate transformations, the Jacobi equation is a tensorial equation and anything we say using it will be valid in an invariant geometric way.

For instance, although the existence of conjugate points (two events each having $\eta = 0$) on two adjacent geodesics follows from an analysis of the Jacobi equation and is true in any system once it is established, properties of solutions of the geodesic equation are not so insensitive.

Suppose we have a Robertson-Walker metric with a true singularity, as, for instance, the initial big bang (collapse) singularity in the standard cosmological models, cf.[17], p. 366. The actual spacetime point, location of the singularity, is then a conjugate point in the sense that there is a non-zero Jacobi field between any later point and the big bang that becomes zero there. Timelike geodesics that end at the big bang (which in this case is an all-encompassing singularity) are still solutions of the geodesic equation (4.4). From standard expressions, the Christoffel symbols diverge at the singularity, in other words the *coefficients* in the geodesic equation (4.4) are diverging. This means that the

standard big bang singularity is a *fixed* singularity in the terminology of the previous Section. It does not depend on the initial conditions (which may be taken to be at any later time), in this case the singularities of the dynamical system are read off from the singularities of its coefficients.

A completely different situation may however arise, and this is reminiscent with what happens to solutions of the very simple nonlinear equation $\dot{x} = x^2$ with initial condition $x(0) = x_0$. There is the solution $x(t) = x_0/(1 - x_0 t)$ and this is defined only on $(-\infty, 1/x_0)$, when $x_0 > 0$, on the whole of $\mathbb{R}$ for $x_0 = 0$, and on $(1/x_0, \infty)$ for $x_0 < 0$. We see that the solution ceases to exist after time $1/x_0$. The coefficient in the equation is, however, regular (in fact equals one). So this equation has singularities whose position depends on the initial conditions and they 'move' accordingly with the initial condition, but the coefficients of the equation don't 'see' them. This is the simplest example of a movable (or sudden) singularity.

We believe that in general relativity we can in principle have a similar situation. For, if a geodesic defined on an interval (a, b) is, say, future incomplete (as that would follow from a tensorial analysis of the invariant Jacobi equation), then it cannot be extended after b to the whole of the real line, but this b may or may not depend on the initial conditions chosen, making the singularity either fixed or movable. This is only to be expected because the geodesic equation (4.4) is nonlinear and so cannot have singularities of only the fixed kind (that is those that are read off from the coefficients of the equation). Singularities in the traditional form of g-incompleteness are usually thought of in the literature only as fixed singularities, no dependence of the singularity on the 'initial' (later for a past, earlier for a future movable singularity) conditions.

4.3   Bel-Robinson energy

It is clear from what has been said earlier that there are several different, inequivalent ways to characterize cosmological singularities. First, there is the traditional way, through differential-geometric methods, as explained in the standard text on this ap-

proach [22]. This set of methods uses a global approach to spacetimes with an emphasis on the causal, topological aspects of singularities. But there is more to characterizing singularities than this, and using *dynamics* one may arrive at a complementary set of tools for such purposes. We saw how, using the method of asymptotic splittings, we can have quantitative information about a found singularity and form a clear picture on the different modes of approach through series expansions of the key unknown functions pertinent to the given problem. Other relevant approaches based on dynamical systems are also of great use, cf. [23]. A third approach is to use the contrapositive of the completeness theorem discussed earlier and end up with *necessary conditions* for singularities. This approach to the nature of cosmological singularities was taken up in [12, 16] for the simple case of Robertson-Walker cosmologies. The result is that we can arrive at an interesting trichotomy of the singularities in various Friedman cosmologies by using the two main functions of the problem, namely, the Hubble expansion rate H and the scale factor a.

To the three main approaches to singularities another must be added. In none of these does the presence of matter fields play a role, at least directly, and the following way to classify cosmological singularities uses exactly this missing piece of information: It is based on the introduction of an invariant geometric quantity, the *Bel-Robinson energy* which takes into account precisely those features of the problem, related to the matter contribution, in which models still differ near the time singularity while having similar behaviours of a and H. In this way, we arrive at a complete classification of the possible cosmological singularities in the isotropic case (for more details see [24, 25, 26]).

The *Bel-Robinson energy at time t* is given by

$$\mathcal{B}(t) = \frac{1}{2} \int_{\mathcal{M}_t} \left(|E|^2 + |D|^2 + |B|^2 + |H|^2 \right) d\mu_{\bar{g}_t}, \qquad (4.6)$$

where by $|X|^2 = g^{ij} g^{kl} X_{ik} X_{jl}$ we denote the spatial norm of the 2-covariant tensor X, and the time-dependent space electric and magnetic tensors comprising a so-called *Bianchi*

24

field are given in tensorial notation by

$$E_{ij} = R^0_{i0j}, \; D_{ij} = \frac{1}{4}\eta_{ihk}\eta_{jlm}R^{hklm}, \; H_{ij} = \frac{1}{2}N^{-1}\eta_{ihk}R^{hk}_{0j}, \; B_{ji} = \frac{1}{2}N^{-1}\eta_{ihk}R^{hk}_{0j}. \quad (4.7)$$

For any Robertson-Walker spacetime, $ds^2 = -dt^2 + a^2(t)d\sigma^2$, the norms of the magnetic parts, $|H|, |B|$, are identically zero while $|E|$ and $|D|$, the norms of the electric parts, reduce to the forms

$$|E|^2 = 3\,(\ddot{a}/a)^2 \quad \text{and} \quad |D|^2 = 3\left((\dot{a}/a)^2 + k/a^2\right)^2. \quad (4.8)$$

Therefore the Bel-Robinson energy becomes

$$\mathcal{B}(t) = \frac{C}{2}\left(|E|^2 + |D|^2\right), \quad (4.9)$$

where C is the constant volume of (or *in* in the case of a non-compact space) the 3-dimensional slice at time t. Thus we find that

$$\mathcal{B}(t) \sim k_u^2(t) + k_\sigma^2(t), \quad (4.10)$$

where k_u, k_σ are the *principal sectional curvatures* of this space. Using this result, it is not difficult to show the following theorem concerning the completeness of Friedmann cosmologies following directly from bounds on their Bel-Robinson energy, and an associated minimum radius.

Theorem 4.4 *A spatially closed, expanding at time t_*, Friedmann universe that satisfies $\gamma < \mathcal{B}(t) < \Gamma$, where γ, Γ are constants, is causally geodesically complete. Further, there is a minimum radius, $a_{min} > \Delta^{-1/2}$, Δ depending on Γ, and these universes are eternally accelerating ($\ddot{a} > 0$).*

The ensuing classification meeting all the different criteria discussed previously results from the possible combinations of the three main functions in the problem, namely, the scale factor a, the Hubble expansion rate H and the Bel Robinson energy $\mathcal{B}$. The singularity types will by necessity entail a possible blow up in the functions $|E|, |D|$. If we suppose that the model has a finite time singularity at time $t = t_s$, then the possible behaviours of the functions in the triplet $(H, a, (|E|, |D|))$ are as follows:

S_1 H non-integrable on $[t_1, t]$ for every $t > t_1$

S_2 $H \to \infty$ at $t_s > t_1$

S_3 H otherwise pathological

N_1 $a \to 0$

N_2 $a \to a_s \neq 0$

N_3 $a \to \infty$

B_1 $|E| \to \infty,\ |D| \to \infty$

B_2 $|E| < \infty,\ |D| \to \infty$

B_3 $|E| \to \infty,\ |D| < \infty$

B_4 $|E| < \infty,\ |D| < \infty.$

The nature of a prescribed singularity is thus described completely by specifying the components in a triplet of the form

$$(S_i, N_j, B_l),$$

with the indices i, j, l taking their respective values as above. Here category S monitors the asymptotic behaviour of the expansion rate, closely related to the extrinsic curvature of the spatial slices, N that of the scale factor, describing in a sense what the whole of space eventually does, while B describes how the matter fields contribute to the evolution of the geometry on approach to the singularity. We know from the completeness theorems that all these quantities need to be uniformly bounded to produce geodesically complete universes. Outside complete universes, the whole situation can be very complicated and the classification above exploits what can happen in such a case when we consider the relatively simple geometry of isotropic cosmologies.

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