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Periodic solution for strongly nonlinear vibration systems by using the homotopy analysis method

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Abstract

In this paper, the periodic solutions for the oscillation of a mass attached to a stretched elastic wire are obtained using the homotopy analysis method (HAM). HAM helps us to obtain square root frequency ($\Omega = \omega^2$) in the form of approximation series of the convergence control parameter $\hbar$. Finally, the so-called valid region of $\hbar$ is determined by plotting the Ω-$\hbar$ curve. Comparison of the obtained results with exact solutions provides confirmation for the validity of HAM.

Keywords: Convergence-controller parameter, Homotopy analysis method, Homotopy-Pade technique, Oscillator

Introduction

It is difficult to solve nonlinear problems especially by analytic technique. There are some analytic techniques for solving nonlinear problems such as the perturbation method [1], the Lyapunov's small parameter method [2], the δ-expansion method [3], and the Adomian decomposition method [4,5] that are well known. These methods cannot always guarantee the convergence of approximation series. In 1992, Liao [6] employed the basic ideas of the homotopy to overcome the restrictions of traditional techniques [7,8], namely, homotopy analysis method (HAM). The topic of HAM has been rapidly growing in recent years and successfully applied to many nonlinear boundary problems such as nonlinear oscillators with discontinuities [9-12]. In this paper, we use HAM to obtain periodic solutions of the oscillation of a mass attached to a stretched elastic wire and afterwards compare the obtained results with exact solutions. The system oscillates between symmetric bounds $[-A, A]$, and its frequency depends on the amplitude A. The oscillation of a mass attached to a stretched elastic wire is as follows:

$$\frac{d^2\nu}{dt^2}+\nu-\frac{\lambda\nu}{\sqrt{1+\nu^2}}=0,\quad 0<\lambda\leq 1,\quad \nu(0)=A,\quad \nu'(0)=0. \tag{1}$$

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Note that for both the small and large ν, (1) becomes [13]

$$\omega \approx \sqrt{1-\lambda} \quad \text{for} \quad A \ll 1 \quad \text{and} \quad \omega \approx 1 \quad \textit{for} \quad A \gg 1. \tag{2}$$

According to (1) which is with odd nonlinearity, we rewrite (1) without including its square root. Then, (1) can be written in the form

$$\left(1+\nu^2\right)\left(\frac{d^2\nu}{dt^2}\nu\right)\left(2\nu+\frac{d^2\nu}{dt^2}\right)+\left(1-\lambda^2+\nu^2\right)\nu^3=0,$$
$$\nu(0)=A,\quad \nu'(0)=0. \tag{3}$$

Assume that the solutions (3) are periodic with the period $T=\frac{2\pi}{\omega}$, where ω is the frequency of oscillation. Substituting $\Omega=\omega^2$, $\tau=\omega t$, and $\nu(t)=V(\tau)$ in (3), we have

$$\left(1+V^2\right)\left(\Omega\frac{d^2V}{d\tau^2}V\right)\left(2V+\Omega\frac{d^2V}{d\tau^2}\right)+(1-\lambda^2+V^2)V^3=0,$$
$$V(0)=A,\; V'(0)=0. \tag{4}$$

Method of solution

The limit cycles of (4) are periodic motions with period $T=\frac{2\pi}{\omega}$, and thus, $V(\tau)$ can be expressed by

$$V(\tau)=\sum_{m=0}^{\infty} c_m\cos(m\tau), \tag{5}$$

where c_m are coefficients to be determined. Under the above rules of solution expression denoted by (5) and the boundary conditions, it is natural to choose

$$V_0(\tau) = A\cos(\tau) \tag{6}$$

as the initial guess to $V(\tau)$. Let Ω_0 denote the initial approximation of the frequency Ω. Under the rules of solution expression, we choose the auxiliary linear operator

$$L[\varphi(\tau;q)] = \Omega_0^2\left[\frac{d^2}{d\tau^2}+1\right]\varphi(\tau;q), \tag{7}$$

with the property

$$L[C_1\cos(\tau)+C_2\sin(\tau)] = 0, \tag{8}$$

where C_1 and C_2 are constants, and $q \in [0,1]$ is the homotopy parameter.

On the basis (4), we define a nonlinear operator

$$N[\varphi(\tau;q),\Omega(q)] = \left(1+\varphi(\tau;q)^2\right)\left(\Omega(q)\frac{d^2\varphi(\tau;q)}{d\tau^2}\varphi(\tau;q)\right) \times\left(2\varphi(\tau;q)\Omega(q)\frac{d^2\varphi(\tau;q)}{d\tau^2}\right) + (1-\lambda^2+\varphi(\tau;q)^2)\varphi(\tau;q)^3. \tag{9}$$

Let $\hbar$ denote a nonzero auxiliary parameter; we construct the zero-order deformation equation

$$(1-q)L[\varphi(\tau;q)-V_0(\tau)] = q\hbar N[\varphi(\tau;q),\Omega(q)],\quad q\in[0,1] \tag{10}$$

such that

$$\varphi(0;q) = A,\quad \frac{\partial\varphi(\tau;q)}{\partial\tau}|_{\tau=0} = 0. \tag{11}$$

When the parameter q increases from 0 to 1, the solution $\varphi(\tau;q)$ varies from $V_0(\tau)$ to $V(\tau)$, and $\Omega(q)$ varies from Ω_0 to Ω. Assume that $\varphi(\tau;q)$ and $\Omega(q)$ are analytic in $q \in [0,1]$ and can be expanded in the Maclaurin series of q as follows:

$$V(\tau) = V_0(\tau)+\sum_{j=0}^{\infty}V_j(\tau)q^j,\quad \Omega = \Omega_0+\sum_{j=0}^{\infty}\Omega_j q^j, \tag{12}$$

where

$$V_m(\tau) = \frac{\partial^m\varphi(\tau;q)}{m!\,\partial q^m}\;|_{q=0}, \tag{13}$$

$$\Omega_m = \frac{\partial^m\Omega(q)}{m!\,\partial q^m}\;|_{q=0}\;. \tag{14}$$

Notice that $V_m(\tau)$ and Ω_m contain the auxiliary parameter $\hbar$, which has influence on their convergence regions. Assume that $\hbar$ is properly chosen such that all of these Maclaurin series are convergent at $q = 1$. Hence, at $q = 1$, we have

$$V(\tau) = V_0(\tau)+\sum_{j=0}^{\infty}V_j(\tau),\quad \Omega = \Omega_0+\sum_{j=0}^{\infty}\Omega_j. \tag{15}$$

Differentiating the zero-order deformation (10) and (11) m times with respect to the embedding parameter q, then dividing them by $m!$, and finally setting $q = 0$, we have the so-called m-th order deformation equation

$$L[V_m(\tau)-\chi_m V_{m-1}(\tau)] = \hbar R_m(V_0,\Omega_0,\ldots,V_{m-1},\Omega_{m-1}),$$
$$V_m(0)=0,\ V_m^{'}(0)=0, m\geq 1, \tag{16}$$

where

$$R_m(V_0,\Omega_0,\ldots,V_{m-1},\Omega_{m-1}) = \frac{\partial^{m-1}N[\varphi(\tau;q),\Omega(q)]}{(m-1)!\,\partial q^{m-1}} \times |_{q=0}\,. \tag{17}$$

Note that V_m and Ω_{m-1} are all unknown; however, we have only (16) for V_m. Thus, an additional algebraic equation is required for determining Ω_{m-1}. It is found that the right-hand side of the m-th order deformation (16) is expressed by

$$R_m(V_0,\Omega_0,\ldots,V_{m-1},\Omega_{m-1}) = \sum_{k=0}^{\psi(m)}c_{m,k}(\Omega_{m-1}) \times\cos((2k+1)\tau), \tag{18}$$

where $c_{m,k}$ is a coefficient and $\psi(m)$ is a positive integer dependent on order m.

To avoid the secular term $\tau\cos(\tau)$ appearing in V_m, we must force the coefficient of $\cos(\tau)$ to be equal to zero. Thus, coefficient $c_{m,0}$ must be enforced to be zero so that this provides with the additional algebraic equation for determining Ω_{m-1}.

Therefore, it is easy to gain the solution of (16)

$$V_m(\tau) = \chi_m V_{m-1}(\tau)+\frac{\hbar}{\Omega_0^2}\sum_{j=1}^{\psi(m)}\frac{c_{m,j}(\Omega_{m-1})}{1-(2j+1)^2} \times\cos((2j+1)\tau)+C_1\cos(\tau)+C_2\sin(\tau), \tag{19}$$

where C_1 and C_2 are two coefficients and to be determined by conditions $V_m(0) = 0$ and $V_m^{'}(0) = 0$.

Thus, the N-th order approximation can be given by

$$V(\tau) \approx V_0(\tau)+\sum_{j=0}^{N}V_j(\tau) \tag{20}$$

and

$$\Omega \approx S_N = \Omega_0+\sum_{j=0}^{N}\Omega_j. \tag{21}$$

Convergence

Theorem 1. *If the solution series* $V_0(\tau) + \sum_{m=1}^{\infty} V_m(\tau)$ *and* $\Omega_0 + \sum_{j=1}^{\infty} \Omega_m$ *are convergent, then they must be the exact solution of Equation (4).*

Proof. Since the solution series

$$V_0(\tau) + \sum_{m=1}^{\infty} V_m(\tau) \tag{22}$$

is convergent, we have

$$\lim_{m\to\infty} V_m(\tau) = 0. \tag{23}$$

Using the left-hand side of high-order deformation equations, we have

$$\sum_{m=1}^{\infty} \left[V_m(\tau) - \chi_m V_{m-1}(\tau)\right] = 0. \tag{24}$$

Then, we have

$$\hbar \sum_{m=1}^{\infty} R_m(V_0, \Omega_0, \ldots, V_{m-1}, \Omega_{m-1}) = 0. \tag{25}$$

Since $h \neq 0$, then the above equation gives

$$\sum_{m=1}^{\infty} R_m(V_0, \Omega_0, \ldots, V_{m-1}, \Omega_{m-1}) = 0. \tag{26}$$

Let

$$\epsilon(\tau; q) = N[\varphi(\tau; q), \Omega(q)] \tag{27}$$

be denoted as the residual error. The residual error at $q = 1$ can be expanded by a Taylor series at $q = 0$ to give

$$\epsilon(\tau; 1) = \sum_{m=0}^{\infty} \frac{\partial^m N[\varphi(\tau; q), \Omega(q)]}{m!\, \partial q^m} |_{q=0} \,. \tag{28}$$

Now, from (26) and (28), we have

$$\epsilon(\tau; 1) = N[V(\tau), \Omega] = 0. \tag{29}$$

This ends the proof. □

The Ω is a function of $\hbar$. In accordance with the $\hbar$-curve of Ω, we can find the valid region of $\hbar$.

Results and discussion

Under the initial conditions mentioned in the previous section, (1) has the exact frequency [13]

$$\omega_E = \frac{\pi}{2} \int_0^1 \frac{A dx}{\sqrt{A^2\left(1 - x^2\right) - 2\lambda\left(\sqrt{1 + A^2} - \sqrt{1 + A^2 x^2}\right)}}. \tag{30}$$

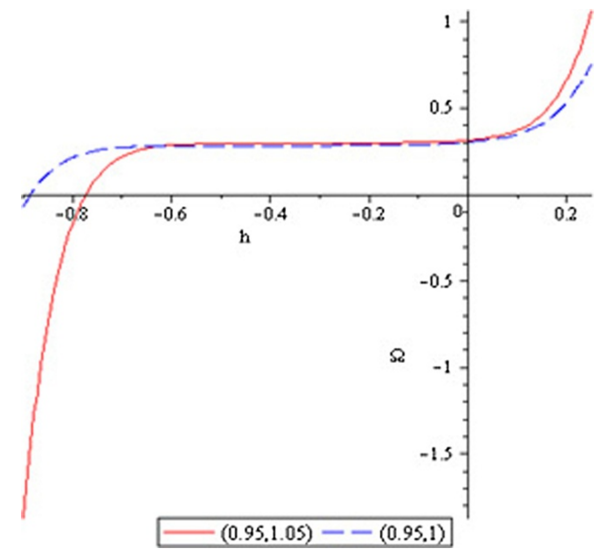

Figure 1 $\hbar$**-curves.** Solid line, $(\lambda, A) = (0.95, 1.05)$. Dash line, $(\lambda, A) = (0.95, 1)$.

Then, we have

$$\Omega_E = \omega_E^2. \tag{31}$$

We determine Ω_{m-1} by the analytic approach mentioned above; thus, for $m = 1$, we have

$$\Omega_0 = 1 \pm \frac{\sqrt{(36 + 30A^2)\lambda^2}}{6 + 5A^2}. \tag{32}$$

Since an increase in the amplitude A results in increasing Ω as discussed earlier in the 'Introduction' section, the negative sign in the equation mentioned above should be selected, i.e.,

$$\Omega_0 = 1 - \frac{\sqrt{(36 + 30A^2)\lambda^2}}{6 + 5A^2}. \tag{33}$$

Table 1 The S_N for $(\lambda, A, \hbar) = (0.95, 1, -0.5)$

N	S_N	$\frac{\lvert S_N - S_{N-1}\rvert}{\lvert S_N\rvert}$
8	0.2804601095	2.1657×10^{-3}
9	0.2801173977	1.2234×10^{-3}
10	0.2797319807	1.2178×10^{-3}
11	0.2794830922	8.9053×10^{-4}
12	0.2792183694	9.4808×10^{-4}
13	0.2790302341	6.7424×10^{-4}
14	0.2788380047	6.8939×10^{-4}
15	0.2786915081	5.2565×10^{-4}

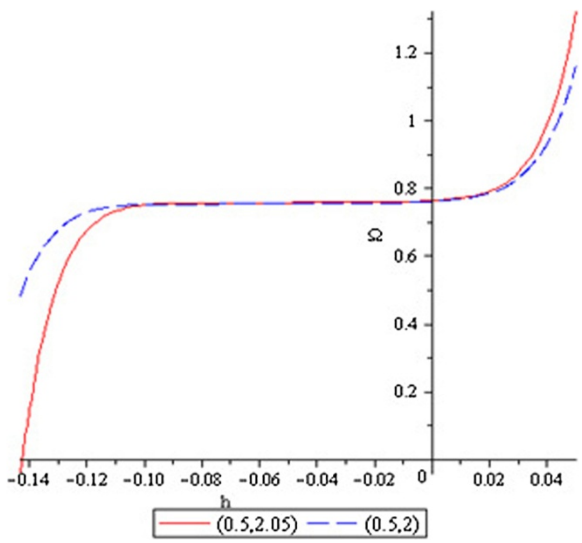

Figure 2 $\hbar$**-curves.** Solid line, $(\lambda, A) = (0.5, 2.05)$. Dash line, $(\lambda, A) = (0.5, 2)$.

Also, for $m = 2$, we have

$$\Omega_1 = \frac{1}{64}\frac{A^4\lambda^2(-654A^2\lambda^2+1,100A^2-636+475A^4)}{\gamma}\hbar, \tag{34}$$

$$\gamma = -150A^4\lambda^2 - 360A^2\lambda^2 - 216\lambda^2 + 450A^4 + 125A^6 + 540A^2 \tag{35}$$

$$+ \left(36\lambda^2 - 60A^2 + 30A^2\lambda^2 - 25A^4 - 36\right)\sqrt{(36+30A^2)\lambda^2}. \tag{36}$$

Note that the obtained results contain the auxiliary parameter $\hbar$. It is found that convergence regions of the approximation series are dependent upon $\hbar$ [6]. For example, we consider the cases of $(\lambda, A) = (0.95, 1)$ and $(\lambda, A) = (0.95, 1.05)$. We can plot the Ω-$\hbar$ curve to determine the so-called valid region of $\hbar$, as shown in Figure 1. Figure 1 indicates that the valid regions of $\hbar$ for $(\lambda, A) = (0.95, 1)$ and $(\lambda, A) = (0.95, 1.05)$ are $-0.7 < \hbar < 0$ and $-0.6 < \hbar < 0$, respectively. For $(\lambda, A, \hbar) = (0.95, 1, -0.5)$, we have the result $\Omega = 0.2786915081$ as shown in Table 1.

Also, we consider the cases of $(\lambda, A) = (0.5, 2)$ and $(\lambda, A) = (0.5, 2.05)$. We can plot the Ω-$\hbar$ curve to determine the so-called valid region of $\hbar$, as shown in Figure 2. Figure 2 indicates that the valid regions of $\hbar$ for $(\lambda, A) = (0.5, 2)$ and $(\lambda, A) = (0.5, 2.05)$ are $-0.11 < \hbar < 0.01$ and $-0.1 < \hbar < 0.01$, respectively.

For a proper value of $\hbar$ chosen in the above-mentioned valid region, the 15th-order approximations of HAM are compared with the exact value given by (31) for the different amplitudes shown in Tables 2 and 3.

Homotopy-Pade technique

There exist some techniques to improve the convergence rate of a given series by HAM. Among these techniques, the so-called Pade technique is widely applied. The so-called homotopy-Pade technique was suggested by the means of combining the Pade technique with HAM [6].

For a given series

$$\Delta_M(q) = \sum_{j=0}^{M} \Omega_j q^j, \tag{37}$$

the corresponding $[k, n]$ Pade approximate is expressed by

$$\Delta_{k+n}(q) = \frac{A_{k,n}(q)}{B_{k,n}(q)} = \frac{\sum_{j=0}^{k} a_j q^j}{1+\sum_{j=1}^{n} b_j q^j}, \tag{38}$$

where a_j and b_j are determined by the coefficients Ω_j, $(j = 0, 1, \ldots, n+k)$. Setting $q = 1$ provides the $[k, n]$ homotopy-Pade approximation

$$\Omega_{[k,n]} = \Delta_{k+n}(1) = \sum_{j=0}^{n+k} \Omega_j = \frac{A_{k,n}(1)}{B_{k,n}(1)} = \frac{\sum_{j=0}^{k} a_j}{1+\sum_{j=1}^{n} b_j}, \tag{39}$$

which accelerates the convergence rate of the solution series of HAM. We have applied the homotopy-Pade technique to accelerate the convergence rate of Mth-order approximations of HAM.

The approximations of homotopy-Pade technique are compared with the exact value given by (31) for different amplitudes shown in Table 4. The comparisons indicate that, even for a rather large amplitude, the homotopy analysis method provides efficient frequency approximations for the oscillation with strong nonlinearity.

Table 2 Comparison of the 15th-order approximations of HAM with the exact solutions for $\lambda = 0.95$

A	HAM	Exact	$\frac{\lvert\text{HAM}-\text{Exact}\rvert}{\lvert\text{Exact}\rvert}$
0.1	0.05393	0.05353	7.4727×10^{-3}
1	0.27869	0.27074	2.8505×10^{-2}
10	0.89466	0.88047	1.5858×10^{-2}
100	0.98957	0.98791	1.6760×10^{-3}

Table 3 Comparison of the 15th-order approximations of HAM with the exact solutions for $\lambda = 0.5$

A	HAM	Exact	$\frac{\lvert\text{HAM}-\text{Exact}\rvert}{\lvert\text{Exact}\rvert}$
0.1	0.50206	0.50186	4.0368×10^{-4}
1	0.62614	0.61806	1.3070×10^{-2}
10	0.94501	0.93722	8.3119×10^{-3}
100	0.99451	0.99363	8.8912×10^{-4}

Table 4 Comparison of the exact solution with the approximation of the homotopy-Pade technique for $\lambda = 0.5$

	$A = 0.1$	$A = 0.4$	$A = 1$	$A = 4$
$\frac{\Omega_{[1,1]}}{\Omega_E}$	1.00041	1.00593	1.0282	1.01942
$\frac{\Omega_{[2,2]}}{\Omega_E}$	1.00013	1.00229	1.01228	1.01794
$\frac{\Omega_{[3,3]}}{\Omega_E}$	1.00012	1.00250	1.01327	1.01755
$\frac{\Omega_{[4,4]}}{\Omega_E}$	1.00014	1.00265	1.01097	1.01758
$\frac{\Omega_{[5,5]}}{\Omega_E}$	1.00013	1.00220	1.01048	1.01750
$\frac{\Omega_{[6,6]}}{\Omega_E}$	1.00010	1.00139	1.00948	1.01739
$\frac{\Omega_{[7,7]}}{\Omega_E}$	1.00010	1.00122	1.00863	1.01734

Conclusions

We have applied HAM to the oscillation of a mass attached to a stretched elastic wire to obtain analytic approximations of the frequency of its limit cycles. According to the obtained results from the previous section, HAM could give efficient frequency approximations for the oscillation with strong nonlinearity. The results obtained with HAM are in excellent agreement with the exact solutions.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors conceived the study, participated in its design and coordination, drafted the manuscript, and participated in the sequence alignment. All authors read and approved the final manuscript.

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