Virtual fundamental classes via deformation theory and perfect obstruction theories

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A thesis submitted for the degree of
Master of Science

at

The University of Melbourne
Department of Mathematics and Statistics

October 2014

Abstract

In this thesis, we give an exposition of the theory of virtual fundamental classes in algebraic geometry introduced by K. Behrend and B. Fantechi in [5]. We place particular emphasis on the notion of obstruction theory, which is an essential ingredient in the construction of a virtual class. We show how obstruction theories can be constructed using fibre sequences of stacks coming from deformation theory. We illustrate the definitions throughout using examples coming from fibre products of schemes.

Contents

Chapter 1

Introduction

1.1 Expected dimensions and virtual fundamental classes

Virtual fundamental classes arise naturally in enumerative geometry as a means of counting in non-transverse situations. A common problem is to count the algebraic curves inside a projective variety Y, where the curves are allowed to degenerate in some well-controlled way. If Y is nice enough (for example, a Calabi-Yau 3-fold), then the space X of such curves has "expected dimension 0", so we expect to be able to calculate the number of such curves as an invariant of Y. However, even when the expected dimension is 0, the actual dimension of the moduli space X is often strictly positive, so there is not a finite number of curves to count.

To illustrate these issues in a simpler case, consider the intersection of the lines

$$Y = \{[x, y, z] \in \mathbb{P}^2 \mid y = 0\}$$

and

$$Z_t = \{[x, y, z] \in \mathbb{P}^2 \mid y = tx\}$$

for t in a base field k. For $t \neq 0$,

$$\begin{aligned} Y \cap Z_t &= \{[x, y, z] \in \mathbb{P}^2 \mid y = 0 \text{ and } y = tx\} \\ &= \{[0, 0, 1]\} \subseteq \mathbb{P}^2 \end{aligned}$$

whereas for $t = 0$

$$\begin{aligned} Y \cap Z_t &= \{[x, y, z] \in \mathbb{P}^2 \mid y = 0 \text{ and } y = 0\} \\ &= \{[x, y, z] \in \mathbb{P}^2 \mid y = 0\} \\ &\cong \mathbb{P}^1. \end{aligned}$$

Since $X_t = Y \cap Z_t$ is defined by two equations in a space of dimension 2, the expected, or virtual, dimension of X_t is $2 - 2 = 0$. When $t = 0$, however, we get a space of dimension 1, since the two defining equations are not independent. In this case, we can count the "generic" number of points in the intersection X_0 by counting the points in the "good" deformation X_t for nonzero t.

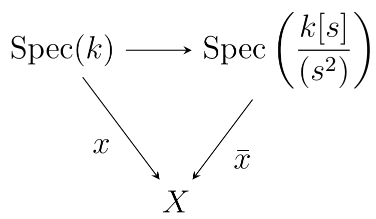
The virtual fundamental class is a generalisation of the generic counting idea above. If X is a scheme with expected dimension n, the virtual fundamental class is a Chow homology class

$$[X]^{vir} \in A_n(X)$$

of the expected dimension. If X can be deformed to a scheme $\tilde{X}$ of dimension n, then the virtual fundamental class of X can be understood in terms of the ordinary fundamental class representing the whole space of $\tilde{X}$. In the example $X_t = Y \cap Z_t$ above, the virtual fundamental class of X_0 is the class $[*] \in A_0(\mathbb{P}^1)$ representing a single point.

Unfortunately, there does not always exist a deformation of X to a scheme of the expected dimension. In such cases, we need extra data to replace the well-behaved deformation $\tilde{X}$.

To motivate where such extra data might come from, consider how we might calculate the expected dimension of a moduli space X. Assuming that X is smooth, the dimension of X is the same as the dimension of the tangent space T_xX at a point $x \in X$. In scheme theory, we can realise the space T_xX as the set of maps $\bar{x} : \text{Spec}(k[s]/(s^2)) \to X$ such that the diagram below commutes.



(Recall that $\text{Spec}(k)$ is a "point scheme" over k, so a map $\text{Spec}(k) \to X$ is the same as a point in X.) Since X is a moduli space, X has a universal property that characterises maps $\text{Spec}(k[s]/(s^2)) \to X$ in terms of the moduli problem. We usually use this universal property to compute the expected dimension of X.

For the example $X_t = Y \cap Z_t$, this plays out as follows. Since X_t is the fibre product of Y and Z_t over $\mathbb{P}^2$, by its universal property, a map $\text{Spec}(k[s]/(s^2)) \to X_t$ is the same as a commutative diagram as below.

$$\begin{array}{ccc} \operatorname{Spec}\left(\dfrac{k[s]}{(s^2)}\right) & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z_t & \longrightarrow & \mathbb{P}^2 \end{array}$$

Hence, the tangent space to X_t at a point x is just the fibre product of T_xY and T_xZ_t over $T_x\mathbb{P}^2$, which we can compute as the kernel of a map

$$T_xY \oplus T_xZ_t \to T_x\mathbb{P}^2. \tag{1.1.1}$$

For example, take $x = [0, 0, 1] \in X_t$. Then $T_xY = k$, $T_xZ_t = k$ and $T_x\mathbb{P}^2 = k^2$, and (1.1.1) is the map $k^2 \to k^2$ is given by the matrix

$$\begin{pmatrix} 1 & -1 \\ 0 & t \end{pmatrix}.$$

For $t \neq 0$, the map $k^2 \to k^2$ is surjective, so its kernel T_xX_t has dimension $2-2 = 0$. For $t = 0$, the map $k^2 \to k^2$ fails to be surjective, so the dimension of T_xX_t is strictly larger than the expected dimension.

This example illustrates that we can extract the virtual dimension of a moduli space X as follows. Using the universal property of X (for example, that it is a fibre product) to study extensions of maps $\operatorname{Spec}(k) \to X$ to maps $\operatorname{Spec}(k[s]/(s^2)) \to X$, we write the tangent space to X at a point x as

$$T_xX = \ker(E_0 \to E_1)$$

for some map of k-vector spaces $E_0 \to E_1$. In the transverse setup, we expect $E_0 \to E_1$ to be surjective, so the virtual dimension is

$$\dim[X]^{vir} = \dim E_0 - \dim E_1.$$

By replacing the map $\operatorname{Spec}(k) \to \operatorname{Spec}(k[s]/(s^2))$ with a more general square-zero extension, we can probe further into the geometry of $[X]^{vir}$. We discuss this more general problem in the next section.

1.2 Deformations and obstructions

Let X be a scheme over a field k, which we think of as a moduli space of some kind. The discussion of the previous section suggests that the extra data needed to define a virtual fundamental on X should come from using the universal property of X to study the following deformation problem.

Consider a square-zero extension $T \to \overline{T}$ of k-schemes (i.e. a closed embedding defined by an ideal of square zero—see Section 3.1) and a map $f : T \to X$. The discussion of Section 1.1 suggests that the extra data needed to define a virtual fundamental class for X should come from using the universal property of X to classify maps $\bar{f} : \overline{T} \to X$ such that the following diagram commutes.

$$\begin{array}{ccc} T & \longrightarrow & \overline{T} \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \bar{f}} & \\ & X & \end{array}$$

Consider again the example $X_t = Y \cap Z_t = Y \times_{\mathbb{P}^2} Z_t$ from Section 1.1. From the universal property, we should think of f as a commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{g} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle i} \\ Z_t & \xrightarrow{j} & \mathbb{P}^2 \end{array}$$

and an extension $\bar{f}$ of f as a commutative diagram

$$\begin{array}{ccc} \overline{T} & \xrightarrow{\bar{g}} & Y \\ {\scriptstyle \bar{h}}\downarrow & & \downarrow{\scriptstyle i} \\ Z_t & \xrightarrow{j} & \mathbb{P}^2 \end{array}$$

of k-schemes. Since Y and Z_t are smooth schemes over k, extensions $\bar{g}$ of g and $\bar{h}$ of h exist so long as T is affine. (This is part of the definition of smoothness for schemes—see Definition A.5.14.) The difference between the maps $i \circ \bar{g}$ and $j \circ \bar{h}$ is an element

$$\mathrm{ob}(\bar{g}, \bar{h}) \in \mathrm{Hom}(g^* i^* \Omega_{\mathbb{P}^2}, J)$$

where J is the ideal sheaf of T in $\overline{T}$, and Ω denotes the cotangent sheaf over k. An extension $\bar{f}$ is a choice of $\bar{g}$ and $\bar{h}$ such that $\mathrm{ob}(\bar{g}, \bar{h}) = 0$. Given arbitrary

$\bar{g}$ and $\bar{h}$, any other choices are obtained via actions of $\mathrm{Hom}(g^*\Omega_Y, J)$ and $\mathrm{Hom}(h^*\Omega_{Z_t}, J)$. Thus, the extensions $\bar{f}$ are classified by pairs

$$(u, v) \in \mathrm{Hom}(g^*\Omega_Y, J) \oplus \mathrm{Hom}(h^*\Omega_{Z_t}, J)$$

which map to $\mathrm{ob}(\bar{g}, \bar{h})$ under the difference map

$$\mathrm{Hom}(g^*\Omega_Y, J) \oplus \mathrm{Hom}(h^*\Omega_{Z_t}, J) \xrightarrow{d} \mathrm{Hom}(g^*i^*\Omega_{\mathbb{P}^2}, J). \tag{1.2.1}$$

Notice that if $T = \mathrm{Spec}(k)$, $\overline{T} = \mathrm{Spec}(k[s]/(s^2))$ then this recovers our computation of the tangent space in Section 1.1.

For T non-affine, it is possible that extensions $\bar{g}$ and $\bar{h}$ may not exist globally. To deal with this, we need to determine whether local choices over affine subsets of T can be glued together to give global extensions. There is a very elegant way to do this using the language of stacks.

First notice that we can rephrase the affine discussion above as follows. The obstruction $\mathrm{ob}(\bar{g}, \bar{h})$ can be thought of as an object in a groupoid

$$\underline{\mathbf{Ob}}_0 = [\mathrm{Hom}(g^*i^*\Omega_{\mathbb{P}^2}, J)/\mathrm{Hom}(g^*\Omega_Y, J) \oplus \mathrm{Hom}(h^*\Omega_{Z_t}, J)].$$

The groupoid $\underline{\mathbf{Ob}}_0$ is the category with objects $\mathrm{Hom}(h^*i^*\Omega_{\mathbb{P}^2}, J)$, and with morphisms $x \to y$ given by

$$\mathrm{Hom}(x, y) = \{(u, v) \in \mathrm{Hom}(g^*\Omega_Y, J) \oplus \mathrm{Hom}(h^*\Omega_{Z_t}, J) \mid d(u, v) = y - x\},$$

where d is the difference map (1.2.1). As an object of $\underline{\mathbf{Ob}}_0$, $\mathrm{ob}(\overline{T}) = \mathrm{ob}(\bar{g}, \bar{h})$ is independent up to canonical isomorphism of the choice of $\bar{g}$ and $\bar{h}$. The extensions $\bar{f}$ are in bijection with the set $\mathrm{Hom}(0, \mathrm{ob})$ of morphisms $0 \to \mathrm{ob}$ in $\underline{\mathbf{Ob}}_0$.

To solve the global problem, we notice that the groupoids $\underline{\mathbf{Ob}}_0$ and obstructions ob glue together to give a *stack*

$$\underline{\mathbf{Ob}} = [\underline{\mathrm{Hom}}(g^*i^*\Omega_{\mathbb{P}^2}, J)/\underline{\mathrm{Hom}}(g^*\Omega_Y, J) \oplus \underline{\mathrm{Hom}}(h^*\Omega_{Z_t}, J)] \tag{1.2.2}$$

and a section $\mathrm{ob}(\overline{T})$ of $\underline{\mathbf{Ob}}$, such that the extensions $\bar{f}$ of f are in bijection with the set $\mathrm{Hom}(0, \mathrm{ob})$ of morphisms $0 \to \mathrm{ob}$ in $\underline{\mathbf{Ob}}$.

We can package this solution very efficiently as follows. Fixing $f : T \to X$ and a sheaf J on T, we can consider stacks

$$\underline{\mathbf{Ext}}_{X_t}(T, J) \quad \text{and} \quad \underline{\mathbf{Ext}}(T, J)$$

of square-zero extensions of k-schemes $T \to \overline{T}$ with ideal sheaf J, with and without extensions $\bar{f} : \overline{T} \to X_t$ of the map f. There is a natural forgetful map

$$\underline{\mathbf{Ext}}_{X_t}(T, J) \to \underline{\mathbf{Ext}}(T, J)$$

and an obstruction map

$$\underline{\mathbf{Ext}}(T,J) \xrightarrow{\text{ob}} \underline{\mathbf{Ob}}$$

such that the sequence

$$\underline{\mathbf{Ext}}_{X_t}(T,J) \to \underline{\mathbf{Ext}}(T,J) \xrightarrow{\text{ob}} \underline{\mathbf{Ob}}$$

identifies $\underline{\mathbf{Ext}}_{X_t}(T,J)$ with the (categorical) fibre of ob over the distinguished section 0 of $\underline{\mathbf{Ob}}$. This is the same information as given above, but in a form which is very convenient to manipulate.

In general, the extra data required to construct a virtual fundamental class on a space X assigns to every map $f : T \to X$ and every sheaf J on T an obstruction stack $\underline{\mathbf{Ob}}$ and a fibre sequence

$$\underline{\mathbf{Ext}}_X(T,J) \to \underline{\mathbf{Ext}}(T,J) \to \underline{\mathbf{Ob}}.$$

The stack $\underline{\mathbf{Ob}}$ should have some nice form similar to (1.2.2). In the next section, we review some different approaches to formulating this precisely.

1.3 Three approaches to obstruction theories and virtual classes

In this section, we discuss three popular approaches to obstruction theories and virtual classes.

Virtual fundamental classes in algebraic geometry were first constructed by K. Behrend and B. Fantechi in [5] and by J. Li and G. Tian in [22]. To produce a virtual fundamental class on a moduli space X, both constructions take as input some kind of well-behaved obstruction theory for X. In [22], the obstruction theory is essentially an assignment of an obstruction space, Ob, and an obstruction map, ob : $\underline{\mathbf{Ext}}(T,J) \to$ Ob, to every scheme T, every sheaf J and every morphism $f : T \to X$, such that f can be extended to a map $\bar{f} : \overline{T} \to X$ if and only if $\text{ob}(\overline{T}) = 0$. These obstructions must be functorial in f, T and J.

In [5], an obstruction theory on X is defined in terms of the cotangent complex of X. To every morphism $\pi : Y \to S$, there is an associated *cotangent complex* $L_{Y/S}$ of sheaves on Y, which is a good generalisation of the cotangent sheaf to singular spaces, and can be used to compute the stack $\underline{\mathbf{Ext}}_S(Y,J)$ of square-zero extensions of Y over S. It follows from functoriality and exactness properties of the cotangent complex that for every $f : T \to X$ and every sheaf J on T, there is a canonical fibre sequence

$$\underline{\mathbf{Ext}}_X(T,J) \to \underline{\mathbf{Ext}}(T,J) \to \underline{\mathbf{Ob}}(f^*L_X, J)$$

where $\underline{\mathbf{Ob}}(f^*L_X, J)$ is a stack constructed from the complex f^*L_X. Thus, the cotangent complex L_X gives a canonical notion of obstruction, which is automatically functorial in f, T and J. If E is another complex of sheaves on X and $E \to L_X$ is a morphism of complexes, then we get functorial sequences of stacks

$$\underline{\mathbf{Ext}}_X(T, J) \to \underline{\mathbf{Ext}}(T, J) \to \underline{\mathbf{Ob}}(f^*E, J).$$

These will be fibre sequences so long as the map $E \to L_X$ satisfies a simple condition on cohomology. With this as motivation, an obstruction theory is defined to be a map $E \to L_X$ of complexes of sheaves on X satisfying this cohomology condition. If the complex E is *perfect*, there is a procedure for constructing a virtual fundamental class for X.

A newer and more refined approach to virtual fundamental classes uses the theory of derived algebraic geometry. In this setup, a virtual fundamental class on a scheme (or Deligne-Mumford stack) X comes from a choice of derived structure on X. The derived structure X^{der} is determined by a sheaf $\mathcal{O}_{X^{der}}$ of simplicial rings on X; the "ring" part of $\mathcal{O}_{X^{der}}$ is the structure sheaf $\mathcal{O}_X$ of X, and the "simplicial" part of $\mathcal{O}_{X^{der}}$ is extra data that keeps track of non-transversality. The sheaf $\mathcal{O}_{X^{der}}$ determines a K-theoretic fundamental class

$$[X]^{K\text{-}vir} = [\mathcal{O}_{X^{der}}] \in G_0(X)$$

where $G_0(X)$ is the K-theory of coherent sheaves on X. If the derived extension X^{der} of X is *quasi-smooth*, the (homological) virtual fundamental class is

$$[X]^{vir} = \mathrm{Td}(T_{X^{der}})^{-1} \cap \tau([X]^{K\text{-}vir}) \in A_*(X)$$

where $\mathrm{Td}(T_{X^{der}})$ is the Todd class of the tangent complex of X^{der} and $\tau : G_0(X) \to A_*(X)$ is the homological Chern character or Grothendieck-Riemann-Roch transformation for X.

The procedure above gives a construction for virtual fundamental classes which avoids the use of an obstruction theory. However, a derived structure X^{der} on X still gives rise to an obstruction theory as follows. The derived scheme X^{der} has a cotangent complex $L_{X^{der}}$, which is a module over the sheaf of simplicial rings $\mathcal{O}_{X^{der}}$. Tensoring $L_{X^{der}}$ with $\mathcal{O}_X$ gives a complex of $\mathcal{O}_X$-modules E with a map $E \to L_X$. Since X^{der} differs from X only in the extra derived structure, the map $E \to L_X$ satisfies the cohomological condition for an obstruction theory. This obstruction theory is perfect if and only if X^{der} is quasi-smooth, in which case the associated virtual fundamental class coincides with the one constructed from $[X]^{K\text{-}vir}$.

This approach also gives a formal interpretation of the philosophy that obstruction theories should come from studying universal properties. A derived extension X^{der} of a moduli space X can be specified by extending the moduli problem, in a more or less natural way, to classify maps $T \to X^{der}$ with T a derived scheme. The cotangent complex of X^{der} can be computed directly from this extended universal property, and therefore appears as a natural obstruction theory for the original universal property of X.

For a good survey of derived algebraic geometry, see [30], and for the rigorous foundational details, see [24] or [31].

Of course, the full power and utility of obstruction theories and virtual classes can only be appreciated by looking at a wide variety of examples. The original applications to Gromov-Witten theory can be found in [3] and [22]. A good survey of applications to curve counting is [27]. For an application using derived algebraic geometry see [28].

1.4 Plan of the thesis

The main aim of this thesis is to rework obstruction theories as defined in [5], using the stack-theoretic approach to deformation theory described in Section 1.2. We also give the construction for virtual fundamental classes, which is the motivation for the whole subject.

In Chapter 2, we recall some useful notions from the theory of sheaves and stacks over a topological space or site. This framework is a convenient way to package information which can be defined locally and satisfies a gluing condition. Important examples include the stacks of square-zero extensions and obstructions discussed in Section 1.2. We also introduce ringed sites, which are a convenient setting for our later study of homological algebra.

In Chapter 3, we review the theory of square-zero extensions. We define obstruction sequences, and demonstrate how they can be constructed from universal properties.

In order to have a framework in which we can easily do calculations, we recall the basics of homological algebra and derived categories in Chapter 4. We show in particular how calculations with derived categories can be converted into calculations with stacks.

In Chapter 5, we introduce obstruction theories as complexes, and show how they give rise to obstruction sequences. The universal example, which makes the whole theory work, is the cotangent complex, which we also discuss here.

In Chapter 6, we define the virtual fundamental class of a scheme X with a perfect obstruction theory. The construction is achieved by comparing a canonical algebraic stack over X, called the intrinsic normal cone of X, to an algebraic stack associated to the obstruction theory. We also compute some explicit examples of virtual classes of fibre products using the obstruction theories of Chapter 5.

We also include two appendices. Appendix A collects important background on algebraic geometry. In particular, we recall the basics of scheme theory, and touch on algebraic stacks and intersection theory. Appendix B collects some material on the cotangent complex, which is needed to deal with some naturality issues in Chapter 5.

Throughout this thesis, we work mainly with obstruction theories and virtual fundamental classes for schemes. In many applications, it is necessary to work in the more general setting of Deligne-Mumford stacks. We indicate at various points what modifications need to be made in this context.

Chapter 2

Sheaves and stacks

In this chapter, we recall some aspects of the theory of sheaves and stacks. Sheaves provide a way to package structures which are defined locally, such as functions or vector fields. Stacks are 2-categorical generalisations of sheaves: they package local structures which may have automorphisms. The ideas and results of this chapter form the basic framework for our study of deformations and obstructions.

2.1 Sheaves on topological spaces

In this section, we review the theory of sheaves on topological spaces. Informally, a sheaf F on a topological space X assigns to every open set $U \subseteq X$ a set $F(U)$ of *sections* over U, and to every inclusion $U \subseteq V$ a restriction map $F(V) \to F(U)$, in such a way that section $s \in F(U)$ can be defined locally by passing to an open cover $\{U_i\}$ of U.

Definition 2.1.1. Let X be a topological space. Denote by $\mathbf{Top}(X)$ the category with objects the open subsets of X and morphisms the inclusions of open sets. A *presheaf (of sets)* on X is a functor

$$F : \mathbf{Top}(X)^{op} \to \mathbf{Set}$$

from the opposite category of $\mathbf{Top}(X)$ to the category of sets. We say that F is a *sheaf* if for every open set $U \subseteq X$ and every open cover $\{U_i\}_{i\in I}$ of U, the diagram

$$F(U) \longrightarrow \prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \cap U_j)$$

is an equaliser. Here the two maps

$$\prod_{i\in I} F(U_i) \to \prod_{i,j\in I} F(U_i \cap U_j)$$

are given by

$$(s_i)_{i\in I} \mapsto (s_i|_{U_i\cap U_j})_{i,j\in I}$$
$$(s_i)_{i\in I} \mapsto (s_j|_{U_i\cap U_j})_{i,j\in I}$$

where $|_{U_i\cap U_j}$ denotes image under F of one of the inclusions $U_i \cap U_j \to U_i$ and $U_i \cap U_j \to U_j$. More explicitly, this means that if we have $s_i \in F(U_i)$ for each i, such that for each $i, j \in I$,

$$s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j},$$

then there exists a unique section $s \in F(U)$ with $s_i = s|_{U_i}$ for each $i \in I$.

Example 2.1.2. Let X be a topological space. The *sheaf of continous real-valued functions* on X is the functor $C_{\mathbb{R}} : \mathbf{Top}(X)^{op} \to \mathbf{Set}$ taking an open set $U \subseteq X$ to the set $C_{\mathbb{R}}(U)$ of continuous real-valued functions defined on U, and a morphism $U \subseteq V$ in $\mathbf{Top}(X)$ to the restriction map $C_{\mathbb{R}}(V) \to C_{\mathbb{R}}(U)$. The sheaf axiom in this case simply states that continuous maps $U \to \mathbb{R}$ can be specified by choosing an open cover $\{U_i\}$ and continuous maps $U_i \to \mathbb{R}$ which agree on the intersections $U_i \cap U_j$.

More generally, we can consider presheaves and sheaves with values in a general category $\mathcal{A}$. We give the definition in the case that $\mathcal{A}$ admits arbitrary products, although it can be generalised to arbitrary categories.

Definition 2.1.3. Let X be a topological space and let $\mathcal{A}$ be a category admitting arbitrary (small) products. A *presheaf with values in* $\mathcal{A}$ is a functor

$$F : \mathbf{Top}(X)^{op} \to \mathcal{A}.$$

We say that F is a *sheaf* if for all open $U \subseteq X$ and all open covers $\{U_i\}_{i\in I}$ of U, the diagram

$$F(U) \longrightarrow \prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \cap U_j)$$

is an equaliser in $\mathcal{A}$.

Example 2.1.4. In Example 2.1.2, the sheaf $C_{\mathbb{R}}$ can be viewed as a sheaf of $\mathbb{R}$-algebras, i.e. a sheaf with values in the category $\mathbb{R}$**-alg** of $\mathbb{R}$-algebras.

Definition 2.1.5. Let X be a topological space, and let $\mathcal{A}$ be a category. A *morphism of $\mathcal{A}$-valued presheaves on X* is a natural transformation of functors $\mathbf{Top}(X)^{op} \to \mathcal{A}$. A *morphism of $\mathcal{A}$-valued sheaves on X* is a morphism of $\mathcal{A}$-valued presheaves. We denote the corresponding categories of presheaves and sheaves by $\mathbf{Pr}(X, \mathcal{A})$ and $\mathbf{Sh}(X, \mathcal{A})$, or $\mathbf{Pr}(X)$ and $\mathbf{Sh}(X)$ in the case $\mathcal{A} = \mathbf{Set}$.

A very useful construction in sheaf theory is the sheaf associated to a presheaf. Roughly speaking, given a presheaf F on X, we can associate a sheaf $\mathrm{sh}(F)$ which is in some sense the best approximation of F by a sheaf. More precisely, we have the following theorem.

Theorem 2.1.6. *Let X be a topological space. Then the inclusion functor $i : \mathbf{Sh}(X) \to \mathbf{Pr}(X)$ has a left adjoint* $\mathrm{sh} : \mathbf{Pr}(X) \to \mathbf{Sh}(X)$*. Moreover,* sh *preserves finite limits.*

Definition 2.1.7. The functor $\mathrm{sh} : \mathbf{Pr}(X) \to \mathbf{Sh}(X)$ is called the *sheafification* functor. If F is a presheaf on X, then $\mathrm{sh}(F)$ is called the *sheafification of F* or the *sheaf associated to F*.

Theorem 2.1.6 is proved, for example, in [2], Exposé II (see Théorème 3.4 and Théorème 4.1) in the more general context of sites (see Section 2.2). The basic idea is to define a functor

$$L : \mathbf{Pr}(X) \to \mathbf{Pr}(X)$$

by setting, for any presheaf F,

$$(LF)(U) = \varinjlim_{\mathfrak{U}} F(\mathfrak{U})$$

where the limit is taken over all open covers $\mathfrak{U} = \{U_i\}_{i \in I}$ of U, and where $F(\mathfrak{U})$ is the limit of the diagram

$$\prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j \in I} F(U_i \cap U_j).$$

One then shows that $L \circ L(F)$ is a sheaf for all presheaves F and that the desired left adjoint is $\mathrm{sh} = L \circ L : \mathbf{Pr}(X) \to \mathbf{Sh}(X)$.

Remark 2.1.8. Theorem 2.1.6 applies equally to sheaves with values in a category $\mathcal{A}$ for many choices of $\mathcal{A}$. For instance, we can define the sheaf associated to an $\mathcal{A}$-valued presheaf so long as there is a faithful limit-preserving functor $F : \mathcal{A} \to \mathbf{Set}$. This holds, for example, for familiar “algebraic” categories, such as groups, abelian groups, rings, modules, algebras over a ring, etc.

Definition 2.1.9. Let $f : F \to G$ be a morphism of presheaves on a topological space X. We say that f is

(1) *locally injective* if for all open $U \subseteq X$ and all sections $s, t \in F(U)$ with $f(s) = f(t)$, there exists an open cover $\{U_i\}_{i \in I}$ of U such that $s|_{U_i} = t|_{U_i}$ for each i,

(2) *locally surjective* if for all open $U \subseteq X$ and all sections $s \in G(U)$, there exists an open cover $\{U_i\}_{i \in I}$ of U and sections $t_i \in F(U_i)$ such that $f(t_i) = s|_{U_i}$,

(3) a *local isomorphism* if f is both locally injective and locally surjective.

Remark 2.1.10. If $f : F \to G$ is a morphism of sheaves, we will usually say that f is *surjective* if f is locally surjective as a map of presheaves. This agrees with category-theoretic notions of surjectivity for, say, categories of sheaves of abelian groups or modules.

The following proposition follows easily from the construction of the sheafification functor.

Proposition 2.1.11. *Let $f : F \to G$ be morphism of presheaves. Then* $\mathrm{sh}(f) : \mathrm{sh}(F) \to \mathrm{sh}(G)$ *is injective (resp. surjective, an isomorphism) if and only if f is locally injective (resp. locally surjective, a local isomorphism).*

If $f : X \to Y$ is a continuous map between topological spaces, there are functors $f_* : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ and $f^{-1} : \mathbf{Sh}(Y) \to \mathbf{Sh}(X)$ called the *pushforward* and *pullback* functors, defined as follows. The pushforward of F is given by

$$(f_*F)(U) = F(f^{-1}(U))$$

for all open $U \subseteq Y$. This defines a presheaf on Y which is easily shown to be a sheaf. The pullback $f^{-1}F$ is the sheaf associated to the presheaf

$$\begin{aligned} \mathbf{Top}(X)^{op} &\to \mathbf{Set} \\ U &\mapsto \varinjlim_{f(U) \subseteq V} F(V). \end{aligned}$$

Proposition 2.1.12. *Let $f : X \to Y$ be a continuous map between topological spaces. Then the functor $f^{-1} : \mathbf{Sh}(Y) \to \mathbf{Sh}(X)$ is left adjoint to the functor $f_* : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$. Moreover, f^{-1} commutes with finite limits.*

Remark 2.1.13. Let X, Y and Z be topological spaces and let $f : X \to Y$ and $g : Y \to Z$ be continuous maps. Then there are canonical isomorphisms of functors $g_* \circ f_* \cong (g \circ f)_*$ and $f^{-1} \circ g^{-1} \cong (g \circ f)^{-1}$.

If X is a topological space, F is a sheaf on X and $x \in X$ is a point, the *stalk* of F at x is

$$F_x = \varinjlim_{x \in U} F(U)$$

where the colimit is taken over all open neighbourhoods U of x. We can also define stalks for $\mathcal{A}$-valued sheaves so long as the necessary colimits exist in $\mathcal{A}$. We can also view this as a special case of the pullback of a sheaf: if $*$ denotes the space with a single point, and $f : * \to X$ is the map sending this point to $x \in X$, then $\mathbf{Sh}(*, \mathcal{A}) = \mathcal{A}$ and $F_x = f^{-1}F$.

2.2 Sheaves on sites

There is a substantially more general setting than topological spaces in which we can study sheaves. The main idea is to replace the category $\mathbf{Top}(X)$ with a more general category $\mathcal{C}$ which retains a notion of open cover.

Definition 2.2.1 (cf. [9], Definition 2.24 and [2], Définition II.1.3)**.** Let $\mathcal{C}$ be a category and suppose that all fibre products exist in $\mathcal{C}$. A *(Grothendieck) topology* on $\mathcal{C}$ assigns to every object $U \in \mathrm{Ob}(\mathcal{C})$ a collection $\mathrm{Cov}(U)$ of families $\{U_i \to U\}_{i \in I}$ of morphisms in $\mathcal{C}$, called *covering families*, satisfying the following conditions.

(1) If $V \to U$ is a morphism in $\mathcal{C}$ and $\{U_i \to U\}_{i \in I}$ is a covering family for U, then $\{U_i \times_U V \to V\}_{i \in I}$ is a covering family for V.

(2) If $\{U_i \to U\}_{i \in I}$ is a covering family for U and $\{U_{ij} \to U_i\}_{j \in J_i}$ is a covering family for U_i for each $i \in I$, then the family of composites $\{U_{ij} \to U\}_{i \in I, j \in J_i}$ is a covering family for U.

(3) If $V \to U$ is an isomorphism, then $\{V \to U\}$ is a covering family.

A *site* is a category equipped with a Grothendieck topology.

Example 2.2.2. If X is a topological space, then the category $\mathbf{Top}(X)$, together with the usual notion of open cover, is a site.

The following examples are the main sites of interest in this thesis.

Example 2.2.3. Let X be a scheme. (See Appendix A, and Section A.1 in particular for a discussion of schemes.) The *Zariski site* of X is the site $X_{Zar} = \mathbf{Top}(|X|)$ associated to the underlying topological space of X.

Example 2.2.4. Let X be a scheme. The *(small) étale site* $X_{ét}$ of X is defined as follows. An object of $X_{ét}$ is an étale map $U \to X$ from a scheme U to X. A morphism in $X_{ét}$ is a commutative diagram of schemes

$$\begin{array}{ccc} U & \longrightarrow & V \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

with $U \to X$ and $V \to X$ étale. A covering in $X_{ét}$ is a family of morphisms $\{f_i : U_i \to U\}_{i \in I}$ such that

$$U = \bigcup_{i \in I} f_i(U_i).$$

The étale site $X_{ét}$ is similar to the Zariski site X_{Zar}, but includes multiple covers among the "open sets". For example, if $X = \operatorname{Spec}(k[x, x^{-1}]) = \mathbb{A}^1_k \backslash \{0\}$ is the punctured affine line over a field k of characteristic 0, then the double cover

$$\begin{aligned} \operatorname{Spec}(k[y, y^{-1}]) &\to \operatorname{Spec}(k[x, x^{-1}]) \\ y^2 &\leftarrow\!\shortmid x \end{aligned}$$

is a covering in $X_{ét}$.

Example 2.2.5. Let X be a scheme. The *big étale site* of X is the category $\mathbf{Sch}_{/X}$ of schemes over X, with the topology for which a family $\{f_i : U_i \to U\}_{i \in I}$ is a covering if each $f_i : U_i \to U$ is étale, and

$$U = \bigcup_{i \in I} f_i(U_i).$$

The main use of the big étale site is in applying sheaf-theoretic methods to construct spaces over X, such as total spaces of vector bundles. We discuss this point in more detail in Section 6.1.

Definition 2.2.6. Let $\mathcal{C}$ be a site, and let $\mathcal{A}$ be a category admitting arbitrary small products. An *$\mathcal{A}$-valued presheaf* on $\mathcal{C}$ is a functor

$$F : \mathcal{C}^{op} \to \mathcal{A}.$$

We say that F is a *sheaf* if for any $U \in \operatorname{Ob}(\mathcal{C})$ and any covering family $\{U_i \to U\}_{i \in I}$, the diagram

$$F(U) \longrightarrow \prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j \in I} F(U_i \times_U U_j)$$

is an equaliser. A *morphism of presheaves* is a natural transformation of functors and a *morphism of sheaves* is a morphism of presheaves. We denote the corresponding categories of $\mathcal{A}$-valued presheaves and sheaves by $\mathbf{Pr}(\mathcal{C}, \mathcal{A})$ and $\mathbf{Sh}(\mathcal{C}, \mathcal{A})$, or $\mathbf{Pr}(\mathcal{C})$ and $\mathbf{Sh}(\mathcal{C})$ in the case $\mathcal{A} = \mathbf{Set}$.

Just as in the case of sheaves on topological spaces, if $\mathcal{C}$ is a "small enough" site, then there is a left exact sheafification functor $\mathrm{sh} : \mathbf{Pr}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{C})$ which is left adjoint to the inclusion $\mathbf{Sh}(\mathcal{C}) \to \mathbf{Pr}(\mathcal{C})$. Here "small enough" means that there exists a small set $G \subseteq \mathrm{Ob}(\mathcal{C})$ such that every object of $\mathcal{C}$ has a covering by elements of G.

Definition 2.2.7 (cf. [9], Definition 2.58)**.** Let $\mathcal{C}$ be a category and let $U \in \mathrm{Ob}(\mathcal{C})$. The *category of objects over* U is the category $\mathcal{C}_{/U}$ with objects the morphisms $V \to U$ in $\mathcal{C}$, and with

$$\mathrm{Hom}_{\mathcal{C}_{/U}}(V \xrightarrow{f} U, W \xrightarrow{g} U) = \{h \in \mathrm{Hom}_{\mathcal{C}}(V, W) \mid g \circ h = f\}.$$

We will often abuse notation and write V for an object $V \to U$ of $\mathcal{C}_{/U}$, with the understanding that a particular map to U has been chosen. If $\mathcal{C}$ is a site, we endow $\mathcal{C}_{/U}$ with the topology in which a family $\{V_i \to V\}_{i \in I}$ of morphisms in $\mathcal{C}_{/U}$ is a covering if and only if it is a covering in $\mathcal{C}$. If F is a sheaf on $\mathcal{C}_{/U}$ we will often say that F is a sheaf on U.

The notion of pushforward and pullback of sheaves along continuous maps of topological spaces can be generalised to the setting of sites as follows.

Definition 2.2.8 ([2], 4.9.1)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be sites. A *morphism of sites* $f : \mathcal{C} \to \mathcal{D}$ is a functor $f^{-1} : \mathcal{D} \to \mathcal{C}$ satisfying

(1) for any sheaf F on $\mathcal{C}$, the presheaf

$$f_* F = F \circ f^{-1} : \mathcal{D}^{op} \to \mathbf{Set}$$

is a sheaf, and

(2) the functor $f_* : \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{D})$ has a left adjoint $f^{-1} : \mathbf{Sh}(\mathcal{D}) \to \mathbf{Sh}(\mathcal{C})$ which commutes with finite limits.

Notice that if $f : X \to Y$ is a continuous map of topological spaces, then we get a functor

$$\begin{aligned} \mathbf{Top}(Y) &\to \mathbf{Top}(X) \\ U &\mapsto f^{-1}(U) \end{aligned}$$

which defines a morphism of sites $f : \mathbf{Top}(X) \to \mathbf{Top}(Y)$ by Proposition 2.1.12. This is the motivation for defining a morphism of sites $\mathcal{C} \to \mathcal{D}$ to be a functor $\mathcal{D} \to \mathcal{C}$ instead of a functor $\mathcal{C} \to \mathcal{D}$.

2.3 Ringed sites

In this section, we recall the definitions of ringed sites and sheaves of modules. This theory is a natural generalisation of the theory of ringed spaces discussed in Section A.1.

Definition 2.3.1 (cf. Definition A.1.1)**.** A *ringed site* is a pair $(\mathcal{C}, A)$ where $\mathcal{C}$ is a site and A is a sheaf of rings on $\mathcal{C}$. If $(\mathcal{C}, A)$ and $(\mathcal{D}, B)$ are ringed sites, a *morphism of ringed sites* $f : (\mathcal{C}, A) \to (\mathcal{D}, B)$ is a pair $(f, f^{\#})$ where $f : \mathcal{C} \to \mathcal{D}$ is a morphism of sites and $f^{\#} : f^{-1}B \to A$ is a morphism of sheaves of rings on $\mathcal{C}$.

Remark 2.3.2. Following the perspective of Section A.1, we should think of a ringed site $(\mathcal{C}, A)$ as some kind of space $\mathcal{C}$, together with some functions A on $\mathcal{C}$.

The following are the main examples of interest in this thesis.

Example 2.3.3 (cf. Example 2.2.3)**.** Let X be a scheme. The structure sheaf $\mathcal{O}_X$ is a sheaf of rings on the Zariski site X_{Zar}, so the pair $(X_{Zar}, \mathcal{O}_X)$ is a ringed site. Every morphism of schemes $f : X \to Y$ induces a morphism of ringed sites $f : (X_{Zar}, \mathcal{O}_X) \to (Y_{Zar}, \mathcal{O}_Y)$.

Example 2.3.4 (cf. Example 2.2.4)**.** Let X be a scheme. The (small) étale site $X_{ét}$ of X has a sheaf of rings $\mathcal{O}_{X_{ét}}$ given by

$$\mathcal{O}_{X_{ét}}(U) = \mathcal{O}_U(U)$$

for $U \to X$ étale. So the pair $(X_{ét}, \mathcal{O}_{X_{ét}})$ is a ringed site. Every morphism of schemes $f : X \to Y$ induces a morphism of ringed sites $f : (X_{ét}, \mathcal{O}_{X_{ét}}) \to (Y_{ét}, \mathcal{O}_{Y_{ét}})$.

Example 2.3.5 (cf. Example 2.2.5)**.** Let X be a scheme. The big étale site $\mathbf{Sch}_{/X}$ of X has a sheaf of rings $\mathcal{O}_{/X}$ given by

$$\mathcal{O}_{/X}(U) = \mathcal{O}_U(U)$$

for $U \to X$ a scheme over X. So the pair $(\mathbf{Sch}_{/X}, \mathcal{O}_{/X})$ is a ringed site.

Example 2.3.6. Let X be a scheme. There is a morphism of ringed sites

$$z : (X_{ét}, \mathcal{O}_{X_{ét}}) \to (X_{Zar}, \mathcal{O}_{X_{Zar}})$$

defined as follows. The morphism of sites $z : X_{ét} \to X_{Zar}$ is given by the functor $z^{-1} : X_{Zar} \to X_{ét}$ which takes an open set $U \subseteq X$ to the inclusion map $U \to X$. (This is indeed a morphism of sites by [2], VII.4.2.2.) The morphism $z^{-1} : \mathcal{O}_X \to \mathcal{O}_{X_{ét}}$ is deduced from the canonical isomorphism $\mathcal{O}_X \to z_*\mathcal{O}_{X_{ét}}$ via the adjunction between z^{-1} and z_*.

Example 2.3.7. Let X be a scheme. There is a morphism of ringed sites

$$e : (\mathbf{Sch}_{/X}, \mathcal{O}_{/X}) \to (X_{\acute{e}t}, \mathcal{O}_{X_{\acute{e}t}})$$

defined as follows. The morphism of sites $e : \mathbf{Sch}_{/X} \to X_{\acute{e}t}$ is given by the functor $e^{-1} : X_{\acute{e}t} \to \mathbf{Sch}_{/X}$ which includes $X_{\acute{e}t}$ as a full subcategory of $\mathbf{Sch}_{/X}$. (This is indeed a morphism of sites by [2], VII.4.0.) The morphism $e^{-1}\mathcal{O}_{X_{\acute{e}t}} \to \mathcal{O}_{/X}$ is deduced from the canonical isomorphism $\mathcal{O}_{X_{\acute{e}t}} \to e_*\mathcal{O}_{/X}$ via the adjunction between e^{-1} and e_*.

Definition 2.3.8. Let $(\mathcal{C}, A)$ be a ringed site. An A-*module* is a sheaf of abelian groups M on $\mathcal{C}$ together with a morphism of sheaves of sets $A \times M \to M$, such that for all $U \in \mathrm{Ob}(\mathcal{C})$, the map $A(U) \times M(U) \to M(U)$ gives $M(U)$ an $A(U)$-module structure. If M and N are A-modules, an A-*module homomorphism* $f : M \to N$ is a morphism of sheaves of abelian groups such that for each $U \in \mathrm{Ob}(\mathcal{C})$, the map $f_U : M(U) \to N(U)$ is an $A(U)$-module homomorphism. We denote by A**-mod** the category of A-modules and A-module homomorphisms.

Example 2.3.9. Let $\mathcal{C}$ be the site associated to the topological space with one point. Then a sheaf A of rings on $\mathcal{C}$ is simply a ring, and A**-mod** as defined above is the usual category of A-modules.

Let $f : (\mathcal{C}, A) \to (\mathcal{D}, B)$ be a morphism of ringed sites. There is a functor

$$A\textbf{-mod} \to f_*A\textbf{-mod}$$
$$M \mapsto f_*M.$$

By the adjunction between f^{-1} and f_*, the morphism $f^{-1}B \to A$ induces a morphism $B \to f_*A$ of sheaves of rings on $\mathcal{D}$, so any f_*A-module has a canonical B-module structure. Combining this with the functor above, we get a functor

$$f_* : A\textbf{-mod} \to B\textbf{-mod}.$$

This functor has a left adjoint given by

$$f^* : B\textbf{-mod} \to A\textbf{-mod}$$
$$M \mapsto f^{-1}M \otimes_{f^{-1}B} A$$

where the tensor product is defined below.

Definition 2.3.10. Let $\mathcal{C}$ be a site, let A be a sheaf of rings on $\mathcal{C}$, and let M and N be A-modules. Then *tensor product* of M and N is the sheaf $M \otimes_A N$ associated to the presheaf

$$U \mapsto M(U) \otimes_{A(U)} N(U).$$

2.4 Stacks

In this section we discuss stacks, which are 2-categorical analogues of sheaves. Roughly speaking, a stack F over a site $\mathcal{C}$ assigns to every object $U \in \mathrm{Ob}(\mathcal{C})$ a category $F(U)$ and to every morphism $f : U \to V$ in $\mathcal{C}$ a functor $F(f) : F(V) \to F(U)$. These functors are required to satisfy $F(f \circ g) \cong F(g) \circ F(f)$ up to certain well-behaved natural isomorphisms. Just as sections of a sheaf may be glued over the covering families of $\mathcal{C}$, we also enforce a gluing condition for the objects of $F(U)$.

We give the definitions in the special case where each category $F(U)$ is a groupoid, as this is a little simpler than the general case and will suffice for our purposes. (Recall that a *groupoid* is a category in which every morphism is invertible.)

First, we formalise the notion of (contravariantly) assigning groupoids and functors to objects and morphisms of $\mathcal{C}$. In what follows, if $p : F \to \mathcal{C}$ is a functor and $U \in \mathrm{Ob}(\mathcal{C})$, we write $F(U)$ for the category with objects

$$\mathrm{Ob}(F(U)) = \{\alpha \in \mathrm{Ob}(\mathcal{C}) \mid p(\alpha) = U\}$$

and morphisms

$$\mathrm{Hom}_{F(U)}(\alpha, \beta) = \left\{ \tilde{f} \in \mathrm{Hom}_F(\alpha, \beta) \;\middle|\; p(\tilde{f}) = \mathrm{id}_U \right\}.$$

Definition 2.4.1 (cf. [9], Definition 3.1 and Proposition 3.22)**.** Let $\mathcal{C}$ be a category. A *category fibred in groupoids* (or *fibred category*) over $\mathcal{C}$ is a category F and a functor $p : F \to \mathcal{C}$ satisfying the following conditions.

(1) For every morphism $f : U \to V$ in $\mathcal{C}$, and every object $\beta \in \mathrm{Ob}(F(U))$, there exists a morphism $\tilde{f} : \alpha \to \beta$ such that $p(\tilde{f}) = f$.

(2) If

$$\begin{array}{ccc} U_1 & & \\ {\scriptstyle g}\downarrow & \overset{f_1}{\searrow} & \\ U_2 & \xrightarrow{f_2} & V \end{array}$$

is a commutative diagram in $\mathcal{C}$ and $\tilde{f}_1 : \alpha_1 \to \beta$ and $\tilde{f}_2 : \alpha_2 \to \beta$ are morphisms in F with $p(\tilde{f}_1) = f_1$ and $p(\tilde{f}_2) = f_2$ then there exists a unique morphism $\tilde{g} : \alpha_1 \to \alpha_2$ such that $p(\tilde{g}) = g$ and such that the diagram

$$\begin{array}{ccc} \alpha_1 & & \\ {\scriptstyle \tilde{g}}\downarrow & \searrow^{\tilde{f}_1} & \\ \alpha_2 & \xrightarrow{\tilde{f}_2} & \beta \end{array}$$

commutes.

If $p : F \to \mathcal{C}$ and $q : G \to \mathcal{C}$ are categories fibred in groupoids over $\mathcal{C}$, a *(1-)morphism of fibred categories* from F to G is a functor $f : F \to G$ such that $q \circ f = p$. If $f, g : F \to G$ are morphisms of fibred categories over $\mathcal{C}$, *2-isomorphism* from f to g is a natural transformation $\eta : f \to g$ such that for all $U \in \mathrm{Ob}(\mathcal{C})$ and all $\alpha \in F(U)$, $q(\eta_\alpha) = \mathrm{id}_U$. We denote the category of morphisms $F \to G$ and 2-isomorphisms by $\mathrm{Hom}(F, G)$. We write $\mathbf{Fib}(\mathcal{C})$ for the 2-category of categories fibred in groupoids over $\mathcal{C}$, and $\mathrm{Ho}(\mathbf{Fib}(\mathcal{C}))$ for the associated "homotopy category" obtained by taking isomorphism classes of 1-morphisms.

Remark 2.4.2. If $p : F \to \mathcal{C}$ is a category fibred in groupoids over $\mathcal{C}$ then $F(U)$ is a groupoid for all $U \in \mathrm{Ob}(\mathcal{C})$.

Definition 2.4.3. Let $p : F \to \mathcal{C}$ be a category fibred in groupoids over $\mathcal{C}$. A *cleavage* for F consists of a choice of morphism $F(f)(\alpha) \to \alpha$ over f for all $\alpha \in \mathrm{Ob}(F(V))$ and all morphisms $f : U \to V$ in $\mathcal{C}$.

Let $p : F \to \mathcal{C}$ be a category fibred in groupoids over $\mathcal{C}$. Given a cleavage for F, and a morphism $f : U \to V$ in $\mathcal{C}$, there is a functor $F(f) : F(V) \to F(U)$ which takes a morphism $\alpha \to \beta$ in $F(V)$ to the unique morphism in $F(U)$ such that the following diagram commutes.

$$\begin{array}{ccc} F(f)(\alpha) & \longrightarrow & \alpha \\ \downarrow & & \downarrow \\ F(f)(\beta) & \longrightarrow & \beta \end{array}$$

One can show that $F(f)$ is independent (up to canonical natural isomorphism) of the choice of cleavage, and that we have $F(g \circ f) \cong F(f) \circ F(g)$ whenever $f : U \to V$ and $g : V \to W$ are morphisms in $\mathcal{C}$.

Remark 2.4.4. We will often write down a fibred category F over $\mathcal{C}$ by specifying the categories $F(U)$ and the functors $F(f) : F(V) \to F(U)$ coming from some choice of cleavage. It is usually clear from these data how to write down the formal definition of F.

Example 2.4.5. Let $F : \mathcal{C}^{op} \to \mathbf{Set}$ be a functor. Define a category $\tilde{F}$ fibred in groupoids over $\mathcal{C}$ as follows. Let

$$\mathrm{Ob}(\tilde{F}) = \coprod_{U \in \mathrm{Ob}(\mathcal{C})} F(U)$$

and if $U, V \in \mathrm{Ob}(\mathcal{C})$ and $\alpha \in F(U)$ and $\beta \in F(V)$, let

$$\mathrm{Hom}_{\tilde{F}}(\alpha, \beta) = \{f \in \mathrm{Hom}_{\mathcal{C}}(U, V) \mid F(f)(\beta) = \alpha\}.$$

There is an obvious functor $p : \tilde{F} \to \mathcal{C}$ which makes $\tilde{F}$ into a category fibred in groupoids over $\mathcal{C}$. In fact, $\tilde{F}$ is an example of a *category fibred in sets* over $\mathcal{C}$. By [9], Proposition 3.26, the construction $F \mapsto \tilde{F}$ defines an equivalence between the category of functors $F : \mathcal{C}^{op} \to \mathbf{Set}$ and the full subcategory of $\mathbf{Fib}(\mathcal{C})$ spanned by the categories fibred in sets.

Let F be a category fibred in groupoids over $\mathcal{C}$ and let $X \in \mathrm{Ob}(\mathcal{C})$. Denote by $j(X)$ the fibred category associated to the functor $j(X) = \mathrm{Hom}(-, X)$. There is a natural equivalence of groupoids

$$F(X) \simeq \mathrm{Hom}(j(X), F).$$

This result is known as the 2-Yoneda Lemma. For a proof, see [9], Section 3.6.2.

Let $F \to \mathcal{C}$ be a category fibred in groupoids over $\mathcal{C}$, and suppose we have chosen a cleavage for F. Fix $U \in \mathrm{Ob}(\mathcal{C})$ and $\alpha, \beta \in F(U)$. Then we have a functor, $\underline{\mathrm{Hom}}_U(\alpha, \beta)$, given by

$$\begin{aligned}(\mathcal{C}_{/U})^{op} &\to \mathbf{Set} \\ (V \xrightarrow{f} U) &\mapsto \mathrm{Hom}_{F(V)}(F(f)(\alpha), F(f)(\beta)).\end{aligned}$$

Definition 2.4.6 (cf. [9], Definition 4.2, Definition 4.6 and Proposition 4.7)**.** Let $\mathcal{C}$ be a site and let $F \to \mathcal{C}$ be a category fibred in groupoids over $\mathcal{C}$. We say that F is a *stack (in groupoids)* if the following conditions hold.

(1) For any $U \in \mathrm{Ob}(\mathcal{C})$ and any $\alpha, \beta \in F(U)$, the presheaf $\underline{\mathrm{Hom}}_U(\alpha, \beta)$ is a sheaf on U.

(2) Let $\{f_i : U_i \to U\}_{i \in I}$ be a covering family for $U \in \mathrm{Ob}(\mathcal{C})$, and let $U_{ij} = U_i \times_U U_j$ and $U_{ijk} = U_i \times_U U_j \times_U U_k$ for all $i, j, k \in I$. Suppose that $\alpha_i \in \mathrm{Ob}(F(U_i))$ for each $i \in I$ and that $\phi_{ij} : F(\pi_2)(\alpha_j) \to F(\pi_1)(\alpha_i)$ is an isomorphism in $F(U_{ij})$ for each $i, j \in I$, where $\pi_1 : U_{ij} \to U_i$ and

$\pi_2 : U_{ij} \to U_j$ are the projection maps. If, for any triple of indices $i, j, k \in I$, we have

$$F(\pi_{13})(\phi_{ik}) = F(\pi_{12})(\phi_{ij}) \circ F(\pi_{23})(\phi_{jk})$$

in $F(U_{ijk})$, then there exists an object $\alpha \in F(U)$ and isomorphisms $\psi_i : F(f_i)(\alpha) \to \alpha_i$ in $F(U_i)$ such that the diagram

$$\begin{array}{ccc} F(\pi_2) \circ F(f_j)(\alpha) & \longrightarrow & F(\pi_1) \circ F(f_i)(\alpha) \\ {\scriptstyle F(\pi_2)(\psi_j)} \big\downarrow & & \big\downarrow {\scriptstyle F(\pi_1)(\psi_i)} \\ F(\pi_2)(\alpha_j) & \xrightarrow{\quad \phi_{ij} \quad} & F(\pi_1)(\alpha_i) \end{array}$$

in $F(U_{ij})$ commutes for all $i, j \in I$.

We write $\mathbf{St}(\mathcal{C})$ for the full 2-subcategory of $\mathbf{Fib}(\mathcal{C})$ spanned by the stacks, and $\mathrm{Ho}(\mathbf{St}(\mathcal{C}))$ for its homotopy category.

Remark 2.4.7. In Definition 2.4.6, condition (1) states that we can glue morphisms, while condition (2) states that we can glue objects.

Example 2.4.8. Let $F : \mathcal{C}^{op} \to \mathbf{Set}$ be a functor. Then F is a sheaf if and only if the associated fibred category $\tilde{F}$ from Example 2.4.5 is a stack.

Definition 2.4.9. Let $\mathcal{C}$ be a site, and let $f : F \to G$ be a morphism of fibred categories over $\mathcal{C}$. We say that f is

(1) *locally fully faithful* if for every $U \in \mathrm{Ob}(\mathcal{C})$ and every $\alpha, \beta \in \mathrm{Ob}(F(U))$, the morphism

$$\mathrm{sh}\underline{\mathrm{Hom}}_U(\alpha, \beta) \to \mathrm{sh}\underline{\mathrm{Hom}}_U(f(\alpha), f(\beta))$$

is an isomorphism of sheaves on $\mathcal{C}_{/U}$,

(2) *locally essentially surjective* if for every $U \in \mathrm{Ob}(\mathcal{C})$ and every $\alpha \in \mathrm{Ob}(G(U))$, there exists a cover $\{g_i : U_i \to U\}_{i \in I}$ of U, objects $\beta_i \in \mathrm{Ob}(F(U_i))$ and morphisms $f(\beta_i) \to \alpha$ in G lying over g_i, and

(3) a *local equivalence* if f is both locally fully faithful and locally essentially surjective.

Proposition 2.4.10. *Let $\mathcal{C}$ be a site, and let $f : F \to G$ be a morphism of stacks over $\mathcal{C}$. Then f is an equivalence if and only if f is a local equivalence.*

Proof. This follows immediately from [29, Tag 04WQ] and [29, Tag 046N]. □

There is an analogue of the sheafification functor for stacks in groupoids, which we recall below.

Proposition 2.4.11 (cf. [29, Tag 02ZN] and [29, Tag 04W9]). *Let $p : F \to \mathcal{C}$ be a category fibred in groupoids over a site $\mathcal{C}$. Then there exists a stack $\mathrm{st}(F)$ and a local equivalence $F \to \mathrm{st}(F)$, which is unique up to unique 2-isomorphism. Moreover, if G is any stack in groupoids over $\mathcal{C}$, then the functor*

$$\mathrm{Hom}_{\mathbf{St}(\mathcal{C})}(\mathrm{st}(F), G) \to \mathrm{Hom}_{\mathbf{Fib}(\mathcal{C})}(F, G)$$

is an equivalence of groupoids.

Corollary 2.4.12. *The construction of Proposition 2.4.11 determines a functor*

$$\mathrm{st} : \mathrm{Ho}(\mathbf{Fib}(\mathcal{C})) \to \mathrm{Ho}(\mathbf{St}(\mathcal{C})).$$

Corollary 2.4.13. *Let $f : F \to G$ be a morphism of fibred categories over a site $\mathcal{C}$. Then f is a local equivalence if and only if*

$$\mathrm{st}(f) : \mathrm{st}(F) \to \mathrm{st}(G)$$

is an equivalence.

Just as for sheaves, we can push fibred categories and stacks forward along morphisms of sites. Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of sites given by a functor $f^{-1} : \mathcal{D} \to \mathcal{C}$, and let $p : F \to \mathcal{C}$ be a fibred category (resp. stack) over $\mathcal{C}$. The *pushforward* of F is the category $f_*F = F$ with

$$\mathrm{Ob}(f_*F) = \left\{ (U, \alpha) \;\middle|\; U \in \mathrm{Ob}\,\mathcal{D}, \alpha \in \mathrm{Ob}\,F(f^{-1}(U)) \right\},$$

and

$$\begin{aligned}\mathrm{Hom}(&(U,\alpha),(V,\beta)) \\ &= \left\{ (g,h) \;\middle|\; g \in \mathrm{Hom}_{\mathcal{D}}(U,V), h \in \mathrm{Hom}_F(\alpha,\beta), p(h) = f^{-1}(g) \right\}\end{aligned}$$

for $(U,\alpha), (V,\beta) \in \mathrm{Ob}\,f_*F$. By [29, Tag 04WB] (resp. [29, Tag 04WD]), f_*F is a fibred category (resp. stack) over $\mathcal{D}$.

2.5 Homotopy coherence for stacks

In this section, we study some of the homotopy (or 2-category) theory of stacks and categories fibred in groupoids. In particular, we study 2-commutative and 2-cartesian squares, and the special cases of prefibre and fibre sequences.

Throughout this section, we fix a site $\mathcal{C}$.

Let K be a small category. Define a topology on the product category $\mathcal{C} \times K^{op}$ by declaring a family $\{(f_i, g_i) : (U_i, k_i) \to (U, k)\}_{i \in I}$ to be a covering if $\{f_i : U_i \to U\}_{i \in J}$ is a covering in $\mathcal{C}$, where

$$J = \{i \in I \mid g_i \text{ is an isomorphism}\}.$$

We can now make the following definition.

Definition 2.5.1. Let K be a small category. A *2-commutative diagram of fibred categories (resp. stacks)* on $\mathcal{C}$ of shape K is a category fibred in groupoids (resp. a stack) over the site $\mathcal{C} \times K^{op}$. We write $\mathrm{Ho}(\mathbf{Fib}(\mathcal{C})^K)$ and $\mathrm{Ho}(\mathbf{St}(\mathcal{C})^K)$ in place of $\mathrm{Ho}(\mathbf{Fib}(\mathcal{C} \times K^{op}))$ and $\mathrm{Ho}(\mathbf{St}(\mathcal{C} \times K^{op}))$.

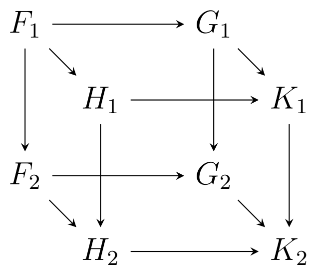
Remark 2.5.2. Choosing cleavages and using the ideas of [9], Section 3.1.2, one can show that a 2-commutative diagram as defined in Definition 2.5.1 is essentially the same as a collection of morphisms and 2-isomorphisms satisfying appropriate compatibility conditions. For example, if $K = \square$ is the commutative square,

$$\begin{array}{ccc} 1 & \longrightarrow & 2 \\ \downarrow & & \downarrow \\ 3 & \longrightarrow & 4 \end{array}$$

then we can equivalently define a 2-commutative diagram of fibred categories (resp. stacks) over $\mathcal{C}$ of shape K to be a diagram

$$\begin{array}{ccc} F & \xrightarrow{f_G} & G \\ {\scriptstyle f_H}\downarrow & & \downarrow{\scriptstyle g} \\ H & \xrightarrow{h} & K \end{array}$$

where F, G, H and K are fibred categories (resp. stacks) over $\mathcal{C}$, together with a 2-isomorphism $\eta : g \circ f_G \to h \circ f_H$. We call such a 2-commutative diagram a *2-commutative square*. A morphism of 2-commutative squares is a diagram



together with 2-isomorphisms making each face 2-commutative, such that the 2-isomorphisms together satisfy a certain compatibility condition.

Remark 2.5.3. Using Definition 2.5.1, we can define 2-commutative squares in the 2-category $\mathbf{St}(\mathcal{C})^{\square} = \mathbf{St}(\mathcal{C} \times \square^{op})$ of 2-commutative squares, as well as more complicated diagrams.

Remark 2.5.4. If K is a small category, we can extend the stackification functor $\mathrm{st} : \mathrm{Ho}(\mathbf{Fib}(\mathcal{C})) \to \mathrm{Ho}(\mathbf{St}(\mathcal{C}))$ to a functor

$$\mathrm{st} : \mathrm{Ho}(\mathbf{Fib}(\mathcal{C})^K) \to \mathrm{Ho}(\mathbf{St}(\mathcal{C})^K).$$

We can see this in the case $K = \square$ by using the universal property of Proposition 2.4.11 to chase around the various 2-isomorphisms. For general K, we can simply apply the stackification functor for the site $\mathcal{C} \times K^{op}$.

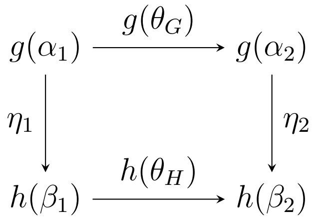
Definition 2.5.5. Let $p_G : G \to \mathcal{C}$, $p_H : H \to \mathcal{C}$ and $p_K : K \to \mathcal{C}$ be categories fibred in groupoids over $\mathcal{C}$, and let $g : G \to K$ and $h : H \to K$ be morphisms. The *(2-)fibre product* of G and H over K is the category $p : G \times_K H \to \mathcal{C}$ over $\mathcal{C}$, with objects

$$\mathrm{Ob}(G \times_K H) = \left\{ (\alpha, \beta, \eta) \;\middle|\; \begin{matrix} \alpha \in \mathrm{Ob}(G), \beta \in \mathrm{Ob}(H), \eta : g(\alpha) \to h(\beta), \\ p_G(\alpha) = p_K(\beta), \quad \text{and} \quad p_H(\eta) = \mathrm{id} \end{matrix} \right\}$$

and morphisms given as follows. If $\gamma_1 = (\alpha_1, \beta_1, \eta_1)$ and $\gamma_2 = (\alpha_2, \beta_2, \eta_2)$ are objects in $G \times_K H$, then a morphism $\gamma_1 \to \gamma_2$ is a pair

$$(\theta_G, \theta_H) \in \mathrm{Hom}_G(\alpha_1, \alpha_2) \times \mathrm{Hom}_H(\beta_1, \beta_2)$$

such that the diagram below commutes.



The functor $p : G \times_K H \to \mathcal{C}$ is given on objects by

$$p(\alpha, \beta, \eta) = p_G(\alpha) = p_H(\beta)$$

and on morphisms by

$$p(\theta_G, \theta_H) = p_G(\theta_G) = p_H(\theta_H).$$

Lemma 2.5.6 (cf. [29, Tag 026G])**.** *In the setup above, if G, H and K are stacks, then so is $G \times_K H$.*

Suppose we have a 2-commutative square

$$\begin{array}{ccc} F & \xrightarrow{f_G} & G \\ {\scriptstyle f_H}\downarrow & & \downarrow{\scriptstyle g} \\ H & \xrightarrow{h} & K \end{array} \tag{2.5.1}$$

of fibred categories over $\mathcal{C}$. Then we get an induced morphism

$$\begin{aligned} F &\to G \times_K H \\ \alpha &\mapsto (f_G(\alpha), f_H(\alpha), \eta(\alpha)) \\ (\theta : \alpha_1 \to \alpha_2) &\mapsto (f_G(\theta), f_H(\theta)). \end{aligned}$$

Definition 2.5.7. We say that a square (2.5.1) of fibred categories (resp. stacks) is *2-cartesian* if the induced morphism $F \to G \times_K H$ is a local equivalence.

Remark 2.5.8. Our definition of 2-cartesian square for fibred categories is not the usual 2-categorical definition of 2-cartesian. The 2-categorical notion can be recovered if we require $F \to G \times_K H$ to be an equivalence instead of a local equivalence. We have chosen our definition so that the following proposition is true.

Proposition 2.5.9. *A 2-commutative square of fibred categories is 2-cartesian if and only if its stackification is 2-cartesian.*

Proof. By [29, Tag 04Y1], the stackification functor st takes 2-fibre products to 2-fibre products. The result now follows from Corollary 2.4.13. □

For our study of deformation theory, there is a special kind of 2-commutative square of particular interest to us. First, observe that the identity functor $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is a stack in groupoids over $\mathcal{C}$, such that every fibre category has exactly one object and one morphism. We call this stack the *point stack over* $\mathcal{C}$ and denote it by $*$. Every category F fibred in groupoids over $\mathcal{C}$ has a unique morphism $F \to *$. We say that F is *locally trivial* if the map $F \to *$ is a local equivalence.

Definition 2.5.10. A *prefibre sequence* of fibred categories (resp. stacks) over $\mathcal{C}$ is a 2-commutative square

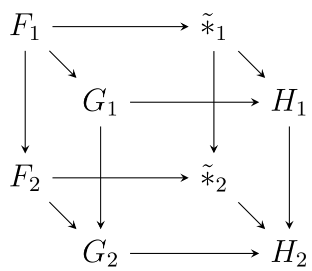
$$\begin{array}{ccc} F & \longrightarrow & \tilde{*} \\ \downarrow & & \downarrow \\ G & \longrightarrow & H \end{array}$$

of fibred categories (resp. stacks) such that $\tilde{*}$ is locally trivial. A *fibre sequence* is a prefibre sequence which is 2-cartesian as a 2-commutative square.

We will usually abbreviate a prefibre sequence as above by

$$F \to G \to H$$

and a morphism



of prefibre sequences by

$$\begin{array}{ccccc} F_1 & \longrightarrow & G_1 & \longrightarrow & H_1 \\ \downarrow & & \downarrow & & \downarrow \\ F_2 & \longrightarrow & G_2 & \longrightarrow & H_2. \end{array}$$

Chapter 3

Square-zero extensions and obstruction sequences

In this chapter, we review the theory of square-zero extensions of schemes. We use the theory of stacks developed in Chapter 2 to construct some obstruction sequences derived from universal properties. These obstruction sequences form the basis for constructing obstruction theories in Chapter 5.

3.1 Square-zero extensions

In this section, we recall the theory of square-zero extensions of schemes. We begin with an algebraic version of the definition.

Definition 3.1.1. Let $\mathcal{C}$ be a site, and let $A \to B$ be a morphism of sheaves of rings on $\mathcal{C}$. A *square-zero extension* of B over A is a surjection $\overline{B} \to B$ such that $J = \ker(\overline{B} \to B)$ satisfies $J^2 = 0$.

If $p : \overline{B} \to B$ is a square-zero extension of B, then the kernel $J = \ker(\overline{B} \to B)$ carries a canonical B-module structure, given by

$$b \cdot j = p(b)j$$

for all $U \in \mathrm{Ob}(\mathcal{C})$, $b \in B(U)$ and $j \in J(U)$.

Definition 3.1.2. If J is a B-module, a *square-zero extension of B by J (over A)* is a square-zero extension $\phi : \overline{B} \to B$ together with an isomorphism of B-modules $J \cong \ker \phi$. We often write this as an exact sequence

$$0 \to J \to \overline{B} \to B \to 0$$

of sheaves on $\mathcal{C}$. A *morphism of square-zero extensions* is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & \overline{B}_1 & \longrightarrow & B & \longrightarrow & 0 \\ & & \Big\downarrow \mathrm{id}_J & & \Big\downarrow & & \Big\downarrow \mathrm{id}_B & & \\ 0 & \longrightarrow & J & \longrightarrow & \overline{B}_2 & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

where $\overline{B}_1 \to \overline{B}_2$ is a morphism of A-algebras. We write $\mathbf{Ext}_A(B, J)$ for the category of square-zero extensions of B by J over A.

Definition 3.1.3. Let S be a scheme and let $\pi : X \to S$ be a scheme over S. A *square-zero extension of X over S* is a square-zero extension of $\mathcal{O}_X$ over $\pi^{-1}\mathcal{O}_S$ on the Zariski site X_{Zar}. If J is an $\mathcal{O}_X$-module, a *square-zero extension of X by J* is a square-zero extension of $\mathcal{O}_X$ by J. We write

$$\mathbf{Ext}_S(X, J) = \mathbf{Ext}_{\pi^{-1}\mathcal{O}_S}(\mathcal{O}_X, J).$$

We will always assume that a square-zero extension of a scheme X has quasi-coherent kernel.

Remark 3.1.4. If X is a scheme over S and J is a quasi-coherent sheaf on X, then a square-zero extension of X by J is the same as a closed embedding $X \to \overline{X}$ of S-schemes with ideal sheaf J satisfying $J^2 = 0$. Notice, however, that if $\overline{X}_1$ and $\overline{X}_2$ are square-zero extensions of X, then a morphism of extensions from $\overline{X}_1$ to $\overline{X}_2$ is a morphism of schemes $\overline{X}_2 \to \overline{X}_1$.

While square-zero extensions of schemes are defined with respect to the Zariski topology, it is often convenient to use the étale topology instead.

Lemma 3.1.5. *Let X be a scheme, and consider the map of ringed sites $z : (X_{ét}, \mathcal{O}_{X_{ét}}) \to (X_{Zar}, \mathcal{O}_X)$ of Example 2.3.6. If M is a quasi-coherent sheaf on X, then the canonical map $M \to z_* z^* M$ is an isomorphism.*

Proof. Fix a quasi-coherent sheaf M and consider the presheaf N on $X_{ét}$ given by

$$N(U \xrightarrow{f} X) = (f^*M)(U)$$

for $U \to X$ étale. By [9], Theorem 4.23, quasi-coherent sheaves form a stack (not in groupoids) over **Sch** for the fpqc topology, and hence for the étale topology. In particular,

$$N = e_* \underline{\mathrm{Hom}}_X(\mathcal{O}_X, M)$$

is a sheaf on $X_{ét}$. Moreover, we have a natural isomorphism

$$\mathrm{Hom}(N, P) \cong \mathrm{Hom}(M, z_* P)$$

for P an $\mathcal{O}_{X_{ét}}$-module. So $z^*M \cong N$ since z^* is left adjoint to z_*. The result now holds by inspection. □

Remark 3.1.6. It follows immediately from Lemma 3.1.5 that the functor

$$z^* : \mathbf{QCoh}(X) \to \mathcal{O}_{X_{ét}}\text{-}\mathbf{mod}$$

is fully faithful. Hence, we can identify quasi-coherent sheaves with sheaves on $\mathcal{O}_{X_{ét}}$-**mod**.

Proposition 3.1.7. *Let $\pi : X \to S$ be a scheme over S, and let J be a quasi-coherent sheaf on X. Pushing forward along $z : X_{ét} \to X_{Zar}$ defines an equivalence of categories*

$$\mathbf{Ext}_{\pi^{-1}\mathcal{O}_{S_{ét}}}(\mathcal{O}_{X_{ét}}, J) \simeq \mathbf{Ext}_S(X, J).$$

Sketch of proof. The inverse functor

$$\mathbf{Ext}_S(X, J) \to \mathbf{Ext}_{\pi^{-1}\mathcal{O}_{S_{ét}}}(\mathcal{O}_{X_{ét}}, J)$$

is constructed as follows. An object of $\mathbf{Ext}_S(X, J)$ is a closed embedding $X \to \overline{X}$ of S-schemes with ideal sheaf J satisfying $J^2 = 0$. By [13], Théorème I.8.3, the induced morphism of sites $X_{ét} \to \overline{X}_{ét}$ is an equivalence. Identifying $\overline{X}_{ét}$ with $X_{ét}$, we get a square-zero extension

$$0 \to J \to \mathcal{O}_{\overline{X}_{ét}} \to \mathcal{O}_{X_{ét}} \to 0$$

over $\pi^{-1}\mathcal{O}_{S_{ét}}$. It is immediate that pushing this forward to X_{Zar} gives back the extension

$$0 \to J \to \mathcal{O}_{\overline{X}} \to \mathcal{O}_X \to 0.$$

Conversely, if

$$0 \to J \to B \to \mathcal{O}_{X_{ét}} \to 0$$

is a square-zero extension of $\mathcal{O}_{X_{ét}}$ over $\pi^{-1}\mathcal{O}_{S_{ét}}$ restricting to X_{Zar} to a square-zero extension $X \to \overline{X}$ of S-schemes, we need to construct a natural isomorphism

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & B & \longrightarrow & \mathcal{O}_{X_{ét}} & \longrightarrow & 0 \\ & & \downarrow \mathrm{id}_J & & \downarrow & & \downarrow \mathrm{id}_{\mathcal{O}_{X_{ét}}} & & \\ 0 & \longrightarrow & J & \longrightarrow & \mathcal{O}_{\overline{X}_{ét}} & \longrightarrow & \mathcal{O}_{X_{ét}} & \longrightarrow & 0 \end{array}$$

of square-zero extensions. It is possible to reduce this to the case where $0 \to J \to \mathcal{O}_{\overline{X}_{ét}} \to \mathcal{O}_{X_{ét}} \to 0$ is the trivial extension, at which point we can construct the isomorphism using the identification on the Zariski site and formal étaleness (see Definition A.5.14). □

Remark 3.1.8. If X is a Deligne-Mumford stack, we need to work with the étale site $X_{ét}$ from the start. For schemes, we have the luxury of working with either the Zariski or the étale topologies.

Example 3.1.9. Let k be a field. Then

$$\frac{k[t]}{(t^2)} \to k$$

together with the map

$$\begin{aligned} k &\to (t) \\ a &\mapsto at \end{aligned}$$

is a square-zero extension of k over k by the k-module k. The associated map of schemes

$$\mathrm{Spec}(k) \to \mathrm{Spec}\left(\frac{k[t]}{(t^2)}\right)$$

is a square-zero extension of $\mathrm{Spec}(k)$ over $\mathrm{Spec}(k)$.

Example 3.1.10. Let $X = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(A)$ be affine schemes. A square-zero extension of X over S is the same as a square-zero extension of B over A.

Example 3.1.11. Let X be an S-scheme and let J be a quasi-coherent sheaf on X. The *trivial extension* of X by J is

$$\mathcal{O}_X \oplus J \to \mathcal{O}_X$$

where $\mathcal{O}_X \oplus J$ is the quasi-coherent $\mathcal{O}_X$-algebra with multiplication

$$(f_1, j_1)(f_2, j_2) = (f_1 f_2, f_1 j_2 + f_2 j_1)$$

for f_1, f_2 sections of $\mathcal{O}_X$ and j_1, j_2 sections of J. The corresponding closed embedding of schemes is

$$X \to \overline{X} = \mathrm{Spec}_X(\mathcal{O}_X \oplus J)$$

where Spec_X is the global Spec on X defined in Section A.3.

Example 3.1.12. Let k be a field. Then

$$\mathrm{Spec}\left(\frac{k[x,y]}{(x^2 - y^3)}\right) \to \mathrm{Spec}\left(\frac{k[x,y,t]}{(t^2, x^2 - y^3 + t)}\right)$$

is a square-zero extension of the affine plane curve $X = \mathrm{Spec}(k[x,y]/(x^2-y^3))$ by $\mathcal{O}_X$ over the base $\mathrm{Spec}(k)$. This extension is not the trivial extension.

Let X be an S-scheme. If J is a quasi-coherent sheaf on X, the groupoid $\mathbf{Ext}_S(X, J)$ captures the global first order deformation theory of X. As suggested in Section 1.2, it is useful to describe how these global deformations are built from deformations of étale opens of X. To do this, we define a category $\underline{\mathbf{Ext}}_S(X, J)$ over $X_{ét}$ as follows. An object of $\underline{\mathbf{Ext}}_S(X, J)$ is a pair $(U, \overline{U})$ where $U \in \operatorname{Ob} X_{ét}$ and $\overline{U} \in \operatorname{Ob} \mathbf{Ext}_S(U, J|_U)$. A morphism $(U, \overline{U}) \to (V, \overline{V})$ in $\underline{\mathbf{Ext}}_S(X, J)$ is a pair (f, g) where $f : U \to V$ is a morphism in $X_{ét}$ and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J|_U & \longrightarrow & \mathcal{O}_{\overline{U}_{ét}} & \longrightarrow & \mathcal{O}_{U_{ét}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}_{J|_U}} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \mathrm{id}_{\mathcal{O}_{U_{ét}}}} & & \\ 0 & \longrightarrow & J|_U & \longrightarrow & \mathcal{O}_{\overline{V}_{ét}}|_U & \longrightarrow & \mathcal{O}_{U_{ét}} & \longrightarrow & 0 \end{array}$$

is a morphism of square-zero extensions of $\mathcal{O}_{U_{ét}}$ over $\pi^{-1}\mathcal{O}_{S_{ét}}$. The functor $\underline{\mathbf{Ext}}_S(X, J) \to X_{ét}$ is the natural forgetful functor.

The following proposition is straightforward given Proposition 3.1.7.

Proposition 3.1.13. *The functor* $\underline{\mathbf{Ext}}_S(X, J) \to X_{ét}$ *is a stack in groupoids over the site* $X_{ét}$.

Remark 3.1.14. The stacks $\underline{\mathbf{Ext}}_S(X, J)$ depend functorially on S. If X, Y and S are schemes, $\pi : X \to Y$ and $f : Y \to S$ are morphisms, then for any quasi-coherent sheaf J on X, there is forgetful a morphism

$$\underline{\mathbf{Ext}}_Y(X, J) \to \underline{\mathbf{Ext}}_S(X, J)$$

of stacks over $X_{ét}$. More generally, given a small category K, a diagram $S_\bullet : K \to \mathbf{Sch}$, and a morphism $X \to S_\bullet$ from the constant diagram X, there is an associated 2-commutative diagram of stacks on $X_{ét}$.

Remark 3.1.15. The stacks $\underline{\mathbf{Ext}}_S(X, J)$ also depend functorially on J. We will not need this functoriality explicitly, so we omit the details.

3.2 Obstruction sequences for fibre products

In this section, we introduce obstruction sequences, and derive a canonical obstruction sequence for a fibre product of schemes.

Definition 3.2.1. Let S be a scheme, and let $f : T \to X$ be a morphism of S-schemes. An *obstruction sequence* for f with respect to a quasi-coherent sheaf J on T is a fibre sequence

$$\underline{\mathbf{Ext}}_X(T, J) \to \underline{\mathbf{Ext}}_S(T, J) \to \underline{\mathbf{Ob}}$$

of stacks over $T_{ét}$, where $\underline{\mathbf{Ext}}_X(T, J) \to \underline{\mathbf{Ext}}_S(T, J)$ is the forgetful morphism of Remark 3.1.14.

For the motivation behind Definition 3.2.1, see Section 1.2 in the introduction.

To illustrate the utility of our stack-theoretic approach to obstructions, consider a square

$$\begin{array}{ccc} X & \xrightarrow{\;j\;} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{\;i\;} & W \end{array}$$

of schemes over S, with $X = Y \times_W Z$. This means that for any S-scheme T, we have a natural isomorphism

$$\mathrm{Hom}_S(T, X) = \mathrm{Hom}_S(T, Y) \times_{\mathrm{Hom}_S(T,W)} \mathrm{Hom}_S(T, Z). \tag{3.2.1}$$

It is helpful to keep the following examples in mind.

Example 3.2.2. Let $f : Y \to W$ and $i : Z \to W$ be closed embeddings. Then the fibre product $X = Y \times_Z W$ is the (scheme-theoretic) intersection $Y \cap Z$. A concrete example is the intersection

$$X_t = Y \cap Z_t = \mathrm{Proj}\left(\frac{k[x, y, z]}{(y, y - tx)}\right) \subseteq \mathbb{P}^2_k = \mathrm{Proj}\,(k[x, y, z])$$

from Section 1.1.

Example 3.2.3. If $Z = \mathrm{Spec}(k)$ is a point, then $i : Z \to W$ picks out a (k-valued) point $w \in W$, and $X = Y \times_W Z$ is the fibre $f^{-1}(w)$. For instance, take

$$W = \mathrm{Spec}(k[s, t]) = \mathbb{A}^2_k$$

and

$$Y = \mathrm{Proj}\left(\frac{k[s, t, x, y]}{(tx - sy)}\right) \subseteq W \times \mathbb{P}^1_k$$

where s and t have degree 0 and x and y have degree 1. (See Section A.3 for a description of Proj.) There is a natural map $f : Y \to W$ induced by the projection $W \times \mathbb{P}^1_k \to W$. The fibre over any point $w \neq 0$ is a single point, whereas the fibre over 0 is

$$X = f^{-1}(0) = \operatorname{Proj}\left(\frac{k[s,t,x,y]}{(tx - sy, s, t)}\right) \cong \operatorname{Proj}(k[x,y]) = \mathbb{P}^1_k.$$

Remark 3.2.4. The "fibre of a family" picture of Example 3.2.3 can be generalised to describe any moduli space X where we can deform the moduli problem to get a space of the expected dimension. For example, if X is a moduli space of maps to a target Y, we can sometimes deform Y to get a moduli space of the correct dimension.

Returning to the general fibre product, we can construct obstruction sequences for X from obstruction sequences for Y, Z and W using the universal property (3.2.1) as follows. Suppose that $h : T \to X$ is a morphism of S-schemes, and that for some quasi-coherent sheaf J we are given fibre sequencs and morphisms between them as follows.

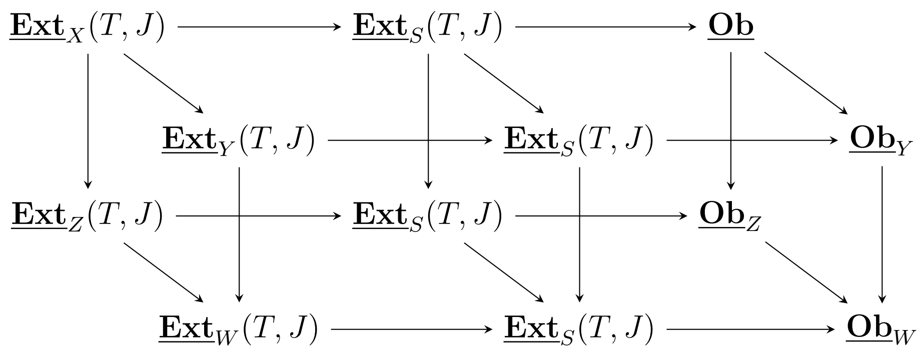
$$\begin{array}{ccccc}
\underline{\mathbf{Ext}}_Y(T,J) & \longrightarrow & \underline{\mathbf{Ext}}_S(T,J) & \longrightarrow & \underline{\mathbf{Ob}}_Y \\
\downarrow & & \downarrow & & \downarrow \\
\underline{\mathbf{Ext}}_W(T,J) & \longrightarrow & \underline{\mathbf{Ext}}_S(T,J) & \longrightarrow & \underline{\mathbf{Ob}}_W \\
\uparrow & & \uparrow & & \uparrow \\
\underline{\mathbf{Ext}}_Z(T,J) & \longrightarrow & \underline{\mathbf{Ext}}_S(T,J) & \longrightarrow & \underline{\mathbf{Ob}}_Z
\end{array} \tag{3.2.2}$$

We will see in Chapter 5 that there are canonical choices of such fibre sequences given by the cotangent complex. To construct a fibre sequence for h with respect to J, we note that (3.2.1) implies that the square

$$\begin{array}{ccc}
\underline{\mathbf{Ext}}_X(T,J) & \longrightarrow & \underline{\mathbf{Ext}}_Y(T,J) \\
\downarrow & & \downarrow \\
\underline{\mathbf{Ext}}_Z(T,J) & \longrightarrow & \underline{\mathbf{Ext}}_W(T,J)
\end{array}$$

is 2-cartesian. Hence, taking the 2-fibre product of the fibre sequences in

(3.2.2), we get a 2-commutative square of prefibre sequences



(3.2.3)

where

$$\underline{\mathbf{Ob}} = \underline{\mathbf{Ob}}_Y \times_{\underline{\mathbf{Ob}}_W} \underline{\mathbf{Ob}}_Z.$$

Since at each place the square is 2-cartesian, a simple, if somewhat lengthy, calculation shows that

$$\underline{\mathbf{Ext}}_X(T,J) \to \underline{\mathbf{Ext}}_S(T,J) \to \underline{\mathbf{Ob}} \tag{3.2.4}$$

is a fibre sequence. Thus, (3.2.4) is an obstruction sequence for h with respect to J.

3.3 Obstruction sequences for mapping spaces

In this section, we construct canonical obstruction sequences for spaces of maps from a fixed domain to a fixed target. These obstruction sequences are the basis for producing the obstruction theories for moduli spaces of maps used in Gromov-Witten theory. (See [3], and Section 6 of [5].)

Fix a base scheme S and schemes C and Y over S. The *(relative) mapping scheme* of maps from C to Y is the scheme,

$$X = \mathrm{Map}_S(C, Y),$$

with the universal property,

$$\mathrm{Hom}_S(T, X) = \mathrm{Hom}_S(T \times_S C, Y) \tag{3.3.1}$$

for all S-schemes T. Fix an S-scheme T, a map $f : T \to X$ and a quasi-coherent sheaf J on T. There is a morphism

$$\begin{aligned} \underline{\mathbf{Ext}}_S(T,J) &\to \pi_* \underline{\mathbf{Ext}}_S(T \times_S C, \pi^* J) \\ \overline{T} &\mapsto \overline{T} \times_S C \end{aligned}$$

of stacks over $T_{\acute{e}t}$, where $\pi : T \times_S C \to T$ is the natural projection. From the universal property (3.3.1), we get a 2-cartesian square as follows.

$$\begin{array}{ccc} \underline{\mathbf{Ext}}_X(T, J) & \longrightarrow & \pi_* \underline{\mathbf{Ext}}_Y(T \times_S C, \pi^* J) \\ \downarrow & & \downarrow \\ \underline{\mathbf{Ext}}_S(T, J) & \longrightarrow & \pi_* \underline{\mathbf{Ext}}_S(T \times_S C, \pi^* J) \end{array} \tag{3.3.2}$$

Suppose we are given an obstruction sequence,

$$\underline{\mathbf{Ext}}_Y(T \times_S C, \pi^* J) \to \underline{\mathbf{Ext}}_S(T \times_S C, \pi^* J) \to \underline{\mathbf{Ob}}_Y,$$

coming, for example, from the cotangent complex for Y. (See Section 5.3.) Then we can extend (3.3.2) to a 2-commutative diagram,

$$\begin{array}{ccccc} \underline{\mathbf{Ext}}_X(T, J) & \longrightarrow & \pi_* \underline{\mathbf{Ext}}_Y(T \times_S C, \pi^* J) & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ \underline{\mathbf{Ext}}_S(T, J) & \longrightarrow & \pi_* \underline{\mathbf{Ext}}_S(T \times_S C, \pi^* J) & \longrightarrow & \pi_* \underline{\mathbf{Ob}}_Y \end{array}$$

such that each square is 2-cartesian. Hence, the outer square is 2-cartesian, which yields an obstruction sequence

$$\underline{\mathbf{Ext}}_X(T, J) \to \underline{\mathbf{Ext}}_S(T, J) \to \pi_* \underline{\mathbf{Ob}}_Y.$$

Remark 3.3.1. In the application to Gromov-Witten theory in [3], S is taken to be some Artin stack of nodal curves, and C is the universal curve over S. For this application, the above calculation still works, although the stacks of square-zero extensions need to be interpreted in the sense of [25], Section 2.26.

Chapter 4

Homological algebra

In Chapter 3, we introduced obstruction sequences and derived some canonical obstruction sequences for fibre products and mapping spaces. As it stands, however, these obstruction sequences are not very useful as we have no concrete means of doing explicit computations with them.

To resolve this issue, we take our cue from the example in Section 1.2. The natural obstruction stack appearing there can be written

$$\underline{\mathbf{Ob}} = [\underline{\mathrm{Hom}}(g^* i^* \Omega_{\mathbb{P}^2}, J) / \underline{\mathrm{Hom}}(g^* \Omega_Y \oplus h^* \Omega_{Z_t}, J)].$$

The data needed to perform this construction is a (two-term) complex

$$\underline{\mathrm{Hom}}(g^* \Omega_Y \oplus h^* \Omega_{Z_t}, J) \to \underline{\mathrm{Hom}}(g^* i^* \Omega_{\mathbb{P}^2}, J)$$

of sheaves on T.

Thus, we should hope that an obstruction sequence

$$\underline{\mathrm{Hom}}_X(T, J) \to \underline{\mathrm{Hom}}_S(T, J) \to \underline{\mathbf{Ob}}$$

can be constructed using tools from homological algebra, and that calculations involving stacks can be reduced to calculations with complexes.

In this chapter, we assemble the tools needed to carry this out. The basic objects of study are, of course, complexes, which we discuss in Section 4.1. In Section 4.2, we describe how to construct stacks and categories fibred in groupoids from complexes of sheaves. Examining which morphisms of complexes give rise to equivalences of stacks leads us naturally to the idea of derived categories, which we introduce in Section 4.3. In order to control 2-commutative diagrams in the world of derived categories, we introduce some homotopy coherence formalism in Section 4.4. Finally, in Section 4.5, we discuss how to use homological algebra to control fibre sequences and 2-cartesian squares.

4.1 Complexes in an abelian category

In this section, we recall the notion of complexes in an abelian category.

Definition 4.1.1 (cf. [11], Section II.5)**.** Let $\mathcal{A}$ be a category. We say that $\mathcal{A}$ is *abelian* if it satisfies the following conditions.

(1) There exists an abelian group structure on every set $\mathrm{Hom}_{\mathcal{A}}(M, N)$, such that the composition maps

$$\mathrm{Hom}_{\mathcal{A}}(N, P) \times \mathrm{Hom}_{\mathcal{A}}(M, N) \to \mathrm{Hom}_{\mathcal{A}}(M, P)$$

are bilinear.

(2) There exists an object $0 \in \mathrm{Ob}(\mathcal{A})$ such that $\mathrm{Hom}_{\mathcal{A}}(0, 0) = 0$ is the zero group.

(3) If $M, N \in \mathrm{Ob}(\mathcal{A})$, then there exists an object $M \oplus N \in \mathrm{Ob}(\mathcal{A})$ and morphisms

$$\begin{aligned} i_1 &: M \to M \oplus N, & p_1 &: M \oplus N \to M, \\ i_2 &: N \to M \oplus N, & p_2 &: M \oplus N \to N, \end{aligned}$$

such that

$$\begin{gathered} p_1 \circ i_1 = \mathrm{id}_M, \qquad p_2 \circ i_1 = 0, \\ p_2 \circ i_2 = \mathrm{id}_N, \qquad p_1 \circ i_2 = 0, \\ i_1 \circ p_1 + i_2 \circ p_2 = \mathrm{id}_{M \oplus N}. \end{gathered}$$

(4) For any morphism $\phi : M \to N$ in $\mathcal{A}$, there exists a sequence,

$$K \xrightarrow{k} M \xrightarrow{i} I \xrightarrow{j} N \xrightarrow{c} C,$$

such that $j \circ i = \phi$, K is the kernel of ϕ, C is the cokernel of ϕ, and I is both the cokernel of k and the kernel of c.

Remark 4.1.2. The key point for us is that the following categories are abelian: abelian groups, sheaves of abelian groups, modules over a ring, modules over a sheaf of rings. In this thesis, we always work in one of these contexts. For ease of exposition, we will often write down morphisms in terms of elements as if $\mathcal{A} = \mathbf{Ab}$ is the category of abelian groups. For the examples of abelian categories above, these definitions can be interpreted directly in terms of elements of an underlying set, or sections of a sheaf. We leave it to the interested reader to think about how the constructions we describe can be written in purely categorical terms.

For the remainder of this section, we fix an abelian category $\mathcal{A}$.

Definition 4.1.3. A *complex* in $\mathcal{A}$ is a sequence

$$M = [\cdots \xrightarrow{d^{-2}} M^{-1} \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \xrightarrow{d^1} M^2 \xrightarrow{d^2} \cdots]$$

of objects M^i in $\mathcal{A}$ and morphisms $d^i : M^i \to M^{i+1}$ for each $i \in \mathbb{Z}$, such that $d^{i+1} \circ d^i = 0$ for all i. We say that M is *bounded below* if there exists $i_0 \in \mathbb{Z}$ such that $M^i = 0$ for all $i < i_0$. We say that M is *bounded above* if there exists $i_0 \in \mathbb{Z}$ such that $M^i = 0$ for all $i > i_0$. If M and N are complexes, a *morphism of complexes* $f : M \to N$ is a collection of morphisms $f^i : M^i \to N^i$ for $i \in \mathbb{Z}$, such that the diagram below commutes.

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & M^{i-1} & \xrightarrow{d_M^{i-1}} & M^i & \xrightarrow{d_M^i} & M^{i+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f^{i-1}} & & \downarrow{\scriptstyle f^i} & & \downarrow{\scriptstyle f^{i+1}} & & \\ \cdots & \longrightarrow & N^{i-1} & \xrightarrow{d_N^{i-1}} & N^i & \xrightarrow{d_N^i} & N^{i+1} & \longrightarrow & \cdots \end{array}$$

We denote the category of complexes in $\mathcal{A}$ by $C(\mathcal{A})$. The full subcategories of complexes bounded below and complexes bounded above are denoted by $C^+(\mathcal{A})$ and $C^-(\mathcal{A})$ respectively.

Definition 4.1.4. Let M be a complex and let $i \in \mathbb{Z}$. The *ith cohomology of* M is

$$H^i(M) = \frac{\ker d^i}{\operatorname{im} d^{i-1}} \in \operatorname{Ob}(\mathcal{A}).$$

We say that M is *acyclic* if $H^i(M) = 0$ for all $i \in \mathbb{Z}$.

If $f : M \to N$ is a morphism of complexes, then there are induced maps $H^i(f) : H^i(M) \to H^i(N)$ for all $i \in \mathbb{Z}$. This determines cohomology functors

$$H^i : C(\mathcal{A}) \to \mathcal{A}$$

for each $i \in \mathbb{Z}$.

Definition 4.1.5. Let $n \in \mathbb{Z}$. Let $C^{\geq n}(\mathcal{A})$ (resp. $C^{\leq n}(\mathcal{A})$) denote the full subcategory of $C(\mathcal{A})$ consisting of complexes M with $H^i(M) = 0$ for $i < n$ (resp. $i > n$). If M is any complex in $\mathcal{A}$, then the *truncations* of M are

$$\tau^{\geq n} M = [\cdots \to 0 \to \operatorname{coker} d^{n-1} \xrightarrow{d^n} M^{n+1} \xrightarrow{d^{n+1}} M^{n+2} \xrightarrow{d^{n+2}} \cdots]$$

and

$$\tau^{\leq n} M = [\cdots \xrightarrow{d^{n-3}} M^{n-2} \xrightarrow{d^{n-2}} M^{n-1} \xrightarrow{d^{n-1}} \ker d^{n+1} \to 0 \to \cdots].$$

This defines a pair of functors,

$$\tau^{\geq n} : C(\mathcal{A}) \to C^{\geq n}(\mathcal{A}) \quad \text{and} \quad \tau^{\leq n} : C(\mathcal{A}) \to C^{\leq n}(\mathcal{A}).$$

If $m, n \in \mathbb{Z}$ with $m \leq n$, then we write $C^{[m,n]}(\mathcal{A})$ for the category of complexes M with $M^i = 0$ for $i < m$ and $i > n$, and we write

$$\tau^{[m,n]} = \tau^{\leq n} \circ \tau^{\geq m} = \tau^{\geq m} \circ \tau^{\leq n}.$$

Remark 4.1.6. If M is a complex in $\mathcal{A}$ and $n \in \mathbb{Z}$, then there is a canonical morphism $M \to \tau^{\geq n} M$ which induces isomorphisms

$$H^i(M) \to H^i(\tau^{\geq n} M)$$

for all $i \geq n$. Similarly, there is a morphism $\tau^{\leq n} M \to M$ inducing isomorphisms on cohomology in degrees $\leq n$.

4.2 From complexes to stacks

Let $(\mathcal{C}, A)$ be a ringed site. For brevity, we write $C(A)$ in place of $C(A\textbf{-mod})$, and similarly for the bounded analogues. The purpose of this section is to define a functor $\mathrm{ch} : C(A) \to \mathrm{Ho}(\mathbf{St}(\mathcal{C}))$, and study a few of its properties.

We build the functor ch in several stages. First, given a complex

$$M = [\cdots \to 0 \to M^{-1} \xrightarrow{d} M^0 \to 0 \to \cdots]$$

in $C^{[-1,0]}(A)$, define a category $\mathrm{pch}(M)$ fibred in groupoids over $\mathcal{C}$ as follows. For each $U \in \mathrm{Ob}(\mathcal{C})$, we set

$$\mathrm{Ob}(\mathrm{pch}(M)(U)) = M^0(U)$$

and

$$\mathrm{Hom}(u, v) = \{w \in M^{-1}(U) \mid dw = v - u\}$$

for $u, v \in M^0(U)$. The composition in $\mathrm{pch}(M)(U)$ is given by addition in $M^{-1}(U)$ and the restriction functors are given on objects and morphisms by the restriction maps for M^0 and M^{-1} respectively. If

$$\begin{array}{ccc} M^{-1} & \xrightarrow{d_M} & M^0 \\ {\scriptstyle f^{-1}}\downarrow & & \downarrow{\scriptstyle f^0} \\ N^{-1} & \xrightarrow{d_N} & N^0 \end{array}$$

is a morphism of complexes, then there is an induced morphism of fibred categories $\mathrm{pch}(f) : \mathrm{pch}(M) \to \mathrm{pch}(N)$ given on objects and morphisms by

$$\mathrm{pch}(f)(u) = f^0(u), \quad \text{and} \quad \mathrm{pch}(f)(w) = f^{-1}(w),$$

for $u \in M^0(U)$ and $w \in M^{-1}(U)$. This defines a functor $\mathrm{pch} : C^{[-1,0]}(A) \to \mathbf{Fib}(\mathcal{C})$ from the category of complexes in degree $[-1,0]$ to the category of fibred categories over $\mathcal{C}$. (Here we regard $\mathbf{Fib}(\mathcal{C})$ as a category by ignoring 2-isomorphisms.)

Composing $\mathrm{pch} : C^{[-1,0]}(A) \to \mathbf{Fib}(\mathcal{C})$ with the truncation functor $\tau^{[-1,0]} : C(A) \to C^{[-1,0]}(A)$ gives a functor

$$\mathrm{pch} : C(A) \to \mathbf{Fib}(\mathcal{C}).$$

Applying the stackification functor $\mathrm{st} : \mathrm{Ho}(\mathbf{Fib}(\mathcal{C})) \to \mathrm{Ho}(\mathbf{St}(\mathcal{C}))$, we get a functor

$$\mathrm{ch} = \mathrm{st} \circ \mathrm{pch} : C(A) \to \mathrm{Ho}(\mathbf{St}(\mathcal{C})).$$

Remark 4.2.1. If

$$M = \left[M^{-1} \to M^0\right]$$

is a complex in $C^{[-1,0]}(A)$, the stack $\mathrm{ch}(M)$ is often called the *stack quotient* of M^0 by M^{-1} and denoted

$$\mathrm{ch}(M) = \left[M^0/M^{-1}\right].$$

This is the notation we have used in Section 1.2.

Proposition 4.2.2. *Let $(\mathcal{C}, A)$ be a ringed site and let $f : M \to N$ be a morphism in $C(A)$. Then*

(1) $\mathrm{pch}(f)$ *is locally fully faithful if and only if $H^{-1}(f)$ is an isomorphism and $H^0(f)$ is injective, and*

(2) $\mathrm{pch}(f)$ *is locally essentially surjective if and only if $H^0(f)$ is surjective.*

In particular, $\mathrm{pch}(f) : \mathrm{pch}(M) \to \mathrm{pch}(N)$ *is a local equivalence if and only if $H^0(f)$ and $H^{-1}(f)$ are isomorphisms of sheaves.*

Corollary 4.2.3. *Let $f : M \to N$ be a morphism in $C(A)$. Then* $\mathrm{ch}(f) : \mathrm{ch}(M) \to \mathrm{ch}(N)$ *is an equivalence if and only if* $H^{-1}(f)$ *and* $H^0(f)$ *are isomorphisms.*

Proof of Proposition 4.2.2. By construction of pch, we may assume without loss of generality that $M, N \in \mathrm{Ob}\, C^{[-1,0]}(A)$.

To prove (2), notice that $H^0(M)$ and $H^0(N)$ are the sheaves associated to the presheaves,

$$U \mapsto \mathrm{coker}(M^{-1}(U) \to M^0(U)) = \pi_0 \mathrm{pch}(M)(U)$$

and

$$U \mapsto \mathrm{coker}(N^{-1}(U) \to N^0(U)) = \pi_0 \mathrm{pch}(N)(U).$$

Here $\pi_0\mathrm{pch}(M)(U)$ (resp. $\pi_0\mathrm{pch}(N)(U)$) denotes the set of isomorphism classes of the groupoid $\mathrm{pch}(M)(U)$ (resp. $\mathrm{pch}(N)(U)$). The map of presheaves,

$$\pi_0\mathrm{pch}(f) : \pi_0\mathrm{pch}(M) \to \pi_0\mathrm{pch}(N),$$

is locally surjective if and only if $\mathrm{pch}(f)$ is locally essentially surjective. Since sheafification preserves local surjectivity, $\mathrm{pch}(f)$ is therefore locally essentially surjective if and only if $H^0(f)$ is surjective.

To prove (1), assume first that $\mathrm{pch}(f)$ is locally fully faithful. Then for every $U \in \mathrm{Ob}\, \mathcal{C}$,

$$H^{-1}(M)|_U = \mathrm{sh}\underline{\mathrm{Hom}}_U(0,0) \xrightarrow{H^{-1}(f)} \mathrm{sh}\underline{\mathrm{Hom}}_U(0,0) = H^{-1}(N)|_U$$

is an isomorphism. So $H^{-1}(f)$ is an isomorphism. Moreover, since $H^0(M)$ and $H^0(N)$ are the sheafifications of $\pi_0\mathrm{pch}(M)$ and $\pi_0\mathrm{pch}(N)$ respectively, by Proposition 2.1.11, $H^0(f)$ is injective if and only if the map of presheaves

$$\pi_0\mathrm{pch}(f) : \pi_0\mathrm{pch}(M) \to \pi_0\mathrm{pch}(N)$$

is locally injective. This is easily seen to hold since $\mathrm{pch}(f)$ is locally fully faithful.

Conversely, assume that $H^{-1}(f)$ is an isomorphism and $H^0(f)$ is injective. Let $U \in \mathrm{Ob}(\mathcal{C})$ and let $u, v \in \mathrm{Ob}\,\mathrm{pch}(M)(U) = M^0(U)$. Then

$$\mathrm{Hom}_{\mathrm{pch}(M)(U)}(u, v) = \{w \in M^{-1}(U) \mid u + d_M w = v\}$$

and

$$\mathrm{Hom}_{\mathrm{pch}(N)(U)}(f^0(u), f^0(v)) = \{w \in N^{-1}(U) \mid f^0(u) + d_N w = f^0(v)\}.$$

To show that $\mathrm{pch}(f)$ is locally fully faithful, it suffices to show that

$$\begin{aligned} \mathrm{Hom}(u, v) &\to \mathrm{Hom}(f^0(u), f^0(v)) \\ w &\mapsto f^{-1}(w) \end{aligned}$$

is a bijection.

To show injectivity, suppose that $w, w' \in \mathrm{Hom}(u, v)$ and $f^{-1}(w) = f^{-1}(w')$. Then $d_M(w - w') = 0$ so $w - w' \in H^{-1}(M)$ and $H^{-1}(f)(w - w') = 0$. But $H^{-1}(f)$ is an isomorphism, so $w = w'$.

To show surjectivity, let $w \in \mathrm{Hom}(f^0(u), f^0(v))$. Then $f^0(v - u) = d_N w$, so

$$H^0(f)(v - u) = 0 \in H^0(N).$$

Since $H^0(M) \to H^0(N)$ is injective, we have $v - u = 0$ in $H^0(M)$. Hence, there exists a covering $\{g_i : U_i \to U\}_{i \in I}$ and sections $x_i \in M^{-1}(U_i)$ such that $d_M x_i = M^0(g_i)(v - u)$. Therefore,

$$f^{-1}(x_i) - N^{-1}(g_i)(w) \in H^{-1}(N)(U_i).$$

Since $H^{-1}(f)$ is an isomorphism, there exists $y_i \in H^{-1}(M)(U_i)$ such that

$$N^{-1}(g_i)(w) - f^{-1}(x_i) = f^{-1}(y_i)$$

and hence

$$N^{-1}(g_i)(w) = f^{-1}(\tilde{w}_i)$$

where $\tilde{w}_i = x_i + y_i$. Moreover, since the $f^{-1}(\tilde{w}_i)$ glue in N^{-1}, we can check using the injectivity of $H^{-1}(f)$ that the $\tilde{w}_i$ glue in M^{-1} to give a section $\tilde{w} \in M^{-1}(U)$ which satisfies $d_M \tilde{w} = v - u$ and $f^{-1}(\tilde{w}) = w$. So the map

$$\mathrm{Hom}_{\mathrm{pch}(M)(U)}(u, v) \to \mathrm{Hom}_{\mathrm{pch}(N)(U)}(f^0(u), f^0(v))$$

is surjective, which completes the proof of (1). □

Remark 4.2.4. The functor $\mathrm{ch} : C(\mathbf{Sh}(\mathcal{C}, \mathbf{Ab})) \to \mathrm{Ho}(\mathbf{St}(\mathcal{C}))$ appears in [2], XVIII.1.4.11. There, it is noted that the stacks $\mathrm{ch}(M)$ naturally have the structure of a *Picard stack*, i.e. a stack with a well-behaved notion of addition. It is shown ([2], Proposition XVIII.1.4.15) that the ch defines an equivalence between the truncated derived category $D^{[-1,0]}(\mathbf{Sh}(\mathcal{C}, \mathbf{Ab}))$ (see Section 4.3) and the category of Picard stacks on $\mathcal{C}$. A similar result should hold for $\mathrm{ch} : C(A) \to \mathrm{Ho}(\mathbf{St}(\mathcal{C}))$, where Picard stacks are replaced with an appropriately defined notion of A-linear Picard stacks. We do not need this extra structure in this work, although it does fit well into the study of obstruction theories.

4.3 Derived categories

Given a ringed site $(\mathcal{C}, A)$ and two complexes M and N, the stacks $\mathrm{ch}(M)$ and $\mathrm{ch}(N)$ can be equivalent even if the truncations $\tau^{[-1,0]}(M)$ and $\tau^{[-1,0]}(N)$ are not isomorphic. This allows us to write down plenty of morphisms $\mathrm{ch}(M) \to \mathrm{ch}(N)$ which do not come from morphisms $M \to N$.

Derived categories resolve this issue as follows: while a morphism $\mathrm{ch}(M) \to \mathrm{ch}(N)$ may not come from a morphism $M \to N$ in $C(A)$, it turns out that all the morphisms of interest to us come from morphisms $M \to N$ in the derived category $D(A)$.

In this section, we introduce the derived category of an abelian category. We fix throughout an abelian category $\mathcal{A}$.

Definition 4.3.1. Let $f : M \to N$ be a morphism of complexes in $\mathcal{A}$. We say that f is a *quasi-isomorphism* if for all $i \in \mathbb{Z}$, the induced map

$$H^i(f) : H^i(M) \to H^i(N)$$

is an isomorphism.

The derived category $D(\mathcal{A})$ is obtained by formally inverting all quasi-isomorphisms in $C(\mathcal{A})$. More precisely, $D(\mathcal{A})$ is characterised by the following universal property.

Definition 4.3.2 (cf. [11], Definition-Theorem III.2.1)**.** The *derived category of $\mathcal{A}$* is a category $D(\mathcal{A})$ equipped with a functor $Q : C(\mathcal{A}) \to D(\mathcal{A})$ satisying the following conditions.

(1) If $f : A \to B$ is a quasi-isomorphism of complexes, then $Q(f)$ is an isomorphism in $D(\mathcal{A})$.

(2) If $F : C(\mathcal{A}) \to \mathcal{C}$ is a functor such that $F(f)$ is an isomorphism whenever f is a quasi-isomorphism, then there exists a unique functor $\tilde{F} : D(\mathcal{A}) \to \mathcal{C}$ such that $\tilde{F} \circ Q = F$.

The *bounded below* and *bounded above* derived categories $D^+(\mathcal{A})$ and $D^-(\mathcal{A})$ are defined similarly by replacing $C(\mathcal{A})$ by $C^+(\mathcal{A})$ and $C^-(\mathcal{A})$ respectively.

Remark 4.3.3. By [11], Definition-Theorem III.2.1, the pair $(D(\mathcal{A}), Q)$ exists and is unique up to isomorphism.

Remark 4.3.4. It follows from the definition that the objects of $D(\mathcal{A})$ are all of the form $Q(M)$ for some $M \in \mathrm{Ob}(\mathcal{A})$. We will usually suppress the functor Q and simply write M instead of $Q(M)$ when there is no danger of confusion.

Remark 4.3.5. The derived category $D(\mathcal{A})$ has a much coarser notion of isomorphism than the category $C(\mathcal{A})$. To emphasise this, we will often use the symbol $\simeq$ instead of $\cong$ to denote isomorphism in $D(\mathcal{A})$.

Remark 4.3.6. Let $(\mathcal{C}, A)$ be a ringed site. It follows immediately from Definition 4.3.2 and Proposition 4.2.2 that $\mathrm{ch} : C(A) \to \mathrm{Ho}(\mathbf{St}(\mathcal{C}))$ descends to a functor

$$\mathrm{ch} : D(A) \to \mathrm{Ho}(\mathbf{St}(\mathcal{C}))$$

where $D(A) = D(A\text{-}\mathbf{mod})$.

The derived category $D(\mathcal{A})$ posesses the following structure. If $M \in \mathrm{Ob}(C(\mathcal{A}))$ is a complex in $\mathcal{A}$ and $n \in \mathbb{Z}$, the *nth translation* of M is the complex $M[n]$ with

$$M[n]^i = M^{n+i} \quad \text{and} \quad d^i_{M[n]} = (-1)^n d^{n+i}_M \quad \text{for all } i \in \mathbb{Z}.$$

If $f : M \to N$ is a morphism of complexes, we set $f^i[n] = f^{i+n}$ for all $i \in \mathbb{Z}$. This defines a functor $-[n] : C(\mathcal{A}) \to C(\mathcal{A})$ for each $n \in \mathbb{Z}$. As these functors send quasi-isomorphisms to quasi-isomorphisms, they descend to functors

$$-[n] : D(\mathcal{A}) \to D(\mathcal{A}).$$

Translation functors for $D^+(\mathcal{A})$ and $D^-(\mathcal{A})$ are defined similarly.

Let $f : M \to N$ be a morphism of complexes in $\mathcal{A}$. The *mapping cone* of f is the complex $\mathrm{Cone}(f)$ given by

$$\mathrm{Cone}(f)^i = M^{i+1} \oplus N^i$$

with differential given by

$$d^i(u, v) = (-d^{i+1}_M(u), f^{i+1}(u) + d^i_N(v)) \quad \text{for } u \in M^{i+1} \text{ and } v \in N^i.$$

(See Remark 4.1.2.) There is a canonical morphism $\mathrm{Cone}(f) \to M[1]$, giving a sequence

$$M \xrightarrow{f} N \to \mathrm{Cone}(f) \to M[1]$$

in $C(\mathcal{A})$, which descends to a corresponding sequence in $D(\mathcal{A})$ via the functor Q.

Definition 4.3.7. A *triangle* in $D(\mathcal{A})$ is a diagram of the form

$$M \to N \to P \to M[1].$$

A *morphism of triangles* is a commutative diagram in $D(\mathcal{A})$ as follows.

$$\begin{array}{ccccccc} M & \longrightarrow & N & \longrightarrow & P & \longrightarrow & M[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ M' & \longrightarrow & N' & \longrightarrow & P' & \longrightarrow & M'[1] \end{array}$$

We write $\Delta D(\mathcal{A})$ for the corresponding category of triangles in $D(\mathcal{A})$. We say that a triangle is *distinguished* if it is isomorphic in $\Delta D(\mathcal{A})$ to a triangle of the form

$$M \xrightarrow{f} N \to \mathrm{Cone}(f) \to M[1],$$

for some morphism of complexes $f : M \to N$.

Remark 4.3.8. Let

$$0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$$

be a sequence of complexes such that $g \circ f = 0$. Then there is a morphism $\phi : \mathrm{Cone}(f) \to P$ given by

$$\begin{aligned} \phi^i : \mathrm{Cone}(f)^i = M^{i+1} \oplus N^i &\to P \\ (u, v) &\mapsto g^i(v). \end{aligned}$$

Moreover, if for each $i \in \mathbb{Z}$ the sequence

$$0 \to M^i \xrightarrow{f^i} N^i \xrightarrow{g^i} P^i \to 0$$

is exact, then $\phi : \mathrm{Cone}(f) \to P$ is a quasi-isomorphism. Composing ϕ^{-1} with the morphism $\mathrm{Cone}(f) \to M[1]$ gives a triangle

$$M \xrightarrow{f} N \xrightarrow{g} P \to M[1]$$

in $D(\mathcal{A})$. This triangle is distinguished since the diagram

$$\begin{array}{ccccccc} M & \longrightarrow & N & \longrightarrow & \mathrm{Cone}(f) & \longrightarrow & M[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ M & \longrightarrow & N & \longrightarrow & P & \longrightarrow & M[1] \end{array}$$

in $D(\mathcal{A})$ commutes, and each vertical arrow is an isomorphism.

Proposition 4.3.9 (cf. [11], Theorem III.4.6)**.** *Let $\mathcal{A}$ be an abelian category, and let*

$$M \xrightarrow{f} N \xrightarrow{g} P \xrightarrow{h} M[1]$$

be a distinguished triangle in $D(\mathcal{A})$. Then the sequence

$$\cdots \to H^i(M) \xrightarrow{H^i(f)} H^i(N) \xrightarrow{H^i(g)} H^i(P) \xrightarrow{H^i(h)} H^{i+1}(M) \to \cdots$$

is exact. Here we note that $H^i(M[1]) = H^{i+1}(M)$.

Remark 4.3.10. The translation functors and distinguished triangles give $D(\mathcal{A})$ (and its bounded analogues) the structure of a *triangulated category.* For a discussion of triangulated categories, see [11], Chapter IV or [17], Chapter I, Section 1.

4.4 Homotopy coherence for complexes

In this section, we set up some formalism for dealing with homotopy coherence of complexes. For convenience we work with the unbounded derived category $D(\mathcal{A})$. The theory for the bounded versions is identical.

To see why such a formalism is useful, suppose that we want to construct an obstruction sequence

$$\underline{\mathbf{Ext}}_X(T, J) \to \underline{\mathbf{Ext}}_S(T, J) \to \underline{\mathbf{Ob}} \tag{4.4.1}$$

by applying the functor ch to some structure defined using complexes. Recall that a fibre sequence (4.4.1) is really a 2-commutative diagram as follows.

$$\begin{array}{ccc} \underline{\mathbf{Ext}}_X(T, J) & \longrightarrow & * \\ \downarrow & & \downarrow \\ \underline{\mathbf{Ext}}_S(T, J) & \longrightarrow & \underline{\mathbf{Ob}} \end{array}$$

This includes a 2-isomorphism as part of the data, so to construct a fibre sequence from objects in a derived category, we need some way of keeping track of 2-isomorphisms. It is straightforward to check that 2-isomorphisms of morphisms $\mathrm{ch}(M) \to \mathrm{ch}(N)$ correspond to chain homotopies in the sense of [11], III.1.2. The tricky part is keeping track of these homotopies when we pass to the derived category $D(A)$.

There are several approaches to homotopy coherence available in the literature. A very natural approach is to use ∞-*categories*, which keep track of homotopies and higher homotopies as well as objects and morphisms. In this framework, we replace the derived category $D(A)$ with an ∞-category $\mathcal{D}(A)$ and the functor $\mathrm{ch} : D(A) \to \mathrm{Ho}(\mathbf{St}(\mathcal{C}))$ with an ∞-functor $\mathcal{D}(A) \to \mathbf{St}(\mathcal{C})$ to the 2-category of stacks over $\mathcal{C}$. These derived ∞-categories are discussed, for example, in [23], Section 1.3.

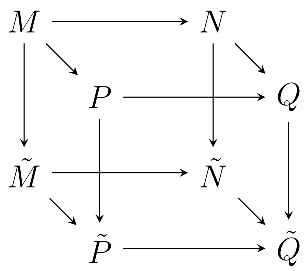
Unfortunately, the theory of ∞-categories is rather technical, and a proper discussion would take us too far afield. There is a gentler approach to homotopy coherence using the theory of *model categories*, which is dicussed, for example, in [12], Chapter VIII. In model category theory, homotopy coherent diagrams can be described using resolutions by strictly commutative diagrams. We use this idea and the formalism of derived categories already at our disposal to capture the homotopy coherence we need.

If $\mathcal{A}$ and K are categories with $\mathcal{A}$ abelian, then the category $\mathcal{A}^K$ of diagrams of shape K in $\mathcal{A}$ is abelian. Hence, we can construct the derived category $D(\mathcal{A}^K)$, which we call the *category of homotopy coherent diagrams of shape K in $D(\mathcal{A})$*. There is a canonical functor $D(\mathcal{A}^K) \to D(\mathcal{A})^K$ from homotopy coherent diagrams to commutative diagrams in $D(\mathcal{A})$.

As a special case, consider the derived category $D(\mathcal{A}^\square)$ of diagrams of the form

$$\begin{array}{ccc} M & \longrightarrow & N \\ \downarrow & & \downarrow \\ P & \longrightarrow & Q \end{array}$$

in $\mathcal{A}$. This is formed from the category $C(\mathcal{A}^\square)$ of commutative squares of complexes by inverting morphisms of diagrams,



for which each vertical arrow is a quasi-isomorphism. We call an object of $D(\mathcal{A}^\square)$ a *homotopy coherent square* in $D(\mathcal{A})$.

Remark 4.4.1. The category of homotopy coherent diagrams $D(\mathcal{A}^K)$ in $D(\mathcal{A})$ cannot be constructed from the category $D(\mathcal{A})$ alone. Instead, we should think of it as extra information about homological algebra that is lost when we pass from $C(\mathcal{A})$ to $D(\mathcal{A})$.

Definition 4.4.2 (cf. Definition 2.5.10)**.** Let $\mathcal{A}$ be an abelian category. The *category of coherent triangles in $D(\mathcal{A})$* is the full subcategory $D(\mathcal{A})_\Delta$ of $D(\mathcal{A}^\square)$ consisting of squares

$$\begin{array}{ccc} M & \longrightarrow & * \\ \downarrow & & \downarrow \\ N & \longrightarrow & P \end{array}$$

such that the complex $*$ is quasi-isomorphic to 0. We will often abuse notation and write a coherent triangle as above by

$$M \to N \to P$$

and a morphism of coherent triangles as follows.

$$\begin{array}{ccccc} M & \longrightarrow & N & \longrightarrow & P \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{M} & \longrightarrow & \tilde{N} & \longrightarrow & \tilde{P} \end{array}$$

Let

$$\begin{array}{ccc} M & \xrightarrow{p} & * \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle q} \\ N & \xrightarrow{g} & P \end{array}$$

be a coherent triangle in $D(\mathcal{A})$. If $* = 0$ as complexes, then there are induced maps

$$\begin{aligned} \mathrm{Cone}(f) &\to P & M &\to \mathrm{Cone}(g)[-1] \\ M^{i+1} \oplus N^i &\to P^i & M^i &\to N^i \oplus P^{i-1} \\ (u, v) &\mapsto g^i(v) & u &\mapsto (-f^i(u), 0) \end{aligned}$$

of complexes. For general $* \simeq 0$, we get

$$\mathrm{Cone}(f) \xleftarrow{\sim} \mathrm{Cone}(M \xrightarrow{(f,-p)} N \oplus *) \to P$$

and

$$M \to \mathrm{Cone}(N \oplus * \xrightarrow{g+q} P)[-1] \xleftarrow{\sim} \mathrm{Cone}(g)[-1]$$

which give morphisms $\mathrm{Cone}(f) \to P$ and $M \to \mathrm{Cone}(g)[-1]$ in $D(\mathcal{A})$.

Proposition 4.4.3. *In the setup above, the map* $\mathrm{Cone}(f) \to P$ *is an isomorphism in* $D(\mathcal{A})$ *if and only if the map* $M \to \mathrm{Cone}(g)[-1]$ *is.*

Definition 4.4.4. We say that a coherent triangle in $D(\mathcal{A})$ is *exact* if it satisfies the equivalent conditions of Proposition 4.4.3. We write $D(\mathcal{A})_{\mathrm{ex}}$ for the full subcategory of $D(\mathcal{A})_{\Delta}$ consisting of exact triangles.

Remark 4.4.5. In the language of [23], Proposition 4.4.3 is essentially the assertion that the derived ∞-category of $\mathcal{A}$ is *stable*.

Let

$$M \xrightarrow{f} N \to P$$

be an exact triangle in $D(\mathcal{A})$. Then the diagram

$$P \xleftarrow{\sim} \mathrm{Cone}(f) \to M[1]$$

gives a triangle,

$$M \xrightarrow{f} N \to P \to M[1],$$

in $D(\mathcal{A})$. This determines a functor,

$$D(\mathcal{A})_{\mathrm{ex}} \to \Delta D(\mathcal{A}),$$

from the category of exact triangles to the category of triangles in $D(\mathcal{A})$. The essential image of this functor is precisely the subcategory of distinguished triangles.

Example 4.4.6. Let

$$0 \to M \to N \to P \to 0$$

be an exact sequence of complexes. Then the associated coherent triangle,

$$\begin{array}{ccc} M & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ N & \longrightarrow & P \end{array}$$

is exact. The image of this exact triangle under the functor $D(\mathcal{A})_{\mathrm{ex}} \to \Delta D(\mathcal{A})$ is the distinguished triangle described in Remark 4.3.8.

Example 4.4.7. Let $f : M \to N$ be a morphism of complexes. Then we have a canonical exact triangle

$$\begin{array}{ccc} M & \longrightarrow & \mathrm{Cone}(M) \\ {\scriptstyle f}\downarrow & & \downarrow \\ N & \longrightarrow & \mathrm{Cone}(f) \end{array}$$

where $\mathrm{Cone}(M) = \mathrm{Cone}(\mathrm{id}_M) \simeq 0$.

It is also useful to generalise the notion of exactness to arbitrary homotopy coherent squares. One way to do this is to fall back on exact triangles as follows. Given a homotopy coherent square,

$$\begin{array}{ccc} M & \xrightarrow{p} & P \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle q} \\ N & \xrightarrow{g} & Q \end{array}$$

there is an associated coherent triangle as follows.

$$\begin{array}{ccc} M & \longrightarrow & 0 \\ {\scriptstyle (f,-p)}\downarrow & & \downarrow \\ N \oplus P & \xrightarrow{g+q} & Q \end{array}$$

This construction preserves quasi-isomorphisms of diagrams, so defines a functor $D(\mathcal{A}^{\square}) \to D(\mathcal{A})_{\Delta}$. We say that a homotopy coherent square is *homotopy cartesian* if the associated coherent triangle is exact. It is clear from the definitions that a coherent triangle is exact if and only if it is homotopy cartesian when considered as a homotopy coherent square.

Remark 4.4.8. Fix a small category K and let $C(A^K) = C(A\text{-}\mathbf{mod}^K)$ denote the category of commutative diagrams of complexes of A-modules. The functor

$$\mathrm{pch} : C(A) \to \mathbf{Fib}(\mathcal{C})$$

determines a functor

$$\mathrm{pch} : C(A^K) \to \mathbf{Fib}(\mathcal{C})^K,$$

which takes a commutative diagram of complexes to the associated strictly commutative diagram of fibred categories. Composing with st, we get a functor,

$$\mathrm{ch} : D(A^K) \to \mathrm{Ho}(\mathbf{St}(\mathcal{C})^K),$$

from homotopy coherent diagrams in $D(A)$ to 2-commutative diagrams in $\mathbf{St}(\mathcal{C})$.

4.5 Exactness of the functor ch

In this section, we prove an exactness result for the functor ch. The main idea is that homotopy cartesian squares in $D^{\geq -1}(A)$ map to 2-cartesian squares in $\mathbf{St}(\mathcal{C})$. This is the basis for using homological methods to compute fibre sequences of stacks.

In fact, since $\mathrm{ch}(M)$ depends only on $\tau^{\leq 0}M$, we can get 2-cartesian squares of stacks from homotopy coherent squares that are not quite homotopy cartesian. This flexibility is very important in the study of obstruction theories: it is the reason why a given space can have more than one obstruction theory, and hence why our formalism can capture virtual fundamental classes which differ from the actual fundamental class.

Definition 4.5.1. Let $\mathcal{A}$ be an abelian category. A $(-\infty, 0]$-*left exact triangle in* $D(\mathcal{A})$ is a coherent triangle,

$$M \to N \xrightarrow{g} P,$$

in $D(\mathcal{A})$ such that the induced map

$$\tau^{\leq 0}M \to \tau^{\leq 0}\mathrm{Cone}(g)[-1]$$

is an isomorphism in $D^{\leq 0}(\mathcal{A})$. A $(-\infty, 0]$-*cartesian square in* $D(\mathcal{A})$ is a homotopy coherent square in $D(\mathcal{A})$ such that the associated coherent triangle is $(-\infty, 0]$-left exact.

Theorem 4.5.2. *Let* $(\mathcal{C}, A)$ *be a ringed site, and let*

$$\begin{array}{ccc} M & \xrightarrow{p} & P \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle q} \\ N & \xrightarrow{g} & Q \end{array} \qquad (4.5.1)$$

be a homotopy coherent square in $D^{\leq 0}(A)$*. Then the 2-commutative square*

$$\begin{array}{ccc} \mathrm{ch}(M) & \xrightarrow{p} & \mathrm{ch}(P) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle q} \\ \mathrm{ch}(N) & \xrightarrow{g} & \mathrm{ch}(Q) \end{array}$$

is 2-cartesian if and only if (4.5.1) *is* $(-\infty, 0]$*-cartesian.*

Proof. Observe that the truncation functor $\tau^{\leq 0} : D(A) \to D^{\leq 0}(A)$ preserves $(-\infty, 0]$-cartesian squares. So we may assume without loss of generality that $M, N, P, Q \in \operatorname{Ob} D^{[-1,0]}(A)$. We need to show that the induced map

$$\mathrm{ch}(M) \to \mathrm{ch}(P) \times_{\mathrm{ch}(Q)} \mathrm{ch}(N)$$

is an equivalence of stacks. Since stackification commutes with 2-fibre products by Proposition 2.5.9, it suffices to show that the map

$$\mathrm{pch}(M) \to \mathrm{pch}(P) \times_{\mathrm{pch}(Q)} \mathrm{pch}(N)$$

is a local equivalence if and only if (4.5.1) is $(-\infty, 0]$-cartesian. For $U \in \mathrm{Ob}(\mathcal{C})$, the groupoid $(\mathrm{pch}(P) \times_{\mathrm{pch}(Q)} \mathrm{pch}(N))(U)$ has objects

$$\left\{(u, v, w) \in N^0 \oplus P^0 \oplus Q^{-1} \,\middle|\, d_Q w = g^0(u) - q^0(v)\right\}$$

and morphisms

$$\begin{aligned} &\mathrm{Hom}((u_1, v_1, w_1), (u_2, v_2, w_2)) \\ &\qquad = \left\{ (x, y) \in N^{-1} \oplus P^{-1} \,\middle|\, \begin{array}{c} u_2 - u_1 = d_N x \\ v_2 - v_1 = d_P y \\ w_2 - w_1 = g^{-1}(x) - q^{-1}(y) \end{array} \right\}. \end{aligned}$$

Observe that this is precisely $\mathrm{pch}(R)$, where R is the complex

$$\begin{array}{ccccc} N^{-1} \oplus P^{-1} & \longrightarrow & N^0 \oplus P^0 \oplus Q^{-1} & \longrightarrow & Q^0 \\ (x, y) & \longmapsto & (d_N x, d_P y, g^{-1}(x) - q^{-1}(y)) & & \\ & & (u, v, w) & \longmapsto & g^0(u) - q^0(v) - d_Q w. \end{array}$$

There is an isomorphism

$$\mathrm{Cone}(N \oplus P \xrightarrow{g+q} Q)[-1] \to R$$

given by negating the summands N^{-1} and N^0. The morphism of fibred categories

$$\mathrm{pch}(M) \to \mathrm{pch}(P) \times_{\mathrm{pch}(Q)} \mathrm{pch}(N) \cong \mathrm{pch}(\mathrm{Cone}(N \oplus P \xrightarrow{g+q} Q)[-1])$$

is induced by the morphism of complexes

$$\begin{aligned} M &\to \mathrm{Cone}(N \oplus P \xrightarrow{g+q} Q)[-1] \\ M^i &\to N^i \oplus P^i \oplus Q^{i-1} \\ u &\mapsto (-f^i(u), p^i(u), 0). \end{aligned}$$

Hence, by Proposition 4.2.2, the map

$$\mathrm{pch}(M) \to \mathrm{pch}(P) \times_{\mathrm{pch}(Q)} \mathrm{pch}(N)$$

is an equivalence of stacks if and only if

$$\tau^{[-1,0]} M \to \tau^{[-1,0]} \mathrm{Cone}(N \oplus P \to Q)[-1]$$

is an isomorphism in $D(A)$. But this is precisely the condition for the square (4.5.1) in $D^{\geq -1}(A)$ to be $(-\infty, 0]$-cartesian, so we are done. □

Corollary 4.5.3. *Let*

$$M \to N \to P \tag{4.5.2}$$

be a coherent triangle in $D^{\geq -1}(A)$. *Then the prefibre sequence*

$$\mathrm{ch}(M) \to \mathrm{ch}(N) \to \mathrm{ch}(P)$$

is a fibre sequence if and only if (4.5.2) *is* $(-\infty, 0]$*-left exact.*

To apply Corollary 4.5.3 in practice, we need a method for constructing $(-\infty, 0]$-left exact triangles. An effective way to do this is the following. Let

$$M \to N \to P$$

be an exact triangle in $D(A)$. If $P \to Q$ is a morphism in $D(A)$, we can form a coherent triangle

$$M \to N \to Q$$

by composition. For reasons of homotopy coherence, this procedure is slightly delicate: we treat it properly in Proposition 4.5.4 below. If we have a distinguished triangle

$$R \to P \to Q \to R[1]$$

with $\tau^{\leq 0} R \simeq 0$, then Proposition 4.5.5 below ensures that the associated coherent triangle is $(-\infty, 0]$-left exact.

Proposition 4.5.4. *Let $\mathcal{A}$ be an abelian category. Let*

$$M \to N \to P \tag{4.5.3}$$

be a coherent triangle in $D(A)$ and let $P \to Q$ be a morphism. Then there exists a coherent triangle

$$\tilde{M} \to \tilde{N} \to Q$$

and a morphism of coherent triangles

$$\begin{array}{ccccc} M & \longrightarrow & N & \longrightarrow & P \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{M} & \longrightarrow & \tilde{N} & \longrightarrow & Q \end{array}$$

such that $M \to \tilde{M}$ and $N \to \tilde{N}$ are isomorphisms in $D(A)$ and $P \to Q$ is the given morphism. Moreover, these data are unique up to isomorphism.

We omit the proof of Proposition 4.5.4, as it is a little involved but not terribly hard. The idea is to use cones and cylinders as in the proof of Theorem III.4.4 of [11] to replace the coherent triangle (4.5.3) with an isomorphic coherent triangle,

$$\tilde{M} \to \tilde{N} \to \tilde{P},$$

for which $P \to Q$ can be represented by a morphism of complexes $\tilde{P} \to Q$.

Proposition 4.5.5. *Let $\mathcal{A}$ be an abelian category, let*

$$M \to N \xrightarrow{g} P \tag{4.5.4}$$

be an exact triangle in $D(\mathcal{A})$, and let

$$R \to P \xrightarrow{h} Q \to R[1]$$

be a distinguished triangle. If $\tau^{\leq 0} R \simeq 0$, then the coherent triangle

$$M \to N \to Q$$

constructed using Proposition 4.5.4 is $(-\infty, 0]$-left exact.

Proof. By replacing the exact triangle $M \to N \to P$ with a quasi-isomorphic one, we can assume without loss of generality that $P \to Q$ is a morphism of complexes. We need to show that the morphism $M \to \mathrm{Cone}(h \circ g)[-1]$ induces isomorphisms on cohomology in degrees ≤ 0. This factors as

$$M \to \mathrm{Cone}(g)[-1] \to \mathrm{Cone}(h \circ g)[-1],$$

so, since (4.5.4) is exact, it suffices to show that the latter morphism induces isomorphisms on cohomology in degrees ≤ 0. Notice that since $\tau^{\leq 0}R \simeq 0$, the long exact sequence associated to

$$R \to P \xrightarrow{h} Q \to R[1]$$

shows that $H^i(h)$ is an isomorphism for $i < 0$ and $H^0(h)$ is an injection. The result now follows by diagram chasing applied to the morphism of long exact sequences associated to the following morphism of distinguished triangles.

$$\begin{array}{ccccccc} N & \xrightarrow{g} & P & \longrightarrow & \mathrm{Cone}(g) & \longrightarrow & N[1] \\ \downarrow{\scriptstyle \mathrm{id}_N} & & \downarrow{\scriptstyle h} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}_N[1]} \\ N & \xrightarrow{h \circ g} & Q & \longrightarrow & \mathrm{Cone}(h \circ g) & \longrightarrow & N[1] \end{array}$$

□

Chapter 5

Obstruction theories

In this chapter, we apply the tools of Chapter 4 to the study of square-zero extensions and obstruction sequences. The key technical addition in this chapter is the cotangent complex, which we introduce in Section 5.2. The cotangent complex can be used to compute stacks of square-zero extensions explicitly, and its exactness properties give us an elegant way to manipulate obstruction sequences. In Section 5.3, we define obstruction theories using the cotangent complex, and characterise them in terms of obstruction sequences.

The cotangent complex of an S-scheme X is only well-defined up to isomorphism in $D(\mathcal{O}_X)$. Thus, it is essential to have versions of usual functors for sheaves (such as pullbacks and Hom sheaves) which descend to the derived category $D(\mathcal{O}_X)$. This is a slightly delicate issue, which is dealt with using the theory of derived functors in Section 5.1.

5.1 Derived functors

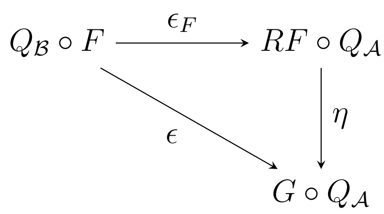
Suppose that $\mathcal{A}$ and $\mathcal{B}$ are abelian categories, and that $F : C(\mathcal{A}) \to C(\mathcal{B})$ is a functor. In this section, we study ways in which F can induce a functor $D(\mathcal{A}) \to D(\mathcal{B})$ between derived categories.

If $F : C(\mathcal{A}) \to C(\mathcal{B})$ takes quasi-isomorphisms to quasi-isomorphisms, then by Definition 4.3.2, F descends to a functor $F : D(\mathcal{A}) \to D(\mathcal{B})$ such that $Q_{\mathcal{B}} \circ F = F \circ Q_{\mathcal{A}}$. This is the case, for example, if F is given by applying an exact functor $\mathcal{A} \to \mathcal{B}$ termwise. Many functors of interest, however, do not preserve quasi-isomorphisms, and therefore do not descend directly to functors $D(\mathcal{A}) \to D(\mathcal{B})$. The idea of a derived functor is to take instead a functor $D(\mathcal{A}) \to D(\mathcal{B})$ which is the best possible approximation to F. We make this precise below.

Definition 5.1.1 (cf. [11], Definition III.6.6 and [17], Section 1.5)**.** Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and let $F : C^+(\mathcal{A}) \to C(\mathcal{B})$ be an exact functor. A *right derived functor* for F is a functor

$$RF : D^+(\mathcal{A}) \to D(\mathcal{B})$$

together with a natural transformation $\epsilon_F : Q_\mathcal{B} \circ F \to RF \circ Q_\mathcal{A}$ such that for any functor $G : D^+(\mathcal{A}) \to D(\mathcal{B})$ and any natural transformation $\epsilon : Q_\mathcal{B} \circ F \to G \circ Q_\mathcal{A}$, there exists a unique natural transformation $\eta : RF \to G$ such that the following diagram commutes.



If $F : C^-(\mathcal{A}) \to C(\mathcal{B})$ is an exact functor, a left derived functor $LF : D^-(\mathcal{A}) \to D(\mathcal{B})$ is defined dually by replacing $+$ with $-$ and reversing all arrows.

Remark 5.1.2. It is immediate from the definition that derived functors are unique up to unique isomorphism if they exist.

Derived functors exist for a fairly large class of functors. To construct, say, a right derived functor for $F : C^+(\mathcal{A}) \to C(\mathcal{B})$, we first find a class of objects $\mathcal{R}$ in $\mathcal{A}$, closed under direct sums, such that F takes acyclic complexes in $C^+(\mathcal{R})$ to acyclic complexes in $C(\mathcal{B})$, and such that every object of $\mathcal{A}$ admits an injection into an object of $\mathcal{R}$. As long as F satisfies the further technical hypothesis that it descends to an exact functor between categories of complexes up to homotopy, the right derived functor $RF : D^+(\mathcal{A}) \to D(\mathcal{B})$ is defined by

$$RF(M) = F(\tilde{M}),$$

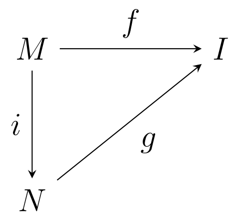
where we choose a quasi-isomorphism $i : M \to \tilde{M}$, with $\tilde{M} \in C^+(\mathcal{R})$. We call $M \to \tilde{M}$ an $\mathcal{R}$*-resolution* of M. The natural transformation $\epsilon_F : Q_\mathcal{B} \circ F \to RF \circ Q_\mathcal{A}$ is given by

$$(\epsilon_F)_M = F(i) : F(M) \to F(\tilde{M}) = RF(M).$$

There is an analogous construction for left derived functors, given by reversing arrows. For the full details of this construction, and the proof that it gives a well-defined functor which satisfies the universal property of Definition 5.1.1, see [11], Theorem III.6.8 or [17], Theorem I.5.1.

As an example, consider the following class of objects in a general abelian category $\mathcal{A}$.

Definition 5.1.3. Let $\mathcal{A}$ be an abelian category. An object $I \in \mathrm{Ob}(\mathcal{A})$ is called *injective* if for any injection $i : M \to N$ in $\mathcal{A}$ and any morphism $f : M \to I$, there exists a morphism $g : N \to I$ such that the diagram below commutes.



Definition 5.1.4. Let $\mathcal{A}$ be an abelian category. We say that $\mathcal{A}$ has *enough injectives* if for every $M \in \mathrm{Ob}\,\mathcal{A}$, there exists an injective object I and an injective map $M \to I$.

Proposition 5.1.5 (cf. [11], Theorem III.6.12)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, and let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. If $\mathcal{A}$ has enough injectives, then $F : C^+(\mathcal{A}) \to C^+(\mathcal{B})$ has a right derived functor, which can be computed using injective resolutions.*

The main examples of derived functors of interest to us are left derived pullbacks and right derived mapping complexes. We introduce these below.

Let $f : (\mathcal{C}, A) \to (\mathcal{C}, B)$ be a morphism of ringed sites. Recall from Section 2.3 that we have a pullback functor

$$f^* : B\text{-}\mathbf{mod} \to A\text{-}\mathbf{mod}.$$

This has a left derived functor

$$Lf^* : D^-(B) \to D^-(A)$$

defined using *flat resolutions*.

Definition 5.1.6. Let $(\mathcal{C}, A)$ be a ringed site, and let M be an A-module. We say that M is *flat* if the functor $M \otimes_A - : A\text{-}\mathbf{mod} \to A\text{-}\mathbf{mod}$ is exact.

Let K be a small category. A morphism of ringed sites $f : (\mathcal{C}, A) \to (\mathcal{D}, B)$ extends in a canonical way to a morphism $f : (\mathcal{C} \times K^{op}, A^K) \to (\mathcal{D} \times K^{op}, B^K)$, where A^K and B^K are the sheaves corresponding to the constant diagrams at A and B. Hence, we get a left derived functor

$$Lf^* : D^-(B^K) \to D^-(A^K).$$

For each $k \in \operatorname{Ob} K$, we have "evaluation at k" functors $D^-(B^K) \to D^-(B)$ and $D^-(A^K) \to D^-(A)$ such that the diagram below commutes.

$$\begin{array}{ccc} D^-(B^K) & \xrightarrow{Lf^*} & D^-(A^K) \\ \downarrow & & \downarrow \\ D^-(A) & \xrightarrow{Lf^*} & D^-(B) \end{array}$$

Derived functors usually extend to homotopy coherent diagrams in this way.

The derived functor Lf^* is exact in the following sense.

Proposition 5.1.7. *Let $f : (\mathcal{C}, A) \to (\mathcal{D}, B)$ be a morphism of ringed sites, and let*

$$M \to N \to P$$

be an exact triangle in $D^-(B)$. Then the coherent triangle

$$Lf^*M \to Lf^*N \to Lf^*P$$

in $D^-(A)$ is exact.

Proof. This follows immediately from the fact that $f^* : C^-(B) \to C^-(A)$ commutes with taking cones, and takes quasi-isomorphisms between complexes of flat modules to quasi-isomorphisms. □

Remark 5.1.8. It follows immediately from Proposition 5.1.7 that $Lf^* : D^-(B) \to D^-(A)$ commutes with the translation functors and takes distinguished triangles to distinguished triangles. This is the exactness property for derived functors used, for example, in [11]. For our purposes, the coherent version of exactness of Proposition 5.1.7 is important for computations with stacks.

The following properties of Lf^* are useful to keep in mind.

Proposition 5.1.9. *Let $f : (\mathcal{C}, A) \to (\mathcal{D}, B)$ be a morphism of ringed sites.*

(1) If $V \in \operatorname{Ob}(\mathcal{D})$ and $U = f^{-1}(V) \in \operatorname{Ob} \mathcal{C}$, then there is a natural isomorphism

$$Lf^*(M|_V) \simeq Lf^*(M)|_U$$

for $M \in \operatorname{Ob} D^-(B)$.

(2) If $n \in \mathbb{Z}$, then there is a natural isomorphism

$$\tau^{\geq n} Lf^* M \simeq \tau^{\geq n} Lf^*(\tau^{\geq n} M)$$

for $M \in \operatorname{Ob} D^-(B)$.

We now turn to the study of derived mapping complexes.

Definition 5.1.10. Let $(\mathcal{C}, A)$ be a ringed site and let M and N be A-modules. The *Hom sheaf* of M and N is the A-module $\underline{\operatorname{Hom}}(M, N)$ given by

$$\underline{\operatorname{Hom}}(M, N)(U) = \operatorname{Hom}(M|_U, N|_U)$$

for $U \in \operatorname{Ob} \mathcal{C}$.

Definition 5.1.11 (cf. [17], Section II.3)**.** Let $(\mathcal{C}, A)$ be a ringed site. If $M, N \in \operatorname{Ob}(C(A))$ are complexes of A-modules, the *mapping complex* of M and N is the complex of A-modules $\underline{\operatorname{Hom}}^\bullet(M, N)$ with ith term

$$\underline{\operatorname{Hom}}^i(M, N) = \prod_{j \in \mathbb{Z}} \underline{\operatorname{Hom}}(M^j, N^{i+j})$$

and differential

$$d^i((f^j)_{j \in \mathbb{Z}}) = (f^{j+1} \circ d_M^j + (-1)^{i+1} d_N^{i+j} \circ f^j)_{j \in \mathbb{Z}}$$

for $i \in \mathbb{Z}$.

Remark 5.1.12. There is a version of Definition 5.1.11 for more general abelian categories which uses the abelian groups $\operatorname{Hom}(M^j, N^{i+j})$ in place of the sheaves $\underline{\operatorname{Hom}}(M^j, N^{i+j})$. This global mapping complex is useful for computing global Ext groups, for example, whereas the local mapping complex we use is more relevant when working with stacks.

The mapping complex defines a functor

$$\underline{\operatorname{Hom}}^\bullet : C(A)^{op} \times C^+(A) \to C(A).$$

This has a right derived functor

$$R\underline{\operatorname{Hom}}^\bullet : D(A)^{op} \times D^+(A) \to D(A)$$

characterised by a universal property similar to Definition 5.1.1. The construction is by taking injective resolutions in the second variable and relies on the following theorem.

Theorem 5.1.13 (cf. [29, Tag 01DU]). *Let $(\mathcal{C}, A)$ be a ringed site. Then the category A-**mod** has enough injectives.*

Just as for Lf^*, the derived functor $R\underline{\mathrm{Hom}}^\bullet$ has good homotopy coherence and exactness properties. If K is a small category, then we have functors

$$\underline{\mathrm{Hom}}^\bullet(-,-) : C(A^{K^{op}})^{op} \times C^+(A) \to C(A^K)$$

and

$$\underline{\mathrm{Hom}}^\bullet(-,-) : C(A)^{op} \times C^+(A^K) \to C(A^K).$$

Since A^K-**mod** has enough injectives by Theorem 5.1.13 applied to the ringed site $(\mathcal{C} \times K^{op}, A^K)$, we can form right derived functors

$$R\underline{\mathrm{Hom}}^\bullet(-,-) : D(A^{K^{op}}) \times D^+(A) \to D(A^K)$$

and

$$R\underline{\mathrm{Hom}}^\bullet(-,-) : D(A) \times D^+(A^K) \to D(A^K)$$

by taking injective resolutions in the second factor. These are compatible with the basic functor $D(A) \times D^+(A) \to D(A)$ under evaluation at each object in K.

The following properties are straightforward and well-known.

Proposition 5.1.14. *The functor $R\underline{\mathrm{Hom}}^\bullet(-,-)$ is exact in each variable in the following sense.*

(1) If

$$M \to N \to P$$

is an exact triangle in $D^+(A)$ and $Q \in \mathrm{Ob}\, D(A)$, then

$$R\underline{\mathrm{Hom}}^\bullet(Q, M) \to R\underline{\mathrm{Hom}}^\bullet(Q, N) \to R\underline{\mathrm{Hom}}^\bullet(Q, P)$$

is an exact triangle in $D(A)$.

(2) If

$$M \to N \to P$$

is an exact triangle in $D(A)$ and $Q \in \mathrm{Ob}\, D^+(A)$, then

$$R\underline{\mathrm{Hom}}^\bullet(P, Q) \to R\underline{\mathrm{Hom}}^\bullet(N, Q) \to R\underline{\mathrm{Hom}}^\bullet(M, Q)$$

is an exact triangle in $D(A)$.

Proposition 5.1.15. *Let $(\mathcal{C}, A)$ be a ringed site.*

(1) If $U \in \operatorname{Ob}\mathcal{C}$, then there is a natural isomorphism

$$R\underline{\mathrm{Hom}}^\bullet(M|_U, N|_U) \simeq R\underline{\mathrm{Hom}}^\bullet(M, N)|_U$$

for $M \in \operatorname{Ob} D(A)$ and $N \in \operatorname{Ob} D^+(B)$.

(2) If $m, n \in \mathbb{Z}$ then there is a natural isomorphism

$$\tau^{\leq m} R\underline{\mathrm{Hom}}^\bullet(M, N) \simeq \tau^{\leq m} R\underline{\mathrm{Hom}}^\bullet(\tau^{\geq n-m} M, N)$$

for $M \in \operatorname{Ob} D(A)$ and $N \in \operatorname{Ob} D^{\geq n}(A)$.

5.2 The cotangent complex

In this section, we give a brief introduction to the theory of cotangent complexes, following the work of L. Illusie in [19]. We give a more detailed treatment in Appendix B.

In what follows, for X a scheme, we write $D(X) = D(\mathcal{O}_{X_{\acute{e}t}})$, and $D_{qc}(X)$ for the full subcategory of $D(X)$ of objects with quasi-coherent cohomology. (See Remark 3.1.6.)

Let $X \to S$ be a morphism of schemes. The cotangent complex $L_{X/S}$ is an object in $D^{\leq 0}_{qc}(X)$, which generalises the cotangent sheaf of a smooth morphism of schemes. When restricted to smooth morphisms, the cotangent sheaf has very good exactness properties, and can be used to classify deformations. The cotangent complex keeps this good behaviour in the singular case by including higher order terms to keep track of the singularities.

The main algebraic properties of the cotangent complex are the following.

Theorem 5.2.1. *Let $X \to S$ be a morphism of schemes.*

(1) We have $L_{X/S} \in \operatorname{Ob} D^{\leq 0}_{qc}(X)$. (See [19], Corollaire II.2.3.7.)

(2) There is a canonical isomorphism $H^0(L_{X/S}) \cong \Omega_{X/S}$, where $\Omega_{X/S}$ is the relative cotangent sheaf. If $X \to S$ is smooth, then this induces an isomorphism $L_{X/S} \simeq \Omega_{X/S}$ in $D^-(X)$.

(3) If $i : X \hookrightarrow Y$ is a closed embedding of S-schemes with Y smooth over S, then there is an isomorphism

$$\tau^{\geq -1} L_{X/S} \simeq [I/I^2 \to i^*\Omega_{Y/S}]$$

where $I \subseteq i^{-1}\mathcal{O}_Y$ is the ideal sheaf of the embedding and the map $I/I^2 \to i^*\Omega_{Y/S}$ is given by differentiation. Furthermore, if i is a regular embedding, then the truncation map

$$L_{X/S} \to \tau^{\geq -1} L_{X/S}$$

is an isomorphism. (See Proposition B.5.2.)

(4) If $f : X \to Y$ is a morphism of schemes over S, then there is a canonical exact triangle

$$Lf^* L_{Y/S} \to L_{X/S} \to L_{X/Y}$$

in $D^-(X)$. (See Corollary B.3.3.)

Remark 5.2.2. Let $\pi : X \to S$ be a morphism of Deligne-Mumford stacks. As discussed in Section A.7, we have associated étale sites $X_{ét}$ and $S_{ét}$ and a morphism of ringed sites

$$\pi : (X_{ét}, \mathcal{O}_{X_{ét}}) \to (S_{ét}, \mathcal{O}_{S_{ét}}).$$

Thus, we can apply Illusie's algebraic construction (Definition B.2.3) to define

$$L_{X/S} = L_{\mathcal{O}_{X_{ét}}/\pi^{-1}\mathcal{O}_{S_{ét}}}.$$

In [3], Gromov-Witten invariants are defined using an obstruction theory for a morphism $\pi : X \to S$, where S is the Artin stack of nodal curves and X is some stack of maps from the universal curve over S to a fixed target variety. In order to carry out such constructions properly, it is necessary to have a theory of cotangent complexes valid for morphisms of Artin stacks. Fortunately, such cotangent complexes have been defined rigorously by M. Olsson in [26]. There is also an approach through derived algebraic geometry (in much greater generality), which can be found, for example, in [24].

For our purposes, the whole point of cotangent complexes is the following result relating $L_{X/S}$ to square-zero extensions of X over S.

Theorem 5.2.3 (cf. Theorem B.4.1 and [19], Theorem III.1.2.3)**.** *Let $\pi : X \to S$ be a morphism of schemes, and let J be a quasi-coherent sheaf on X. Then there is an equivalence*

$$\mathrm{ch}(R\underline{\mathrm{Hom}}^\bullet(L_{X/S}, J[1])) \simeq \underline{\mathbf{Ext}}_S(X, J)$$

of stacks over $X_{ét}$.

Remark 5.2.4. The equivalence of Theorem 5.2.3 is natural in S in a very precise way. By the arguments of Section B.3, any diagram of schemes $S_\bullet$ of shape K with a morphism $X \to S_\bullet$ gives rise to a homotopy coherent diagram

$$L_{X/S_\bullet} \in \mathrm{Ob}\, D\left(\mathcal{O}_{X_{ét}}^{K^{op}}\right)$$

giving the appropriate cotangent complex at each place. Hence we have a 2-commutative diagram

$$\mathrm{ch}(R\underline{\mathrm{Hom}}^\bullet(L_{X/S_\bullet}, J[1])) \in \mathrm{Ob}\,\mathrm{Ho}(\mathbf{St}(\mathcal{C})^K).$$

By Remark B.4.2, the equivalence of Theorem 5.2.3 extends to an equivalence of 2-commutative diagrams

$$\mathrm{ch}(R\underline{\mathrm{Hom}}^\bullet(L_{X/S_\bullet}, J[1])) \simeq \underline{\mathbf{Ext}}_{S_\bullet}(X, J)$$

where $\underline{\mathbf{Ext}}_{S_\bullet}(X, J)$ is the 2-commutative diagram obtained from the forgetful morphisms of Remark 3.1.14.

5.3 Obstruction theories

Throughout this section, we fix a scheme S and an S-scheme X.

Definition 5.3.1. An *obstruction theory* for X over S is an object $E \in \mathrm{Ob}(D_{qc}^{\leq 0}(X))$, together with a morphism $\phi : E \to L_{X/S}$ such that

(1) the morphism $H^0(\phi) : H^0(E) \to H^0(L_{X/S})$ is an isomorphism of sheaves, and

(2) the morphism $H^{-1}(\phi) : H^{-1}(E) \to H^{-1}(L_{X/S})$ is a surjection of sheaves.

Remark 5.3.2. Let $\phi : E \to L_{X/S}$ be a morphism in $D_{qc}(X)$, and let $\mathrm{Cone}(\phi)$ be a mapping cone of ϕ. From the long exact sequence associated to the distinguished triangle

$$E \to L_{X/S} \to \mathrm{Cone}(\phi) \to E[1]$$

we see that E is an obstruction theory if and only if $\tau^{\geq -1}(\mathrm{Cone}(\phi)) \simeq 0$.

Example 5.3.3. The *trivial obstruction theory* is the identity map $L_{X/S} \to L_{X/S}$.

Obstruction theories are algebraic structures which control obstruction sequences of maps to X in the sense of Definition 3.2.1. Recall that an obstruction sequence for a morphism $f : T \to X$ relative to a quasi-coherent sheaf J consists of an obstruction stack $\underline{\mathbf{Ob}}$ on T and a fibre sequence

$$\underline{\mathbf{Ext}}_X(T, J) \to \underline{\mathbf{Ext}}_S(T, J) \to \underline{\mathbf{Ob}}.$$

The cotangent complex gives a universal choice of obstruction sequence as follows. Recall from Theorem 5.2.1, (4) that there is an exact triangle

$$Lf^* L_{X/S} \to L_{T/S} \to L_{T/X}$$

in $D(T)$. Hence, by Proposition 5.1.14 and Corollary 4.5.3, we get a fibre sequence

$$\mathrm{ch}(R\underline{\mathrm{Hom}}^\bullet(L_{T/X}, J[1])) \to \mathrm{ch}(R\underline{\mathrm{Hom}}^\bullet(L_{T/S}, J[1])) \\ \to \mathrm{ch}(R\underline{\mathrm{Hom}}^\bullet(Lf^*L_{X/S}, J[1]))$$

of stacks over $T_{\acute{e}t}$. Applying Theorem 5.2.3 and Remark 5.2.4 to interpret the first two terms, this yields an obstruction sequence

$$\underline{\mathbf{Ext}}_X(T, J) \to \underline{\mathbf{Ext}}_S(T, J) \xrightarrow{\mathrm{ob}} \mathrm{ch}(R\underline{\mathrm{Hom}}^\bullet(Lf^*L_{X/S}, J[1])).$$

Now suppose that $\phi : E \to L_{X/S}$ is a morphism in $D^{\leq 0}_{qc}(X)$. Using Proposition 4.5.4, we obtain a well-defined coherent triangle

$$Lf^*E \to L_{T/S} \to L_{T/X}$$

and hence a prefibre sequence

$$\underline{\mathbf{Ext}}_X(T, J) \to \underline{\mathbf{Ext}}_S(T, J) \xrightarrow{\mathrm{ob}_E} \mathrm{ch}(R\underline{\mathrm{Hom}}^\bullet(Lf^*E, J[1]).$$

Theorem 5.3.4 (cf. [5], Theorem 4.5). *Let $\phi : E \to L_{X/S}$ be a morphism in $D^{\leq 0}_{qc}(X)$. Then E is an obstruction theory for X if and only if for all schemes T, all morphisms $f : T \to X$ and all quasi-coherent sheaves J on T, the prefibre sequence*

$$\underline{\mathbf{Ext}}_X(T, J) \to \underline{\mathbf{Ext}}_S(T, J) \xrightarrow{\mathrm{ob}_E} \mathrm{ch}(R\underline{\mathrm{Hom}}^\bullet(f^*E, J[1])) \tag{5.3.1}$$

is a fibre sequence.

Proof. Let Cone(ϕ) denote the mapping cone of ϕ. By Remark 5.3.2, $\phi : E \to L_{X/S}$ is an obstruction theory if and only if $\tau^{\geq -1}\mathrm{Cone}(\phi) \simeq 0$.

Suppse that $\tau^{\geq -1}\mathrm{Cone}(\phi) \simeq 0$. Then for any T, f and J, we have a distinguished triangle

$$R\underline{\mathrm{Hom}}^\bullet(Lf^*\mathrm{Cone}(\phi), J[1]) \to R\underline{\mathrm{Hom}}^\bullet(Lf^*L_{X/S}, J[1]) \\ \to R\underline{\mathrm{Hom}}^\bullet(Lf^*E, J[1]) \to R\underline{\mathrm{Hom}}^\bullet(Lf^*\mathrm{Cone}(\phi), J[1])[1]$$

in $D(T)$, and

$$\begin{aligned} \tau^{\leq 0}R\underline{\mathrm{Hom}}^\bullet(Lf^*\mathrm{Cone}(\phi), J[1]) &\simeq \tau^{\leq 0}R\underline{\mathrm{Hom}}^\bullet(\tau^{\geq -1}Lf^*\mathrm{Cone}(\phi), J[1]) \\ &\simeq \tau^{\leq 0}R\underline{\mathrm{Hom}}^\bullet(\tau^{\geq -1}Lf^*\tau^{\geq -1}\mathrm{Cone}(\phi), J[1]) \\ &\simeq 0 \end{aligned}$$

by Propositions 5.1.9 and 5.1.15. So by Proposition 4.5.5 and Corollary 4.5.3, (5.3.1) is a fibre sequence.

Conversely, suppose that (5.3.1) is a fibre sequence for all T, f and J. By Corollary 4.5.3, the coherent triangle

$$R\underline{\mathrm{Hom}}^\bullet(Lf^*L_{T/X}, J[1]) \to R\underline{\mathrm{Hom}}^\bullet(Lf^*L_{T/S}, J[1]) \to R\underline{\mathrm{Hom}}^\bullet(Lf^*E, J[1])$$

is $(-\infty, 0]$-left exact. Setting $T = X$ and $f = \mathrm{id}_X$, this implies that

$$0 \to R\underline{\mathrm{Hom}}^\bullet(L_{X/S}, J[1]) \to R\underline{\mathrm{Hom}}^\bullet(E, J[1])$$

is $(-\infty, 0]$-left exact, which is equivalent to

$$\tau^{\leq 0} R\underline{\mathrm{Hom}}^\bullet(\mathrm{Cone}(\phi), J[1]) \simeq 0.$$

Hence,

$$\begin{aligned}\underline{\mathrm{Hom}}(H^0(\mathrm{Cone}(\phi)), H^0(\mathrm{Cone}(\phi)))& \\ &= \left(\tau^{\leq -1} R\underline{\mathrm{Hom}}^\bullet(\mathrm{Cone}(\phi), H^0(\mathrm{Cone}(\phi))[1])\right)[-1] \\ &\simeq 0,\end{aligned}$$

so $H^0(\mathrm{Cone}(\phi)) = 0$, and

$$\begin{aligned}\underline{\mathrm{Hom}}(H^{-1}(\mathrm{Cone}(\phi)), H^{-1}(\mathrm{Cone}(\phi)))& \\ &= \tau^{\leq 0} R\underline{\mathrm{Hom}}^\bullet(\mathrm{Cone}(\phi), H^{-1}(\mathrm{Cone}(\phi))[1]) \\ &\simeq 0,\end{aligned}$$

so $H^{-1}(\mathrm{Cone}(\phi)) = 0$. Hence $\tau^{\geq -1}\mathrm{Cone}(\phi) \simeq 0$ so $E \to L_{X/S}$ is an obstruction theory. □

Remark 5.3.5. If X is a Deligne-Mumford stack instead of a scheme, we cannot set $T = X$ in the proof of Theorem 5.3.4. Since the statement is local in the étale topology, we can take T to be an étale cover of X by a scheme, and leave the rest of the proof unchanged.

5.4 Obstruction theories for fibre products

In this section, we compute some explicit examples of obstruction theories on fibre products, based on the obstruction sequences of Section 3.2.

Fix a scheme S and consider a diagram of schemes

$$\begin{array}{ccc} X & \xrightarrow{j} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{i} & W \end{array}$$

with $X = Y \times_W Z$. Starting from obstruction sequences for Y, Z and W, in Section 3.2 we constructed an obstruction sequence

$$\underline{\mathbf{Ext}}_X(T, J) \to \underline{\mathbf{Ext}}_S(T, J) \to \underline{\mathbf{Ob}}$$

for a given $h : T \to X$ and given quasi-coherent sheaf J on T. Take the obstruction sequences for Y, Z and W to be given by the trivial obstruction theories, so that, in the notation of Section 3.2, we have

$$\begin{aligned} \underline{\mathbf{Ob}}_Y &= \mathrm{ch}(R\underline{\mathrm{Hom}}^\bullet(h^* j^* L_{Y/S}, J[1])), \\ \underline{\mathbf{Ob}}_Z &= \mathrm{ch}(R\underline{\mathrm{Hom}}^\bullet(h^* g^* L_{Z/S}, J[1])), \\ \underline{\mathbf{Ob}}_W &= \mathrm{ch}(R\underline{\mathrm{Hom}}^\bullet(h^* g^* i^* L_{W/S}, J[1])), \end{aligned}$$

where for brevity we write h^* in place of Lh^* and so on. Using the functoriality of the cotangent complex (see Section B.3), we have canonical morphisms of fibre sequences as in (3.2.2). So we get an obstruction sequence

$$\underline{\mathbf{Ext}}_X(T, J) \to \underline{\mathbf{Ext}}_S(T, J) \to \underline{\mathbf{Ob}},$$

where

$$\underline{\mathbf{Ob}} = \underline{\mathbf{Ob}}_Y \times_{\underline{\mathbf{Ob}}_W} \underline{\mathbf{Ob}}_Z.$$

Let

$$E = \mathrm{Cone}(g^* i^* L_{W/S} \xrightarrow{(f^*, -i^*)} j^* L_{Y/S} \oplus g^* L_{Z/S}) \tag{5.4.1}$$

where

$$\begin{array}{ccc} g^* i^* L_{W/S} & \xrightarrow{i^*} & g^* L_{Z/S} \\ {\scriptstyle f^*}\downarrow & & \downarrow \\ j^* L_{Y/S} & \longrightarrow & L_{X/S} \end{array} \tag{5.4.2}$$

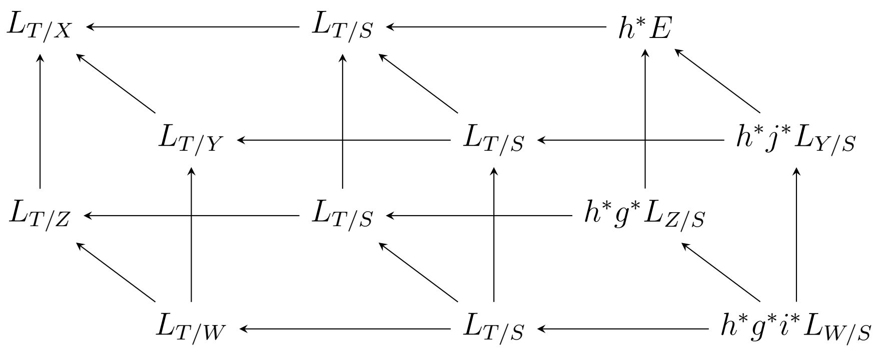
is the canonical homotopy coherent square constructed using the methods of Section B.3. Note that the square

$$
\begin{array}{ccc}
R\underline{\mathrm{Hom}}^\bullet(h^*E, J[1]) & \longrightarrow & R\underline{\mathrm{Hom}}^\bullet(j^*L_{Y/S}, J[1]) \\
\downarrow & & \downarrow \\
R\underline{\mathrm{Hom}}^\bullet(g^*L_{Z/S}, J[1]) & \longrightarrow & R\underline{\mathrm{Hom}}^\bullet(g^*i^*L_{W/S}, J[1])
\end{array}
$$

is homotopy cartesian, and therefore

$$\mathrm{ch}(R\underline{\mathrm{Hom}}^\bullet(h^*E, J[1])) \simeq \underline{\mathbf{Ob}}$$

by Theorem 4.5.2. Using the homotopy coherent diagram (5.4.2), we get a morphism $E \to L_{X/S}$. This induces a homotopy coherent diagram



(5.4.3)

where the top row is the coherent triangle of Proposition 4.5.4. Applying $\mathrm{ch}(R\underline{\mathrm{Hom}}^\bullet(-, J[1]))$ to (5.4.3) gives (3.2.3). In particular, the prefibre sequence

$$\underline{\mathbf{Ext}}_X(T, J) \to \underline{\mathbf{Ext}}_S(T, J) \to \mathrm{ch}(R\underline{\mathrm{Hom}}^\bullet(h^*E, J[1]))$$

induced by $E \to L_{X/S}$ is a fibre sequence. Applying Theorem 5.3.4, we get the following result.

Proposition 5.4.1. *The map $E \to L_{X/S}$ is an obstruction theory for the fibre product $X = Y \times_W Z$.*

So far, we have constructed the obstruction theory $E \to L_{X/S}$ abstractly in terms of cotangent complexes. It is often convenient to have a more explicit description.

Assme now that Y and W are smooth over S and that $i : Z \to W$ is a regular embedding. In this case, Theorem 5.2.1, (3), gives us explicit representatives the cotangent complexes of Y, Z and W. We can use these, or more precisely the more refined statements in Section B.5, to compute the complex E and the truncation $E \to \tau^{\geq -1} L_{X/S}$.

Proposition 5.4.2. *Under the hypotheses above, let $I \subseteq i^{-1}\mathcal{O}_W$ and $J \subseteq j^{-1}\mathcal{O}_Y$ be the ideal sheaves of the closed embeddings $i : Z \to W$ and $j : X \to Y$. Then there is a morphism of homotopy coherent squares*

$$\begin{array}{ccccccc}
g^*i^*L_{W/S} & & \longrightarrow & & g^*L_{Z/S} & & \\
\downarrow & \searrow & & & \downarrow & \searrow & \\
 & & j^*L_{Y/S} & \longrightarrow & & & L_{X/S} \\
 & & \downarrow & & & & \downarrow \\
[0 \to g^*i^*\Omega_{W/S}] & & \longrightarrow & & [g^*I/I^2 \to g^*i^*\Omega_{W/S}] & & \\
 & \searrow & & & & \searrow & \\
 & & [0 \to j^*\Omega_{Y/S}] & \longrightarrow & & & [J/J^2 \to j^*\Omega_{Y/S}]
\end{array} \tag{5.4.4}$$

such that

$$\begin{array}{rr}
g^*i^*L_{W/S} \to [0 \to g^*i^*\Omega_{W/S}], & j^*L_{Y/S} \to [0 \to j^*\Omega_{Y/S}], \\
g^*L_{Z/S} \to [g^*I/I^2 \to g^*\Omega_{Z/S}], & \tau^{\geq -1}L_{X/S} \to [J/J^2 \to j^*\Omega_{Y/S}]
\end{array} \tag{5.4.5}$$

are isomorphisms in $D(X)$.

Proof. Recall from Section B.3 that the homotopy coherent diagram

$$\begin{array}{ccc}
g^*i^*L_{W/S} & \longrightarrow & g^*L_{Z/S} \\
\downarrow & & \downarrow \\
j^*L_{Y/S} & \longrightarrow & L_{X/S}
\end{array} \tag{5.4.6}$$

is constructed as follows. Consider the diagram

$$B_\bullet = \begin{array}{ccc}
g^{-1}i^{-1}\mathcal{O}_W & \longrightarrow & g^{-1}\mathcal{O}_Z \\
\downarrow & & \downarrow \\
j^{-1}\mathcal{O}_Y & \longrightarrow & \mathcal{O}_X,
\end{array}$$

regarded as a sheaf of rings over the site $X_{ét} \times K^{op}$, where K is the square (B.3.1). Writing $\pi : X \to S$ for the structure map, the square (5.4.6) is the image of $L_{B_\bullet/\pi^{-1}\mathcal{O}_S}$ under the functor $D(B_\bullet) \to D(\mathcal{O}_X{}^\square)$ given by (B.3.3).

We have a surjection $\phi : C_\bullet \to B_\bullet$ of diagrams of $\pi^{-1}\mathcal{O}_S$-algebras, where

$$C_\bullet = \begin{array}{ccc} g^{-1}i^{-1}\mathcal{O}_W & \longrightarrow & g^{-1}i^{-1}\mathcal{O}_W \\ \downarrow & & \downarrow \\ j^{-1}\mathcal{O}_Y & \longrightarrow & j^{-1}\mathcal{O}_Y. \end{array}$$

The kernel of ϕ is the diagram

$$I_\bullet = \begin{array}{ccc} 0 & \longrightarrow & g^{-1}I \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & J, \end{array}$$

so by Theorem B.5.1, we have a morphism

$$L_{B_\bullet/\pi^{-1}\mathcal{O}_S} \to [I_\bullet/I_\bullet^2 \to \Omega_{C_\bullet/\pi^{-1}\mathcal{O}_S} \otimes B_\bullet], \tag{5.4.7}$$

where

$$[I_\bullet/I_\bullet^2 \to \Omega_{C_\bullet/\pi^{-1}\mathcal{O}_S} \otimes B_\bullet]$$

is the diagram below.

$$\begin{array}{ccc} [0 \to g^{-1}i^{-1}\Omega_{W/S}] & \longrightarrow & [g^{-1}I/I^2 \to g^{-1}i^*\Omega_{W/S}] \\ \downarrow & & \downarrow \\ [0 \to j^{-1}\Omega_{Y/S}] & \longrightarrow & [J/J^2 \to j^*\Omega_{Y/S}] \end{array}$$

The image of (5.4.7) under the functor $D(B_\bullet) \to D(\mathcal{O}_X{}^\square)$ is the desired morphism of diagrams (5.4.4). It follows immediately from the construction and Proposition B.5.2 that the maps (5.4.5) are isomorphisms in $D(X)$. $\square$

Corollary 5.4.3. *Suppose that W and Y are smooth over S and that $i : Z \to W$ is a regular embedding. Then there is a commutative diagram in $D(X)$,*

$$\begin{array}{ccc} E & \longrightarrow & [g^*I/I^2 \to j^*\Omega_{Y/S}] \\ \downarrow & & \downarrow \\ L_{X/S} & \longrightarrow & [J/J^2 \to j^*\Omega_{Y/S}], \end{array} \tag{5.4.8}$$

such that the morphisms

$$E \to [g^*I/I^2 \to j^*\Omega_{Y/S}], \qquad \tau^{\geq -1}L_{X/S} \to [J/J^2 \to j^*\Omega_{Y/S}] \tag{5.4.9}$$

are isomorphisms.

Proof. The morphism (5.4.4) of Proposition 5.4.2 induces a commutative diagram

$$\begin{array}{ccc} E & \longrightarrow & C \\ \downarrow & & \downarrow \\ L_{X/S} & \longrightarrow & [J/J^2 \to j^*\Omega_{Y/S}] \end{array} \tag{5.4.10}$$

where

$$C = \mathrm{Cone}([0 \to g^*i^*\Omega_{W/S}] \to [g^*I/I^2 \to j^*\Omega_{Y/S} \oplus g^*i^*\Omega_{W/S}]).$$

But there is an exact sequence of complexes

$$0 \to [0 \to g^*i^*\Omega_{W/S}] \to [g^*I/I^2 \to j^*\Omega_{Y/S} \oplus g^*i^*\Omega_{W/S}] \\ \to [g^*I/I^2 \to j^*\Omega_{W/S}] \to 0,$$

so

$$C \simeq [g^*I/I^2 \to j^*\Omega_{W/S}].$$

Combining this with (5.4.10), we get a diagram of the form (5.4.8) as required. Since the maps (5.4.5) are isomorphisms, it follows directly that the maps (5.4.9) are isomorphisms as well. □

Example 5.4.4 (Self-intersection)**.** Let W be a smooth variety over k and let X be a smooth subvariety of W of codimension d. Then we have

$$X = X \times_W X$$

as schemes so Proposition 5.4.1 gives an obstruction theory $E \to L_X$ on X. Since X is smooth, it follows from Proposition A.5.10 that the inclusion $X \to W$ is a regular embedding of codimension d, so the obstruction theory $E \to L_X$ can be computed using Corollary 5.4.3. Explicitly, we have

$$E = [I/I^2 \xrightarrow{0} \Omega_X] \to [0 \to \Omega_X] = L_X,$$

where I is the ideal sheaf of X in W.

Example 5.4.5. As a special case of Example 5.4.4, consider the self-intersection

$$X = X \times_{\mathbb{P}^2} X,$$

where

$$X = \mathbb{P}^1 = \mathrm{Proj}(k[x,z]) = \mathrm{Proj}\left(\frac{k[x,y,z]}{(y)}\right) \subseteq \mathbb{P}^2 = \mathrm{Proj}(k[x,y,z]).$$

(This is precisely the intersection $X_0 = Y \cap Z_0$ of Section 1.1 and Example 3.2.2.) In this case, I/I^2 is the quasi-coherent sheaf associated to the graded $k[x,z]$-module

$$\frac{(x_2)}{(x_2)^2} \cong k[x,z] \cdot v,$$

where the generator v has degree 1, and Ω_X is the quasi-coherent sheaf associated to the graded $k[x,z]$-module

$$k[x,z] \cdot (zdx - xdz) \cong k[x,z] \cdot w,$$

where the generator w has degree 2. So the obstruction theory $E \to L_X$ is

$$[\mathcal{O}(-1) \xrightarrow{0} \mathcal{O}(-2)] \to [0 \to \mathcal{O}(-2)].$$

Example 5.4.6 (Fibre of a map between smooth schemes)**.** Let Y and W be smooth schemes over k and let $f : Y \to W$ be a morphism. If $w : \mathrm{Spec}(k) \to W$ is any k-point in W, then the fibre of f over w is the k-scheme

$$X = Y \times_W \mathrm{Spec}(k) = f^{-1}(w).$$

Since W is smooth, every map from a point to W is a regular embedding. So the obstruction theory $E \to L_X$ of Proposition 5.4.1 is given by

$$E = [g^* I/I^2 \to j^*\Omega_Y] \to [J/J^2 \to j^*\Omega_Y] = \tau^{\geq -1} L_X,$$

where $g : X \to \mathrm{Spec}(k)$ and $j : X \to Y$ are the projection maps and I is the ideal sheaf of the closed embedding w. A simple computation shows that the morphism $I/I^2 \to w^*\Omega_W$ is an isomorphism. Hence, the obstruction theory E is

$$E = [j^* f^* \Omega_W \to j^*\Omega_Y] \to [J/J^2 \to j^*\Omega_Y]$$

where the differential of E is obtained from the usual pullback map $f^*\Omega_W \to \Omega_Y$ by applying j^*.

Example 5.4.7. As a concrete case of Example 5.4.6, consider the fibre

$$X = \mathbb{P}^1 = \mathrm{Proj}(k[x, y])$$

of the map

$$f : Y = \mathrm{Proj}\left(\frac{k[s, t, x, y]}{(tx - sy)}\right) \to \mathrm{Spec}(k[s, t]) = W$$

of Example 3.2.3 over the point $0 \in W = \mathbb{A}^2$. The sheaf $j^*f^*\Omega_W$ is the quasi-coherent sheaf on X associated to the graded module

$$j^*f^*\Omega_W = k[x, y]ds \oplus k[x, y]d, t$$

where ds and dt have degree 0. The sheaf $j^*\Omega_Y$ is the quasi-coherent sheaf associated to

$$j^*\Omega_Y = k[x, y]w \oplus k[x, y]v,$$

where w has degree -1 and v has degree 2. So the obstruction theory E is the complex

$$\left[\mathcal{O}^{\oplus 2} \to \mathcal{O}(1) \oplus \mathcal{O}(-2)\right].$$

given in terms of graded modules by

$$\begin{aligned} k[x, y]ds \oplus k[x, y]dt &\to k[x, y]w \oplus k[x, y]v \\ ds &\mapsto xw \\ dt &\mapsto yw. \end{aligned}$$

Chapter 6

Virtual fundamental classes

The main point of this chapter is to define and give examples of virtual fundamental classes associated to perfect obstruction theories.

Let X be a scheme over a base scheme S, and let $E \to L_{X/S}$ be a perfect obstruction theory for X. (See Definition 6.3.1.) By Theorem 5.3.4, for any S-scheme T and any morphism $f : T \to X$, there is a fibre sequence

$$\underline{\mathbf{Ext}}_X(T, \mathcal{O}_T) \to \underline{\mathbf{Ext}}_S(T, \mathcal{O}_T) \to \mathrm{ch}((Lf^*E)^\vee[1]),$$

where

$$(Lf^*E)^\vee = R\underline{\mathrm{Hom}}^\bullet(Lf^*E, \mathcal{O}_T).$$

In the special case where $T = X$ and $f : X \to X$ is the identity, this gives us a fibre sequence

$$\underline{\mathbf{Ext}}_X(X, \mathcal{O}_X) \to \underline{\mathbf{Ext}}_S(X, \mathcal{O}_X) \to \mathrm{ch}(E^\vee[1]) \qquad (6.0.1)$$

of stacks over $X_{ét}$.

If $Y \to X$ is a scheme over X, then there is a sheaf of sections

$$(X_{Zar})^{op} \to \mathbf{Set}$$
$$U \mapsto \mathrm{Hom}_X(U, Y)$$

of Y over $X_{ét}$. Similarly, if $Y \to X$ is an algebraic stack over X (see Section A.7), then there is an associated stack of sections over $X_{ét}$. In Section 6.2, we show that for S locally Noetherian and X locally of finite type over S, the fibre sequence (6.0.1) can be obtained from a fibre sequence,

$$X \to N_{X/S} \to \underline{\mathbf{Ob}}, \qquad (6.0.2)$$

of algebraic stacks over X by taking stacks of sections. Over S, this means that we have a 2-cartesian square

$$\begin{array}{ccc} X & \longrightarrow & X \\ \Big\downarrow & & \Big\downarrow \\ N_{X/S} & \longrightarrow & \underline{\mathbf{Ob}} \end{array}$$

of algebraic S-stacks. So the obstruction theory E tells us to regard X as the intersection of the stack $N_{X/S}$ with a given section of the obstruction stack $\underline{\mathbf{Ob}}$. Since the virtual fundamental class $[X]^{vir}$ is supposed to capture the behaviour of X when we deform to a transverse setup, it is not unreasonable to define $[X]^{vir}$ by computing this intersection in by first "deforming" to a transverse intersection. (We make precise the meaning of "deforming" in Section 6.3—see Remark 6.3.4 in particular.)

This construction gives a well-behaved virtual fundamental class so long as X itself is smooth over S. In general, to obtain good functoriality properties for the virtual classes, it is necessary to replace the stack $N_{X/S}$ with a refinement $C_{X/S}$, called the *intrinsic normal cone* of X over S.

In Section 6.1, we descibe the methods behind the construction of the sequence (6.0.2). In Section 6.2 we study the geometry of the stack $N_{X/S}$ and define its refinement $C_{X/S}$. In Section 6.3, we give the construction for virtual fundamental classes and describe some simple techniques for calculating them in special cases. In Section 6.4, we compute examples of virtual fundamental classes using the obstruction theories constructed in Section 5.4.

6.1 Schemes, sheaves and algebraic stacks

Let X be a scheme. An X-scheme Y determines a functor

$$\begin{aligned} j(Y) : \left(\mathbf{Sch}_{/X}\right)^{op} &\to \mathbf{Set} \\ Z &\mapsto \mathrm{Hom}_X(Z, Y). \end{aligned}$$

The functor $j(Y)$ is called the *functor of points* of Y over X. If we endow the category $\mathbf{Sch}_{/X}$ with the étale topology, then $j(Y)$ is a sheaf. Thus, we have a functor

$$j : \mathbf{Sch}_{/X} \to \mathbf{Sh}(\mathbf{Sch}_{/X}),$$

which is fully faithful by the Yoneda Lemma. Hence, we can regard schemes over X as certain sheaves over the big étale site $\mathbf{Sch}_{/X}$.

This point of view has many advantages. For example, it is often easier and more natural to specify a scheme Y by its functor of points than by writing down a locally ringed space $(|Y|, \mathcal{O}_Y)$. We did this, for example, to define mapping spaces in Section 3.3.

As a further example, consider the global Spec functor of Section A.3. Let X be a scheme and let A be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras, which we regard as a sheaf on $X_{ét}$ via Remark 3.1.6. The scheme $\mathrm{Spec}_X(A)$ is the X-scheme Y with functor of points

$$\begin{aligned} j(Y) : \left(\mathbf{Sch}_{/X}\right)^{op} &\to \mathbf{Set} \\ (f : Z \to X) &\mapsto \mathrm{Hom}_{\mathcal{O}_Z\textbf{-alg}}(f^*A, \mathcal{O}_Z), \end{aligned}$$

where

$$f^*A = f^{-1}A \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_Z,$$

and where we have dropped the notation $\mathcal{O}_{X_{ét}}$ in favour of $\mathcal{O}_X$ for the sake of readability.

We can also describe this construction using morphisms of ringed sites. Endowing $\mathbf{Sch}_{/X}$ with the étale topology, recall from Example 2.3.7 that we have a morphism of ringed sites

$$e : (\mathbf{Sch}_{/X}, \mathcal{O}_{/X}) \to (X_{ét}, \mathcal{O}_{X_{ét}}).$$

If $Y = \mathrm{Spec}_X(A)$ for some quasi-coherent sheaf of $\mathcal{O}_X$-algebras A, then $j(Y)$ is the sheaf Hom

$$j(Y) = \underline{\mathrm{Hom}}_{\mathcal{O}_{/X}\textbf{-alg}}(e^*A, \mathcal{O}_{/X}),$$

where

$$\underline{\mathrm{Hom}}_{\mathcal{O}_{/X}\textbf{-alg}}(e^*A, \mathcal{O}_{/X})(Z) = \mathrm{Hom}_{\mathcal{O}_{/X}|_Z\textbf{-alg}}(e^*A|_Z, \mathcal{O}_{/X}|_Z).$$

The morphism $e : (\mathbf{Sch}_{/X}, \mathcal{O}_{/X}) \to (X_{ét}, \mathcal{O}_X)$ also captures the relationship between X-schemes and their sheaves of sections over $X_{ét}$. More precisely, if Y is an X-scheme, then $e_*j(Y)$ is the sheaf

$$\begin{aligned} e_*j(Y) : (X_{Zar})^{op} &\to \mathbf{Set} \\ U &\mapsto j(Y)(e^{-1}U) = \mathrm{Hom}_X(U, Y). \end{aligned}$$

So $e_*j(Y)$ is the sheaf of sections of Y.

Example 6.1.1. Let E be a vector bundle on X, viewed as a locally free sheaf of finite rank. (See Section A.2.) The *total space* of E is the X-scheme

$$E = \mathrm{Spec}_X(\mathrm{Sym}_{\mathcal{O}_X}(E^\vee))$$

where $E^\vee = \underline{\mathrm{Hom}}(E, \mathcal{O}_X)$. The functor of points of E is

$$\begin{aligned} j(E) &= \underline{\mathrm{Hom}}_{\mathcal{O}_{/X}\text{-}\mathbf{alg}}(e^*\mathrm{Sym}_{\mathcal{O}_X}(E^\vee), \mathcal{O}_{/X}) \\ &= \underline{\mathrm{Hom}}_{\mathcal{O}_{/X}\text{-}\mathbf{mod}}(e^*(E^\vee), \mathcal{O}_{/X}) \\ &= e^*E. \end{aligned}$$

More explicitly,

$$j(E)(U) = (f^*E)(U)$$

for $f : U \to X$ an object of $\mathbf{Sch}_{/X}$. The sheaf of sections of the X-scheme E is

$$e_*e^*E \cong E.$$

Example 6.1.2. Let M be a quasi-coherent sheaf on X. The *abelian cone of M* is the X-scheme

$$\mathrm{Spec}_X(\mathrm{Sym}_{\mathcal{O}_X}(M)).$$

The associated sheaf on $\mathbf{Sch}_{/X}$ is

$$\underline{\mathrm{Hom}}(e^*M, \mathcal{O}_{/X}) = (e^*M)^\vee.$$

Suppose that S is a scheme and X is an S-scheme. In Section 6.2, we are interested in the *intrinsic normal sheaf* $N_{X/S}$ of X over S. This is an algebraic stack over X such that the "sheaf of sections"

$$e_*N_{X/S} = \mathrm{ch}((L_{X/S})^\vee[1]) \simeq \underline{\mathbf{Ext}}_S(X, \mathcal{O}_X)$$

is the stack on $X_{\acute{e}t}$ of square-zero extensions of X by $\mathcal{O}_X$. We construct $N_{X/S}$ using the following generalisation of the abelian cone construction of Example 6.1.2.

Proposition 6.1.3 (cf. [5], Proposition 2.4)**.** *Let X be a scheme and let $M \in \mathrm{Ob}\, D^{\leq 0}_{qc}(\mathcal{O}_X)$. If $H^0(M)$ and $H^{-1}(M)$ are coherent sheaves on X, then*

$$\mathrm{ch}((Le^*M)^\vee[1])$$

is an algebraic X-stack, satisfying

$$e_*\mathrm{ch}((Le^*M)^\vee[1]) \simeq \mathrm{ch}(M^\vee[1]).$$

Proof. The claim that $\mathrm{ch}((Le^*M)^\vee[1])$ is an algebraic stack is [5], Proposition 2.4. We do not strictly need the claim about e_*, so we omit the proof. □

6.2 The intrinsic normal cone

In this section, we recall from [5] the definitions and main properties of the intrinsic normal sheaf and intrinsic normal cone of a scheme X. Throughout this section, we restrict to the situation where X is locally of finite type over a locally Noetherian scheme S. (See Section A.5.)

Definition 6.2.1 (cf. [5], Definition 3.6)**.** Let S be a locally Noetherian scheme, and let X be an S-scheme locally of finite type. The *intrinsic normal sheaf* of X over S is the stack

$$N_{X/S} = \mathrm{ch}((Le^*L_{X/S})^\vee[1]).$$

Remark 6.2.2. By [19], Corollaire II.2.3.7, $L_{X/S}$ has coherent cohomology under our hypotheses on X and S. So, by Proposition 6.1.3, $N_{X/S}$ is an algebraic stack over X, with

$$e_*N_{X/S} \simeq \mathrm{ch}((L_{X/S})^\vee[1]) \simeq \underline{\mathbf{Ext}}_S(X, \mathcal{O}_X).$$

Suppose for the moment that there exists a closed embedding $f : X \to Y$ of X into a smooth S-scheme Y. By Theorem 5.2.1, (3), we have

$$\tau^{\geq -1}L_{X/S} \simeq [I/I^2 \to f^*\Omega_{Y/S}],$$

where $I \subseteq f^{-1}\mathcal{O}_Y$ is the ideal sheaf of the embedding f. Since Y is smooth over S, $f^*\Omega_{Y/S}$ is locally free, and hence flat. So

$$\begin{aligned} N_{X/S} &= \mathrm{ch}(\tau^{\leq 0}R\underline{\mathrm{Hom}}^\bullet(Le^*L_{X/S}, \mathcal{O}_{/X}[1])) \\ &= \mathrm{ch}(\tau^{\leq 0}R\underline{\mathrm{Hom}}^\bullet([e^*I/I^2 \to e^*f^*\Omega_{Y/S}], \mathcal{O}_{/X}[1])). \end{aligned}$$

Notice that we have a distinguished triangle

$$\begin{aligned} &R\underline{\mathrm{Hom}}^\bullet([e^*I/I^2 \to e^*f^*\Omega_{Y/S}], \mathcal{O}_{/X}[1]) \to R\underline{\mathrm{Hom}}^\bullet(e^*f^*\Omega_{Y/S}, \mathcal{O}_{/X}[1]) \\ &\quad \to R\underline{\mathrm{Hom}}^\bullet(e^*I/I^2, \mathcal{O}_{/X}[1]) \to R\underline{\mathrm{Hom}}^\bullet([e^*I/I^2 \to e^*f^*\Omega_{Y/S}], \mathcal{O}_{/X}[1])[1]. \end{aligned}$$

Using the associated long exact sequence and the fact that $\underline{\mathrm{Hom}}^\bullet(e^*f^*\Omega_{Y/S}, -)$ takes quasi-isomorphisms to quasi-isomorphisms, we find that

$$\begin{aligned} \tau^{\leq 0}R\underline{\mathrm{Hom}}^\bullet([e^*I/I^2 \to e^*f^*\Omega_{Y/S}], &\mathcal{O}_{/X}[1]) \\ &\simeq \underline{\mathrm{Hom}}^\bullet([e^*I/I^2 \to e^*f^*\Omega_{Y/S}], \mathcal{O}_{/X}[1]) \\ &= \left[\left(e^*f^*\Omega_{Y/S}\right)^\vee \to \left(e^*I/I^2\right)^\vee\right]. \end{aligned}$$

Recall that the *normal sheaf* of X in Y is the X-scheme

$$N_{X/Y} = \operatorname{Spec}_X \left(\operatorname{Sym}_{\mathcal{O}_X}(I/I^2)\right).$$

The *tangent bundle* of Y over S is the Y-scheme

$$T_{Y/S} = \operatorname{Spec}_Y \left(\operatorname{Sym}_{\mathcal{O}_Y}(\Omega_{Y/S})\right),$$

which pulls back along f to the X-scheme

$$f^*T_{Y/S} = \operatorname{Spec}_X \left(\operatorname{Sym}_{\mathcal{O}_X}(f^*\Omega_{Y/S})\right).$$

By Example 6.1.2, the sheaves on $\mathbf{Sch}_{/X}$ represented by $N_{X/S}$ and $f^*T_{Y/S}$ are $(e^*I/I^2)^\vee$ and $\left(e^*f^*\Omega_{Y/S}\right)^\vee$. In particular, $f^*T_{X/Y}$ is a group scheme over Y acting on $N_{X/S}$ via the map $f^*T_{Y/S} \to N_{X/S}$, and we have

$$\begin{aligned} N_{X/S} &= \operatorname{ch}\left(\left[\left(e^*f^*\Omega_{Y/S}\right)^\vee \to \left(e^*I/I^2\right)^\vee\right]\right) \\ &= \left[N_{X/Y}/f^*T_{Y/S}\right]. \end{aligned}$$

Recall from Definition A.5.6 that $N_{X/Y}$ has a canonical closed subscheme

$$C_{X/Y} = \operatorname{Spec}_X \left(\bigoplus_{n\geq 0} I^n/I^{n+1}\right),$$

called the *normal cone* of X in Y. By [5], Lemma 3.2, $C_{X/Y}$ is invariant under the action of $f^*T_{X/Y}$ on $N_{X/Y}$, so we get a closed substack

$$C_{X/S} = \left[C_{X/Y}/f^*T_{Y/S}\right] \subseteq N_{X/S}.$$

In general, X may not admit a closed embedding into a smooth S-scheme Y. However, since we are assuming that X is locally of finite type over S, it is immediate from Definition A.5.2 that there exists a (Zariski) open cover $\{U_i\}_{i\in I}$ of X and closed embeddings $f_i : U_i \to Y_i$ for some smooth S-schemes Y_i. (In fact, we can take each Y_i to be an affine space over an open subset of S.) So we can apply the above analysis locally on X to conclude that $N_{X/S}$ is an algebraic X-stack satisfying

$$N_{X/S}|_{U_i} = \left[N_{U_i/Y_i}/f^*T_{Y_i/S}\right]$$

with closed substacks $C_{U_i/S} \subseteq N_{X/S}|_{U_i}$ for each $i \in I$. The following theorem ensures that the stacks $C_{U_i/S}$ glue to give a globally defined substack $C_{X/S} \subseteq N_{X/S}$.

Theorem 6.2.3 (cf. [5], Corollary 3.9). *Let X be a scheme locally of finite type over a locally Noetherian scheme S. Then there exists a unique closed substack $C_{X/S}$ of $N_{X/S}$ such that for any open set $U \subseteq X$ and any closed embedding $f : U \to Y$ with Y a smooth S-scheme, we have*

$$C_{X/S}|_U = \left[N_{U/Y}/f^*T_{U/Y}\right]$$

as substacks of $N_{X/S}|_U = N_{U/S}$.

Definition 6.2.4. The X-stack $C_{X/S}$ of Theorem 6.2.3 is called the *intrinsic normal cone* of X over S.

The following result follows immediately from Definition 6.2.4 and well-known properties of normal cones.

Theorem 6.2.5 (cf. [5], Theorem 3.11). *Let S be a locally Noetherian scheme of pure dimension n. Then $C_{X/S}$ is n-dimensional as an algebraic stack.*

Example 6.2.6. Suppose that $X \to S$ is smooth. Then $L_{X/S} = \Omega_{X/S}$, so

$$N_{X/S} = \mathrm{ch}([(e^*\Omega_{X/S})^\vee \to 0]) = [0/T_{X/S}].$$

If $S = \mathrm{Spec}(k)$ for k a field, then every k-point of $N_{X/S}$ is a k-point x of X, with automorphism group equal to the tangent space T_xX to X at x. In this case, $N_{X/S}$ has dimension 0 since there is a $\dim X$-dimensional space of points, each with a $\dim X$-dimensional automorphism group.

6.3 Virtual fundamental classes

In this section, we describe how a virtual fundamental class can be constructed from a perfect obstruction theory. For the purposes of this section, we assume that S is a smooth scheme of finite type of dimenison n over a ground field k and that X is a connected scheme, separated and of finite type over S. We first recall the definition of a perfect obstruction theory.

Definition 6.3.1 (cf. [5], Definition 5.1). Let X be an S-scheme. We say that an obstruction theory $E \to L_{X/S}$ is *perfect* if there exists an étale cover $\{U_i\}_{i\in I}$ of X and complexes $[E_i^{-1} \to E_i^0]$ of vector bundles such that $E|_{U_i} \simeq [E_i^{-1} \to E_i^0]$ for each $i \in I$. If E is perfect, the *rank* of E is the Euler characteristic

$$\mathrm{rank}\, E = \mathrm{rank}\, E_i^0 - \mathrm{rank}\, E_i^{-1}.$$

Remark 6.3.2. If $E \to L_{X/S}$ is a perfect obstruction theory, by Remark 6.3.6 it is convenient to have vector bundles E^{-1} and E^0 and a map $E^{-1} \to E^0$ such that $E \simeq [E^{-1} \to E^0]$ globally on X. In fact, Lemma 2.5 of [4] ensures that such global resolutions exist for all perfect obstruction theories on X where $S = \mathrm{Spec}(\mathbb{C})$ and X is a quasi-projective over S.

Let $\phi : E \to L_{X/S}$ be a perfect obstruction theory on X, and write

$$\underline{\mathbf{Ob}} = \mathrm{ch}((Le^*E)^\vee).$$

By Proposition 6.1.3, $\underline{\mathbf{Ob}}$ is an algebraic X-stack. Since ϕ is an obstruction theory, by [5], Proposition 2.6, the induced map $N_{X/S} \to \underline{\mathbf{Ob}}$ is a closed embedding. Hence, we have a closed embedding

$$C_{X/S} \hookrightarrow N_{X/S} \hookrightarrow \underline{\mathbf{Ob}}.$$

Since S is smooth of dimension n, by Theorem 6.2.3, $C_{X/S}$ is an Artin stack of pure dimension n, and thus defines a Chow homology class

$$[C_{X/S}] \in A_n(\underline{\mathbf{Ob}}).$$

At this point, we want to define the virtual fundamental class $[X]^{vir}$ by intersecting the class $[C_{X/S}]$ with the zero section of $\underline{\mathbf{Ob}}$. One way to make this precise is the following.

Since the obstruction theory E is perfect, we can cover X with étale neighbourhoods U such that $E|_U \cong [E^{-1} \to E^0]$ for some vector bundles E^{-1} and E^0 on U. So

$$\underline{\mathbf{Ob}}|_U \simeq \mathrm{ch}((Le^*[E^{-1} \to E^0])^\vee) \simeq [(E^{-1})^\vee/(E_0)^\vee],$$

where $(E^{-1})^\vee$ and $(E^0)^\vee$ are the dual vector bundles of E^{-1} and E^0 respectively. Hence, the stack $\underline{\mathbf{Ob}}$ is a vector bundle stack in the sense of [5], Definition 1.9, of rank

$$\mathrm{rank}\,\underline{\mathbf{Ob}} = \mathrm{rank}(E^{-1})^\vee - \mathrm{rank}(E^0)^\vee = \mathrm{rank}\,E^{-1} - \mathrm{rank}\,E^0 = -\,\mathrm{rank}\,E,$$

where $\mathrm{rank}\,E$ is defined as in Definition 5.3.1. In particular, the morphism $p : \underline{\mathbf{Ob}} \to X$ is flat of relative dimension $-\,\mathrm{rank}\,E$, so for all $i \in \mathbb{Z}$ we have a pullback morphism

$$p^* : A_i(X) \to A_{i-\mathrm{rank}\,E}(\underline{\mathbf{Ob}}).$$

Since X is a scheme, by [21], Proposition 3.5.5 and Proposition 4.3.2, p^* is an isomorphism for all $i \in \mathbb{Z}$.

Definition 6.3.3. Let $\phi : E \to L_{X/S}$ be a perfect obstruction theory for X over S. The *virtual fundamental class* of X with respect to E is the Chow homology class

$$[X]^{vir} = (p^*)^{-1}([C_{X/S}]) \in A_{n+\operatorname{rank} E}(X).$$

The *virtual dimension* of X is the dimension $n + \operatorname{rank} E$ of the class $[X]^{vir}$.

Remark 6.3.4. According to Definition 6.3.3, to compute $[X]^{vir}$ we first replace the cycle $[C_{X/S}]$ with a rationally equivalent cycle of the form

$$\sum_i m_i p^*[Y_i] = \sum_i m_i [Y_i \times_X \underline{\mathbf{Ob}}],$$

for some $m_i \in \mathbb{Z}$ and some integral closed subschemes Y_i of X. (See Section A.6.) Each substack $Y_i \times_X \underline{\mathbf{Ob}}$ intersects the zero section of $\underline{\mathbf{Ob}}$ transversely in the scheme Y_i, and

$$[X]^{vir} = \sum_i m_i [Y_i].$$

Thus, the virtual fundamental class $[X]^{vir}$ is the result of deforming the intersection of $C_{X/S}$ with the zero section of $\underline{\mathbf{Ob}}$ to a transverse intersection via rational equivalence.

Remark 6.3.5. One reason for using $C_{X/S}$ instead of $N_{X/S}$ in Definition 6.3.3 is the following. In [5], Proposition 5.10, it is shown that given a sufficiently nice morphism of schemes (or Deligne-Mumford stacks) $f : X \to Y$, together with compatible perfect obstruction theories on X and Y, the virtual fundamental classes are related by

$$[X]^{vir} = f^![Y]^{vir},$$

where $f^! : A_*(Y) \to A_*(X)$ is a homomorphism obtained by combining Gysin homomorphisms and flat pullbacks. This result, which is important in applications to Gromov-Witten theory, for example, relies on the use of $C_{X/S}$ instead of $N_{X/S}$.

Remark 6.3.6. A perfect obstruction theory E is required to be *locally* isomorphic in $D_{qc}(X)$ to a two term complex of vector bundles. In the special case that E is globally isomorphic to such a complex,

$$E \simeq [E^{-1} \to E^0],$$

we can avoid working with Chow groups for Artin stacks as follows. There is a smooth surjection

$$r : (E^{-1})^\vee \to \underline{\mathbf{Ob}} \simeq [(E^{-1})^\vee/(E^0)^\vee]$$

of relative dimension rank E^0, from which we can form a closed subscheme

$$C = (E^{-1})^\vee \times_{\underline{\mathbf{Ob}}} C_{X/S}$$

of $(E^{-1})^\vee$. We then have a commutative diagram,

$$\begin{array}{ccc} (E^{-1})^\vee & \xrightarrow{\ r\ } & \underline{\mathbf{Ob}} \\ & {}_{q}\searrow \quad \swarrow_{p} & \\ & X & \end{array}$$

so the flat pullbacks satisfy $r^* p^* = q^*$. Hence,

$$[X]^{vir} = (p^*)^{-1}[C_{X/S}] = (q^*)^{-1} r^*[C_{X/S}] = (q^*)^{-1}[C],$$

where the last equality follows by definition of flat pullback. Hence, we can compute $[X]^{vir}$ working entirely with the vector bundle $(E^{-1})^\vee$ and the closed subscheme C.

Remark 6.3.7 (cf. [5], Proposition 5.6)**.** Suppose now that X is smooth over S and that E is a perfect obstruction theory for X over S. In this case

$$H^0(E) = H^0(L_{X/S}) = \Omega_{X/S}$$

is a vector bundle on X, so the coherent sheaf $H^{-1}(E)$ is also a vector bundle on X. In what follows, we denote its dual $H^{-1}(E)^\vee = H^0(E^\vee[1])$ by Ob. We claim that in this case the virtual fundamental class is

$$[X]^{vir} = e(\mathrm{Ob}) \cap [X],$$

where $e(\mathrm{Ob}) = c_{\mathrm{rank}\,\mathrm{Ob}}(\mathrm{Ob})$ is the top Chern class of the vector bundle Ob, as defined in [10], Section 3.2, and $[X]$ is the usual fundamental class of X. By [10], Proposition 3.3, it suffices to show that

$$[X]^{vir} = (q^*)^{-1}[0],$$

where $q : \mathrm{Ob} \to X$ is the projection and $[0] \in A_{\dim X}(\mathrm{Ob})$ is the class represented by the zero section. To see this, we first show that the quotient map $\underline{\mathbf{Ob}} \to \mathrm{Ob}$ is flat of relative dimension $\dim S - \dim X$, and that the diagram

$$\begin{array}{ccc} C_{X/S} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \underline{\mathbf{Ob}} & \longrightarrow & \mathrm{Ob} \end{array} \tag{6.3.1}$$

is 2-cartesian, where X includes into Ob as the zero section. Notice that it suffices to check both claims étale locally on X. Since E is perfect, there exists an open cover $\{U_i\}_{i\in I}$ of X and two term complexes of vector bundles $[E_i^{-1} \xrightarrow{d_i} E_i^0]$ on U_i such that

$$E|_{U_i} \simeq [E_i^{-1} \xrightarrow{d_i} E_i^0].$$

Since $\Omega_{X/S}$ is a vector bundle, by possibly refining this cover, we can assume that U_i is affine and $\Omega_{X/S}|_{U_i}$ is a free $\mathcal{O}_{U_i}$-module for each i. Fixing $i \in I$, there exists a section $\Omega_{X/S}|_{U_i} \to E_i^0$ of the map $E_i^0 \to H^0(E_i^0) = \Omega_{X/S}|_{U_i}$. This defines a quasi-isomorphism

$$[\mathrm{Ob}^\vee|_{U_i} \xrightarrow{0} \Omega_{X/S}|_{U_i}] \to [E_i^{-1} \to E_i^0]$$

and hence an equivalence of stacks

$$\underline{\mathbf{Ob}}|_{U_i} \to [\mathrm{Ob}/T_{X/S}]|_{U_i}.$$

So $r : \underline{\mathbf{Ob}}|_{U_i} \to \mathrm{Ob}|_{U_i}$ is flat of relative dimension $-\operatorname{rank} T_{X/S} = \dim S - \dim X$. Moreover, we have a 2-commutative diagram

$$\begin{array}{ccccc} C_{X/S}|_{U_i} & \longrightarrow & [0/T_{X/S}]|_{U_i} & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \underline{\mathbf{Ob}}|_{U_i} & \longrightarrow & [\mathrm{Ob}/T_{X/S}]|_{U_i} & \longrightarrow & \mathrm{Ob}|_{U_i}, \end{array}$$

where the morphism $C_{X/S}|_{U_i} \to [0/T_{X/S}]|_{U_i}$ is an equivalence since X is smooth. Moreover, the square on the right is 2-cartesian, so (6.3.1) is 2-cartesian. So we have

$$r^*[0] = [C_{X/S}] \in A_n(\underline{\mathbf{Ob}}).$$

Hence, the commutative diagram

$$\begin{array}{ccc} \underline{\mathbf{Ob}} & \xrightarrow{r} & \mathrm{Ob} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & X & \end{array}$$

gives

$$[X]^{vir} = (p^*)^{-1}[C_{X/S}] = (q^*)^{-1}(r^*)^{-1}[C_{X/S}] = (q^*)^{-1}[0] = e(\mathrm{Ob}) \cap [X]$$

as claimed. Notice that in this case, the virtual dimension of X is

$$\dim[X]^{vir} = \dim X - \operatorname{rank} \mathrm{Ob}.$$

6.4 Examples

In this section, we compute some examples of virtual fundamental classes.

Example 6.4.1. Let X any scheme over S. Recall that the *trivial obstruction theory* is the identity map $L_{X/S} \to L_{X/S}$. If X is smooth over S then this obstruction theory is perfect, and the associated virtual fundamental class is the usual fundamental class of X.

Example 6.4.2. Consider the self-intersection of a smooth subvariety X of a smooth variety W over k, with the obstruction theory,

$$E = [I/I^2 \xrightarrow{0} \Omega_X] \to [0 \to \Omega_X] = L_X,$$

of Example 5.4.4. Since the embedding of a smooth subvariety into a smooth variety is always regular, I/I^2 is a vector bundle, so E is a perfect obstruction theory. Since X is smooth, we can use Remark 6.3.7 to compute the virtual fundamental class $[X]^{vir}$. The obstruction bundle,

$$\mathrm{Ob} = H^{-1}(E)^\vee = \left(I/I^2\right)^\vee = N_{X/W},$$

is the normal bundle of X in W. So

$$[X]^{vir} = e\left(N_{X/W}\right) \cap [X]$$

is the Euler class of the normal bundle of X in W. The virtual dimension is $2\dim X - \dim W$.

Example 6.4.3. As a special case, consider the self-intersection

$$X = X \times_{\mathbb{P}^2} X,$$

where

$$X = \mathbb{P}^1 = \mathrm{Proj}(k[x,z]) = \mathrm{Proj}\left(\frac{k[x,y,z]}{(y)}\right) \subseteq \mathbb{P}^2 = \mathrm{Proj}(k[x,y,z])$$

as in Example 5.4.5. The normal bundle of X in $\mathbb{P}^2$ is

$$N_{X/W} = \mathcal{O}(1),$$

so the virtual fundamental class is the class,

$$[X]^{vir} = e(\mathcal{O}(1)) \cap \mathbb{P}^1 = [*] \in A_0(X),$$

represented by a point in $X = \mathbb{P}^1$.

Example 6.4.4. Let $f : Y \to W$ be a morphism of smooth k-schemes, and let $X = f^{-1}(w)$ be the fibre over a k-point of W. As shown in Example 5.4.6, we have an obstruction theory

$$E = [j^* f^* \Omega_W \to j^* \Omega_Y] \to [J/J^2 \to j^* \Omega_Y] = L_X.$$

Since Y and W are smooth, Ω_Y and Ω_W are vector bundles, so E is perfect. By Remark 6.3.6, the virtual fundamental class of X is the class

$$[X]^{vir} = (q^*)^{-1}([C_{X/Y}]) \in A_*(X),$$

where $q : j^* f^* T_W \to X$ is the projection, and

$$C_{X/Y} = \mathrm{Spec}_X \left(\bigoplus_{n \geq 0} J^n / J^{n+1} \right)$$

is the normal cone of X in Y.

Example 6.4.5. As a special case, consider the fibre

$$X = \mathbb{P}^1 = \mathrm{Proj}(k[x, y])$$

of the family

$$\mathrm{Proj} \left(\frac{k[s, t, x, y]}{(tx - sy)} \right) \to \mathrm{Spec}(k[s, t])$$

of Examples 5.4.6 and 5.4.7. In this case, the total space of $j^* f^* T_W$ is

$$j^* f^* T_W = \mathrm{Spec}_X(k[x, y, ds, dt]),$$

where $k[x, y, ds, dt]$ denotes the quasi-coherent sheaf of $\mathcal{O}_X$-algebras associated to the graded $k[x, y]$-algebra $k[x, y, ds, dt]$, where ds and dt have degree 0. The normal cone $C_{X/Y}$ is the subscheme

$$C_{X/Y} = \mathrm{Spec}_X \left(\frac{k[x, y, ds, dt]}{(xdt - yds)} \right) \subseteq j^* f^* T_W.$$

We have a globally defined rational function

$$r = \frac{xdt - yds}{x}$$

on the total space $j^* f^* T_W$, which gives a rational equivalence

$$[C_{X/Y}] = \left[\mathrm{Spec}_X \left(\frac{k[x, y, ds, dt]}{(x)} \right) \right] = q^*[*]$$

between $C_{X/Y}$ and the pullback of a point in $X = \mathbb{P}^1$. Hence,

$$[X]^{vir} = (q^*)^{-1}([C_{X/Y}]) = [*] \in A_0(\mathbb{P}^1).$$

Appendix A

Algebraic geometry

In this appendix, we review some of the basic objects and constructions in algebraic geometry. The main objects of study are schemes and algebraic structures defined on them such as quasi-coherent sheaves. We also touch on intersection theory and algebraic stacks.

A.1 Schemes

Let k be an algebraically closed field, and let $f_1, f_2, \ldots, f_m \in k[x_1, \ldots, x_n]$ be polynomials over k. The *affine variety* defined by $f_1, \ldots, f_m$ is

$$X = \{x \in k^n \mid f_i(x) = 0, i = 1, 2, \ldots, m\}.$$

To the affine variety X, we can associate a ring $k[X]$, called the *coordinate ring* of X as follows. Intuitively, $k[X]$ is the ring of polynomial functions restricted to X. We can obtain this as the quotient ring,

$$k[X] = \frac{k[x_1, x_2, \ldots, x_n]}{I(X)},$$

where

$$I(X) = \{f \in k[x_1, \ldots, x_n] \mid f(x) = 0 \text{ for all } x \in X\}.$$

By Hilbert's Nullstellensatz, we have

$$I(X) = \sqrt{(f_1, f_2, \ldots, f_m)}$$

where $(f_1, \ldots, f_m)$ is the ideal generated by the f_i and $\sqrt{J}$ denotes the radical of an ideal $J \subseteq k[x_1, \ldots, x_n]$. So the coordinate ring of X is

$$k[X] = \frac{k[x_1, \ldots, x_n]}{\sqrt{(f_1, \ldots, f_m)}}.$$

It is a beautiful fact from classical algebraic geometry that all information about the affine variety X (up to polynomial changes of coordinates) is captured by the k-algebra $k[X]$ (up to isomorphism). To summarise, we have a correspondence

$$\left\{\begin{matrix}\text{Affine varieties} \\ \text{over } k\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{finitely generated} \\ \text{reduced } k\text{-algebras}\end{matrix}\right\}$$

where we recall that a ring A is *reduced* if it has no nonzero nilpotent elements.

The main aim of scheme theory is to extend the notion of affine variety to give a geometric interpretation of all commutative rings, not just reduced rings which are finitely generated algebras over an algebraically closed field. This is not such an unnatural thing to do. For example, in the setup above, there is another ring floating around in addition to $k[X]$, namely

$$\frac{k[x_1, \ldots, x_n]}{(f_1, \ldots, f_m)}.$$

This possibly non-reduced ring contains information about the equations $f_i(x) = 0$ used to describe X which may not be captured simply by looking at their solution set over k. For example, if we consider the equation $x^2 = 0$ in one variable x, the corresponding affine variety is $X = \{0\} \subseteq k$, with coordinate ring $k[X] = k[x]/(x) = k$. Geometrically, the affine variety given by this equation is simply a point, while the corresponding affine scheme associated to the ring $k[x]/(x^2)$ can be thought of as a "first order neighbourhood" of $0 \in k$. Such non-reduced schemes play a pivotal role in our study of deformations and obstructions.

Given a ring A, we want to define some kind of space $\mathrm{Spec}(A)$ so that the elements of A act in some sense like functions on $\mathrm{Spec}(A)$. More precisely, $\mathrm{Spec}(A)$ will be a ringed space in the following sense.

Definition A.1.1. A *ringed space* is a pair $X = (|X|, \mathcal{O}_X)$ where $|X|$ is a topological space and $\mathcal{O}_X$ is a sheaf of rings on $|X|$. If $X = (|X|, \mathcal{O}_X)$ and $Y = (|Y|, \mathcal{O}_Y)$ are ringed spaces, a *morphism of ringed spaces* $f : X \to Y$ is a continuous map $|f| : |X| \to |Y|$ together with a morphism $f^\# : |f|^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ of sheaves of rings on $|X|$ (or equivalently, a morphism $f^\# : \mathcal{O}_Y \to |f|_*\mathcal{O}_X$ of sheaves of rings on $|Y|$).

If $X = (|X|, \mathcal{O}_X)$, is a ringed space, we will often write X instead of $|X|$ where no confusion is likely to arise. Similarly, if $f = (|f|, f^\#) : X \to Y$ is a morphism of ringed spaces, we will often write f in place of $|f|$.

Example A.1.2. Let X be a smooth manifold. Then $(|X|, \mathcal{O}_X)$ is a ringed space, where $|X|$ is the underlying topological manifold of X and $\mathcal{O}_X$ is the sheaf of smooth real-valued functions on X. If $f : X \to Y$ is a smooth map between smooth manifolds, then there is an induced map $f : (|X|, \mathcal{O}_X) \to (|Y|, \mathcal{O}_Y)$ of ringed spaces, with $|f| : |X| \to |Y|$ given by the map f and $f^\# : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ given by $f^\#(\alpha) = \alpha \circ f$ for any (local) smooth function α on Y.

Let A be a ring. We define a ringed space $\mathrm{Spec}(A)$ as follows. First, as a set, we have

$$\mathrm{Spec}(A) = \{\mathfrak{p} \subseteq A \mid \mathfrak{p} \text{ is a prime ideal in } A\}.$$

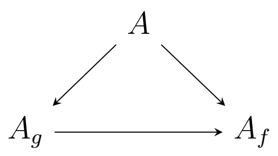
To ease notation, we will often distinguish between prime ideals of A and points in $\mathrm{Spec}(A)$. If $x \in \mathrm{Spec}(A)$, we denote the corresponding prime ideal by $\mathfrak{p}_x$, and if $\mathfrak{p}$ is a prime ideal in A, we denote the corresponding point in $\mathrm{Spec}(A)$ by $[\mathfrak{p}]$. For each $f \in A$, set

$$D(f) = \{x \in \mathrm{Spec}(A) \mid f \notin \mathfrak{p}_x\}.$$

As a topological space, $\mathrm{Spec}(A)$ has basis $\{D(f) \mid f \in A\}$ for the open sets. This topology on $\mathrm{Spec}(A)$ is called the *Zariski topology*. Notice that we have $D(f) \subseteq D(g)$ if and only if $\sqrt{(f)} \subseteq \sqrt{(g)}$. The structure sheaf $\mathcal{O}_{\mathrm{Spec}(A)}$ on $\mathrm{Spec}(A)$ is determined by

$$\mathcal{O}_{\mathrm{Spec}(A)}(D(f)) = A_f,$$

where $A_f = A[f^{-1}]$ is the localisation of A obtained by formally inverting f. Notice that this is well-defined, since whenever $D(f) = D(g)$, we have $\sqrt{(f)} = \sqrt{(g)}$, so there is a canonical isomorphism $A_f \cong A_g$. If $D(f) \subseteq D(g)$, the restriction map $\mathcal{O}_{\mathrm{Spec}(A)}(D(g)) \to \mathcal{O}_{\mathrm{Spec}(A)}(D(f))$ is the unique ring homomorphism $A_g \to A_f$ such that the diagram



commutes, where the homomorphisms $A \to A_f$ and $A \to A_g$ are the natural localisation maps. This defines a ringed space $\mathrm{Spec}(A)$.

Definition A.1.3. An *affine scheme* is a ringed space X which is isomorphic to $\mathrm{Spec}(A)$ for some commutative ring A.

Example A.1.4. Let k be a field. Then the affine scheme $\mathrm{Spec}(k)$ is a point, and the structure sheaf assigns the ring k to that point.

Example A.1.5. Let k be a field. Then the affine scheme $\mathrm{Spec}(k[t]/(t^2))$ is a point, and the structure sheaf assings the ring $k[t]/(t^2)$ to that point.

Example A.1.6. Let A be any ring, and let $A[x_1, \ldots, x_n]$ be a polynomial ring in n variables over A. The scheme $\mathbb{A}^n_A = \mathrm{Spec}(A[x_1, \ldots, x_n])$ is called *n-dimensional affine space* or *affine n-space* over A.

Before we give the definition of morphism between affine schemes, first notice that if $\phi : A \to B$ is a ring homomorphism, then there is an induced map

$$\begin{aligned} \mathrm{Spec}(\phi) : \mathrm{Spec}(B) &\to \mathrm{Spec}(A) \\ [\mathfrak{p}] &\mapsto [\phi^{-1}(\mathfrak{p})]. \end{aligned}$$

Note that $\mathrm{Spec}(\phi)$ is continuous since $\mathrm{Spec}(\phi)^{-1}(D(f)) = D(\phi(f))$ for all $f \in A$. Moreover, if $f \in A$, there is a unique homomorphism $A_f \to B_{\phi(f)}$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\ \phi\ } & B \\ \downarrow & & \downarrow \\ A_f & \longrightarrow & B_{\phi(f)} \end{array}$$

commutes. These homomorphisms define a morphism $\mathrm{Spec}(\phi)^{\#} : \mathcal{O}_{\mathrm{Spec}(A)} \to \mathrm{Spec}(\phi)_* \mathcal{O}_{\mathrm{Spec}(B)}$ of sheaves of rings on $\mathrm{Spec}(A)$, and hence a morphism $\mathrm{Spec}(\phi) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ of ringed spaces. The definition below is cooked up so that the morphisms of affine schemes $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ are precisely the morphisms of ringed spaces of the form $\mathrm{Spec}(\phi)$ for some ring homomorphism $\phi : A \to B$.

If $x \in X = \mathrm{Spec}(A)$, then the stalk $\mathcal{O}_{X,x}$ of the structure sheaf of X at x can be identified with the localisation $A_{\mathfrak{p}_x}$ of A at the prime ideal $\mathfrak{p}_x$. This is a local ring with maximal ideal $\mathfrak{m}_x = \mathfrak{p}_x A_{\mathfrak{p}_x}$. If $\phi : A \to B$ is a ring homomorphism, then the induced map

$$f = \mathrm{Spec}(\phi) : Y = \mathrm{Spec}(B) \to X = \mathrm{Spec}(A)$$

induces ring homomorphisms

$$f^{\#}_y : \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y},$$

such that

$$f_y^\#(\mathfrak{m}_{f(y)}) \subseteq \mathfrak{m}_y$$

for all $y \in Y$. (We call such homomorphisms between local rings *local ring homomorphisms.*) This motivates the following definition.

Definition A.1.7. A *locally ringed space* is a ringed space $X = (|X|, \mathcal{O}_X)$ such the stalk $\mathcal{O}_{X,x}$ of the structure sheaf at x is a local ring for all $x \in |X|$. If X and Y are locally ringed spaces, a *morphism of locally ringed spaces* $f : X \to Y$ is a morphism of ringed spaces such that, for all $x \in X$, the induced homomorphism $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local ring homomorphism.

Example A.1.8. Let X be a smooth manifold. Then the ringed space associated to X in Example A.1.2 is a locally ringed space. If X and Y are smooth manifolds, the construction of Example A.1.2 gives a bijection between the set of smooth maps $X \to Y$ and the set of morphisms $X \to Y$ of locally ringed spaces.

If X and Y are affine schemes, we define a morphism of affine schemes $f : X \to Y$ to be a morphism of locally ringed spaces.

Proposition A.1.9 (cf. [8], Theorem I-40 and Corollary I-41)**.** *Let A and B be commutative rings. Then the map*

$$\mathrm{Spec} : \mathrm{Hom}(A, B) \to \mathrm{Hom}(\mathrm{Spec}(B), \mathrm{Spec}(A))$$

is a bijection. Hence, the category **Aff** *of affine schemes is equivalent to the opposite category* $\mathbf{CR}^{op}$ *of the category* **CR** *of commutative rings.*

Definition A.1.10. A locally ringed space X is a *scheme* if there exists an open cover $\{U_i\}_{i \in I}$ of X such that each U_i is an affine scheme. A *morphism of schemes* is a morphism of locally ringed spaces. We denote the category of schemes by **Sch**.

Example A.1.11. Let $X = \mathrm{Spec}(A)$ be an affine scheme, and let $f \in A$. Then the locally ringed space $D(f)$ is isomorphic to $\mathrm{Spec}(A_f)$ and is therefore an affine scheme.

Example A.1.12. More generally, let X be a scheme. Since X has a basis for the open sets consisting of affine schemes, every open subset of X is a scheme.

Just as varieties are defined relative to some base field, there is a notion of schemes defined over a fixed base scheme S.

Definition A.1.13. Let S be a scheme. The *category of schemes over S* is the category $\mathbf{Sch}_{/S}$ with

$$\mathrm{Ob}(\mathbf{Sch}_{/S}) = \{X \to S \text{ a morphism of schemes}\}$$

and

$$\mathrm{Hom}(X \xrightarrow{\pi_X} S, Y \xrightarrow{\pi_Y} S) = \{f : X \to Y \mid \pi_Y \circ f = \pi_X\}.$$

We often write X instead of $X \to S$ for an object of $\mathbf{Sch}_{/S}$, and implicitly assume that a map to S has been chosen. We refer to objects of $\mathbf{Sch}_{/S}$ as *schemes over S* or *S-schemes*. In the special case that $S = \mathrm{Spec}(A)$ is affine, we also refer to S-schemes as A-schemes, and write $\mathbf{Sch}_{/A}$ in place of $\mathbf{Sch}_{/S}$.

Example A.1.14. Let A and B be rings. Then the structure of an A-scheme on $\mathrm{Spec}(B)$ is the same as an A-algebra structure on B.

Example A.1.15. More generally, if A is a ring and X is a scheme, an A-scheme structure on X is the same as an A-algebra structure on the structure sheaf $\mathcal{O}_X$.

All fibre products exist in the category of schemes. These are easiest to describe in the affine case: if $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$ and $Z = \mathrm{Spec}(C)$, and $X \to Z$, $Y \to Z$ are morphisms of schemes corresponding to ring homomorphisms $C \to A$ and $C \to B$, then

$$X \times_Z Y = \mathrm{Spec}(A \otimes_C B).$$

More generally, suppose that X, Y and Z are schemes and $f : X \to Z$ and $g : Y \to Z$ are morphisms. Let $\{Z_i\}_{i\in I}$ be an open cover of Z by affine schemes, and for each $i \in I$, let $\{X_j\}_{j\in J_i}$ and $\{Y_k\}_{k\in K_i}$ be open covers of $f^{-1}(Z_i)$ and $g^{-1}(Z_i)$ by affine schemes. Then the affine schemes $\{X_j \times_{Z_i} Y_k\}_{i\in I, j\in J_i, k\in K_i}$ form an open cover for $X \times_Z Y$. For more details on this construction, see [8], Section I.3.1.

A.2 Sheaves of modules

In this section we recall the theory of sheaves of modules over a scheme X. These include locally free sheaves, which play the role of vector bundles over X in algebraic geometry, and more general objects called quasi-coherent sheaves. We start with the following general setting.

Definition A.2.1 (cf. Definition 2.3.8)**.** Let X be a topological space and let A be a sheaf of rings on X. An *A-module* is a sheaf M of abelian groups on X, together with a morphism $A \times M \to M$ of sheaves on X giving $M(U)$ the structure of an $A(U)$-module for each open set $U \subseteq X$. If M and N are A-modules, an A-module homomorphism from M to N is a morphism $\phi : M \to N$ of sheaves of abelian groups such that each map ϕ_U is an $A(U)$-module homomorphism. We denote the category of A-modules by **A-mod**.

Definition A.2.2. Let X be a topological space, let A be a sheaf of rings on X and let M, N and P be A-modules.

(1) The *direct sum* $M \oplus N$ of M and N is the sheaf

$$U \mapsto M(U) \oplus N(U).$$

(2) The *tensor product* $M \otimes N$ of M and N is the sheaf associated to the presheaf

$$U \mapsto M(U) \otimes_{A(U)} N(U).$$

(3) The *Hom sheaf* $\underline{\mathrm{Hom}}(M, N)$ of M and N is the sheaf

$$U \mapsto \mathrm{Hom}_{A|_U}(M|_U, N|_U).$$

(4) If $M \subseteq N$ is a submodule (subsheaf with inherited operations), then the *quotient* N/M is the sheaf associated to the presheaf

$$U \mapsto N(U)/M(U).$$

(5) If $f : M \to N$ is an A-module homomorphism, the *kernel* $\ker(f)$ of f is the sheaf

$$U \mapsto \ker(f_U : M(U) \to N(U)).$$

(6) If $f : M \to N$ is an A-module homomorphism, the *image* $\mathrm{im}(f)$ of f is the sheaf associated to the presheaf

$$U \mapsto \mathrm{im}(f_U : M(U) \to N(U)).$$

(7) If $f : M \to N$ is an A-module homomorphism, the *cokernel* $\mathrm{coker}(f)$ of f is the sheaf associated to the presheaf

$$U \mapsto \mathrm{coker}(f_U : M(U) \to N(U)).$$

This is canonically isomorphic to quotient $N/\mathrm{im}(f)$.

(8) A sequence

$$M \xrightarrow{f} N \xrightarrow{g} P$$

is *exact* if $\mathrm{im}(f) = \ker(g)$.

Remark A.2.3. All the sheaves in the definition above carry natural A-module structures, and satisfy the usual universal properties from commutative algebra.

Let $X = \mathrm{Spec}(A)$ be an affine scheme. If M is an A-module, there is an associated $\mathcal{O}_X$-module $\tilde{M}$ with

$$\tilde{M}(D(f)) = M_f = M \otimes_A A_f$$

for all $f \in A$. The sheaf $\tilde{M}$ is the basic example of a quasi-coherent sheaf.

Definition A.2.4. Let X be a scheme. A *quasi-coherent sheaf* is an $\mathcal{O}_X$-module such that there is an open cover $\{U_i\}_{i \in I}$ of X by affine schemes $U_i = \mathrm{Spec}(A_i)$ and A_i-modules M_i such that $M|_{U_i} \cong \tilde{M}_i$ for each $i \in I$. We denote by $\mathbf{QCoh}(X)$ the full subcategory of $\mathcal{O}_X$**-mod** spanned by the quasi-coherent sheaves.

Remark A.2.5. One can check that the functor $M \mapsto \tilde{M}$ commutes with all the operations of Definition A.2.2. Thus, in particular, the subcategory $\mathbf{QCoh}(X) \subseteq \mathcal{O}_X$**-mod** is closed under all these operations.

Proposition A.2.6 (cf. [18], Proposition II.5.4)**.** *Let X be a scheme, and let M be an $\mathcal{O}_X$-module. Then M is quasi-coherent if and only if for every affine open subset $U = \mathrm{Spec}(A) \subseteq X$ there exists an A-module M_0 such that $M|_U \cong \tilde{M}_0$.*

Corollary A.2.7 (cf. [18], Corollary II.5.5)**.** *Let A be a ring. The construction $M \mapsto \tilde{M}$ defines an equivalence of categories*

$$A\text{-}\mathbf{mod} \simeq \mathbf{QCoh}(\mathrm{Spec}(A)).$$

Let $f : X \to Y$ be a morphism of schemes. Then pushing forwards sheaves of abelian groups gives a functor $f_* : \mathcal{O}_X$**-mod** $\to \mathcal{O}_Y$**-mod**, where, for M an $\mathcal{O}_X$-module, the $\mathcal{O}_Y$-module structure on f_*M is inherited from the $f_*\mathcal{O}_X$-module structure via the homomorphism $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$. This functor has a left adjoint $f^* : \mathcal{O}_Y$**-mod** $\to \mathcal{O}_X$**-mod** given by

$$f^*M = f^{-1}M \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

for all $\mathcal{O}_Y$-modules M. For any f, the functor f^* preserves quasi-coherence, and under certain finiteness assumptions, (such as X Noetherian: see [18], Proposition II.5.8), the pushforward f_* also preserves quasi-coherence.

Let X be a scheme. There is special class of quasi-coherent sheaves on X called locally free sheaves.

Definition A.2.8. An $\mathcal{O}_X$-module E is called *free (of rank n)* if it is isomorphic to an n-fold direct sum $\mathcal{O}_X^{\oplus n}$ of the $\mathcal{O}_X$-module $\mathcal{O}_X$ for some $n \geq 0$. We say that E is *locally free (of rank n)* if there is an open cover $\{U_i\}$ of X such that $E|_{U_i}$ is free (of rank n) for each i. We also refer to locally free sheaves as *vector bundles*.

If $f : X \to Y$ is a morphism of schemes and E is a locally free sheaf of rank n on Y, then f^*E is a locally free sheaf of rank n on X.

A.3 Functors for constructing schemes

In this section, we look at modifications of the functor Spec : $\mathbf{CR} \to \mathbf{Sch}$, which take in algebraic objects and produce schemes which are not necessarily affine. There are two main ideas: constructing schemes over S from quasi-coherent $\mathcal{O}_S$-algebras, and constructing projective schemes from graded rings. We start with some preliminaries on algebras over a scheme.

Definition A.3.1. Let S be a scheme. A *quasi-coherent $\mathcal{O}_S$-algebra* is a sheaf B of rings on S together with a map $\mathcal{O}_S \to B$ such that B is quasi-coherent as an $\mathcal{O}_S$-module. We denote by $\mathbf{QCoh^{alg}}(S)$ the category of quasi-coherent $\mathcal{O}_S$-algebras.

Just as for modules, if A is any ring, there is a functor

$$(-)^\sim : A\textbf{-alg} \to \mathbf{QCoh^{alg}}(\mathrm{Spec}(A))$$

which takes an A-algebra B to the quasi-coherent sheaf $\tilde{B}$ with the natural ring structure. Again, this functor is an equivalence of categories.

Let S be a scheme. There is a fully faithful functor

$$\mathrm{Spec}_S : \mathbf{QCoh^{alg}}(S) \to \mathbf{Sch}_{/S}.$$

We can describe this functor as follows. If $S = \mathrm{Spec}(A)$ is affine, and B is an A-algebra, then $\mathrm{Spec}_S(\tilde{B}) = \mathrm{Spec}(B)$, with the map to S given by the homomorphism $A \to B$. In general, if S is any scheme and B is a quasi-coherent $\mathcal{O}_S$-algebra, we can obtain $\mathrm{Spec}_S(B)$ by taking an open cover

$\{U_i\}_{i\in I}$ of S by affine open subschemes U_i and gluing together the U_i-schemes $\mathrm{Spec}_{U_i}(B|_{U_i})$. For more details, see [15], Section 1.3. For a discussion of an alternative approach, see [8], Section I.3.3.

Our next functor takes as input graded rings. We recall the definition below.

Definition A.3.2. A $\mathbb{Z}$-*graded ring* is a ring A together with a decomposition,

$$A = \bigoplus_{n\in\mathbb{Z}} A_n,$$

of A as a direct sum of abelian subgroups such that $A_m A_n \subseteq A_{m+n}$ for all $m, n \in \mathbb{Z}$. We say that A is *non-negatively graded*, or simply a *graded ring*, if $A_n = 0$ for all $n < 0$.

There is a construction, Proj, taking graded rings to schemes defined as follows. Given a graded ring A, $n \geq 0$, we call an element $f \in A$ *homogeneous (of degree n)* if $f \in A_n$ for some n. If we fix a homogeneous element $f \in A_n$, the localisation $A_f = A[f^{-1}]$ has a natural $\mathbb{Z}$-grading with

$$(A_f)_m = \sum_{k\geq 0} f^{-k} A_{m+kn}$$

for all $m \in \mathbb{Z}$. If f is homogeneous of degree $n > 0$, we can form the affine scheme $\mathrm{Spec}((A_f)_0)$. The scheme $\mathrm{Proj}(A)$ can be obtained by gluing together the schemes $\mathrm{Spec}((A_f)_0)$ for each in a natural way. There is a natural structure map $\mathrm{Proj}(A) \to \mathrm{Spec}(A_0)$, coming from the ring homomorphisms $A_0 \to (A_f)_0$. For more details, see [18], Proposition 2.5, or [8], Section III.2.1.

There is a global version of Proj, just as there is a global version of the functor Spec. If A is a ring, a *graded A-algebra* is a graded ring B together with a ring homomorphism $A \to B_0$. More generally, if S is a scheme, a *quasi-coherent graded $\mathcal{O}_S$-algebra* is a quasi-coherent $\mathcal{O}_S$-algebra B together with decomposition

$$B = \bigoplus_{n\geq 0} B_n$$

of B as a direct sum of (necessarily quasi-coherent) $\mathcal{O}_S$-modules such that $B(U)$ is a ring for each open $U \subseteq S$. We denote by $\mathbf{QCoh}^{\mathbf{gralg}}(S)$ the category of quasi-coherent graded $\mathcal{O}_S$-algebras. Just as with other quasi-coherent objects, there is an equivalence of categories

$$(-)^{\sim} : \mathbf{QCoh}^{\mathbf{gralg}}(\mathrm{Spec}(A)) \simeq A\text{-}\mathbf{gralg}$$

for any ring A, where A-**gralg** denotes the category of graded A-algebras. We can define a construction Proj_S taking quasi-coherent $\mathcal{O}_S$-algebras to schemes over S as follows. If $S = \mathrm{Spec}(A)$ is affine, then we set $\mathrm{Proj}_S(\tilde{B}) = \mathrm{Proj}(B)$ for B a graded A-algebra, with structure map $\mathrm{Proj}(B) \to \mathrm{Spec}(B_0) \to \mathrm{Spec}(A)$. For general S, we choose an open cover $\{U_i\}_{i\in I}$ of S by affine open subsets U_i and glue together the U_i-schemes $\mathrm{Proj}_{U_i}(B|_{U_i})$ to obtain $\mathrm{Proj}_S(B)$. For more details, see [8], Section III.2.3 or [15], Section 3.1.

Example A.3.3. Let A be a ring. *Projective n-space over A* is the A-scheme

$$\mathbb{P}^n_A = \mathrm{Proj}(A[x_0, x_1, \ldots, x_n])$$

where $A[x_0, x_2, \ldots, x_n]$ has the grading with x_i homogeneous of degree 1 for each i.

Example A.3.4. More generally, let S be any scheme. *Projective n-space over S* is the S-scheme

$$\mathbb{P}^n_S = \mathrm{Proj}(\mathcal{O}_S[x_1, x_1, \ldots, x_n]).$$

Just as there is a correspondence between A-modules and quasi-coherent sheaves on $\mathrm{Spec}(A)$ for any ring A, there is a correspondence between graded A-modules and quasi-coherent sheaves on $\mathrm{Proj}(A)$ for any graded ring A.

Definition A.3.5. Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a $\mathbb{Z}$-graded ring. A *graded A-module* is an A-module M together with a decomposition,

$$M = \bigoplus_{n\in\mathbb{Z}} M_n,$$

of M into subgroups such that $A_m M_n \subseteq M_{m+n}$ for all $m, n \in \mathbb{Z}$. If M and N are graded A-modules, a *graded A-module homomorphism* from M to N is an A-module homomorphism $f : M \to N$ such that $f(M_n) \subseteq N_n$ for all $n \in \mathbb{Z}$. We denote by A-**grmod** the category of graded A-modules.

Let A be a (positively) graded ring and let M be a graded A-module. We can associate a quasi-coherent sheaf $\tilde{M}$ on $\mathrm{Proj}(A)$ as follows. For each homogeneous element $f \in A$ of positive degree, there is a natural grading on the A_f-module $M_f = M \otimes_A A_f$ so that $(M_f)_0$ is an $(A_f)_0$-module. The sheaf $\tilde{M}$ is obtained by gluing together the quasi-coherent sheaves $(\tilde{M}_f)_0$ over the affine open subsets $\mathrm{Spec}((A_f)_0)$. This defines a functor

$$(-)^\sim : A\text{-}\mathbf{grmod} \to \mathbf{QCoh}(\mathrm{Proj}(A)).$$

Remark A.3.6. This functor enjoys all the exactness properties of the analogous functor for Spec (since localisation and taking graded pieces are exact), but unlike the functor for Spec, it is not an equivalence of categories. In fact, if M is a graded A-module, then for any $n_0 \in \mathbb{Z}$, we have a graded A-module

$$M_{\geq n_0} = \bigoplus_{n \geq n_0} M_n.$$

It is an immediate consequence of the construction above that the map $\tilde{M}_{\geq n_0} \to \tilde{M}$ induced by the inclusion $M_{\geq n_0} \to M$ is an isomorphism.

Example A.3.7. Let A be a graded ring. For any $k \in \mathbb{Z}$, let $A(k)$ be free graded A-module generated by a single generator in degree $-k$, so that $A(k)_n = A_{n+k}$ for all $n \in \mathbb{Z}$. We denote by $\mathcal{O}(k)$ the quasi-coherent sheaf $\tilde{A}(k)$ on $\mathrm{Proj}(A)$. This is an example of a locally free sheaf of rank 1.

Example A.3.8. Let A be a ring and $n \in \mathbb{Z}$. The sheaf $\mathcal{O}(-1)$ on $\mathbb{P}^n_A$ is called the *tautological line bundle*.

A.4 The cotangent sheaf

In this section, we will review the theory of Kähler differentials of rings and cotangent sheaves of schemes.

Definition A.4.1. Let A be a ring, let B be an A-algebra and let M be a B-module. An A-*derivation* from B to M is an A-linear map $\partial : B \to M$ such that

$$\partial(b_1 b_2) = b_1 \partial(b_2) + b_2 \partial(b_1)$$

for all $b_1, b_2 \in B$. We denote the set of A-derivations from B to M by $\mathrm{Der}_A(B, M)$.

Remark A.4.2. With A, B and M as above, we have a ring structure on the A-module $B \oplus M$ with product given by

$$(b_1, m_1)(b_2, m_2) = (b_1 b_2, b_1 m_2 + b_2 m_1).$$

The projection map $B \oplus M \to B$ is an A-algebra homomorphism. If we let $\mathrm{Hom}_{A/-/B}(B, B \oplus M)$ be the set of A-algebra homomorphisms $B \to B \oplus M$ such that the composite $B \to B \oplus M \to B$ is the identity map, then there is a canonical isomorphism

$$\mathrm{Der}_A(B, M) \cong \mathrm{Hom}_{A/-/B}(B, B \oplus M).$$

Observe that if $f : M \to N$ is a morphism of B-modules, then there is a function

$$\begin{aligned} \mathrm{Der}_A(B, M) &\to \mathrm{Der}_A(B, N) \\ \partial &\mapsto f \circ \partial. \end{aligned}$$

Thus, we have a functor

$$\mathrm{Der}_A(B, -) : B\text{-}\mathbf{mod} \to \mathbf{Set}.$$

Proposition A.4.3. *Let A be a ring and let B be an A-algebra. Then there exists a B-module $\Omega_{B/A}$ and a natural isomorphism*

$$\mathrm{Hom}_B(\Omega_{B/A}, M) \cong \mathrm{Der}_A(B, M)$$

for M a B-module.

Proof. An explicit construction is given in [7], Chapter 16. □

The module $\Omega_{B/A}$ is called the *module of Kähler differentials* of B over A. From the universal property of $\Omega_{B/A}$, there is a universal derivation $d = d_B : B \to \Omega_{B/A}$ associated to the identity map $\Omega_{B/A} \to \Omega_{B/A}$. We will often refer to this derivation as the *derivative map*, and for $b \in B$ we call $db \in \Omega_{B/A}$ the *derivative* of b.

Suppose that we have a commutative diagram of rings as shown below.

$$\begin{array}{ccc} A & \longrightarrow & B \\ {\scriptstyle \phi_A}\downarrow & & \downarrow{\scriptstyle \phi_B} \\ A' & \longrightarrow & B' \end{array}$$

Then there is an A-derivation $d_{B'} \circ \phi_B : B \to \Omega_{B'/A'}$, which induces a B-module homomorphism $\Omega_{B/A} \to \Omega_{B'/A'}$.

Now let X be a topological space, and let $A \to B$ be a morphism of sheaves of rings on X. Then for any open sets $U, V \subseteq X$ with $U \subseteq V$, the restriction maps $A(V) \to A(U)$ and $B(V) \to B(U)$ induce a map $\Omega_{B(V)/A(V)} \to \Omega_{B(U)/A(U)}$.

Definition A.4.4. Let X be a topological space, and let A be a sheaf of rings on X. If B is an A-algebra, the *module of Kähler differentials of B over A* is the sheaf associated to the presheaf

$$U \mapsto \Omega_{B(U)/A(U)}.$$

This has a canonical B-module structure induced by the $B(U)$-module structure on $\Omega_{B(U)/A(U)}$ for each open $U \subseteq X$.

Definition A.4.5. Let S be a scheme and let $\pi : X \to S$ be a scheme over S. The *cotangent sheaf* of X over S is the $\mathcal{O}_X$-module

$$\Omega_X = \Omega_{X/S} = \Omega_{\mathcal{O}_X/\pi^{-1}\mathcal{O}_S}.$$

Proposition A.4.6. *Let S be a scheme and let X be a scheme over S. Then $\Omega_{X/S}$ is a quasi-coherent sheaf. More precisely, if $X = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(A)$, then*

$$\Omega_{X/S} \cong \tilde{\Omega}_{B/A}$$

is the $\mathcal{O}_X$-module associated to the B-module $\Omega_{B/A}$.

Example A.4.7. Let A be a ring and let $B = A[x_1, \ldots, x_n]$ be a polynomial ring over A. Then

$$\Omega_{B/A} = Bdx_1 \oplus Bdx_2 \oplus \ldots \oplus Bdx_n$$

is a free B-module with basis $\{dx_1, \ldots, dx_n\}$.

Example A.4.8. More generally, if M is an A-module and $B = \mathrm{Sym}_A(M)$ is the symmetric algebra of M, then

$$\Omega_{B/A} \cong M \otimes_A B.$$

The main tool for computing the modules $\Omega_{B/A}$ is the following.

Proposition A.4.9 (cf. [7], Propositions 16.2 and 16.3)**.** *Let $A \to B \to C$ be morphisms of rings. Then there is an exact sequence C-modules*

$$\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0.$$

Moreover, if $B \to C$ is surjective with kernel I, then $\Omega_{C/B} = 0$ and we have an exact sequence of C-modules,

$$I/I^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0,$$

where the map $I/I^2 \to \Omega_{B/A} \otimes_B C$ is induced by the derivative map $d : I \to \Omega_{B/A}$.

A.5 Properties of schemes and morphisms

In this section, we collect a glossary of properties of schemes and morphisms between them.

Definition A.5.1. Let X be a scheme. We say that X is *locally Noetherian* if there is an open cover $\{U_i = \mathrm{Spec}(A_i)\}_{i \in I}$ of X by affine open subsets such that each ring A_i is Noetherian. We say that X is *Noetherian* if we can find such a cover with I a finite set.

Definition A.5.2. Let $f : X \to Y$ be a morphism of schemes. We say that f is *locally of finite type*, or that X *is locally of finite type over* S, if there exists an open cover $\{V_i = \mathrm{Spec}(A_i)\}$ of Y by affine schemes, and for each i an open cover $\{U_{ij} = \mathrm{Spec}(B_{ij})\}$ of $f^{-1}(V_i)$ by affine schemes, such that each B_{ij} is a finitely generated A_i-algebra. We say that f is *locally of finite presentation* if each B_{ij} is a finitely presented A_i-algebra. We say that f is *of finite type* if each $f^{-1}(V_i)$ can be covered by finitely many U_i with B_{ij} finitely generated over A_i.

Definition A.5.3. A morphism $f : X \to Y$ is an *open immersion* if f is a homeomorphism of X with a Zariski open subset of Y, and the induced map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is an isomorphism.

Definition A.5.4. Let $i : X \to Y$ be a morphism of schemes. We say that i is a *closed embedding* (or *closed immersion*) if i is a homeomorphism onto a closed subset of Y and the map $i^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is a surjection of sheaves on Y. If $i : X \to Y$ is a closed embedding, we often refer to X as a *closed subscheme* of Y.

Let $i : X \hookrightarrow Y$ be a closed embedding of schemes. The kernel I of the map $\mathcal{O}_Y \to i_*\mathcal{O}_X$ is a quasi-coherent sheaf on Y called the *ideal sheaf* of the embedding i. To ease notation, we also write I for the $i^{-1}\mathcal{O}_Y$-module $i^{-1}I$ on X and call this the ideal sheaf where there is no danger of confusion. The sheaf $I/I^2 = i^*(I)$ is a quasi-coherent sheaf on X, called the *conormal sheaf* of X in Y.

Example A.5.5. Let $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$ be affine schemes. Then a closed embedding $i : X \to Y$ is the same thing as a surjective ring homomorphism $\phi : A \to B$. If $I = \ker(\phi)$, then the ideal sheaf of i is the quasi-coherent sheaf $\tilde{I}$ on Y. The conormal sheaf of X in Y is

$$i^*(\tilde{I}) = (I/I^2)^{\sim} \in \mathrm{Ob}(\mathbf{QCoh}(X)).$$

The following plays a very important role in this work. A good reference (albeit with slightly different language) is [16], Section 16.1. Let $i : X \to Y$ be a closed embedding of schemes with ideal sheaf I. There are associated quasi-coherent sheaves of $\mathcal{O}_X$-algebras

$$\mathrm{Sym}_{\mathcal{O}_X}(I/I^2)$$

and

$$\bigoplus_{n\geq 0} i^*(I^n) = \bigoplus_{n\geq 0} I^n/I^{n+1}.$$

Here the symmetric algebra functor is obtained by taking symmetric algebras over open subsets and taking the associated sheaf. For affine schemes $S = \mathrm{Spec}(A)$, it agrees with the symmetric algebra functor for A-modules.

Definition A.5.6. Let $i : X \to Y$ be a closed embedding of schemes with ideal sheaf I. The *normal sheaf* of X in Y is the X-scheme

$$N_{X/Y} = \mathrm{Spec}_X \left(\mathrm{Sym}_{\mathcal{O}_X}(I/I^2)\right).$$

The *normal cone* of X in Y is the X-scheme

$$C_{X/Y} = \mathrm{Spec}_X \left(\bigoplus_{n\geq 0} I^n/I^{n+1}\right).$$

Observe that the multiplication maps

$$\mathrm{Sym}^n_{\mathcal{O}_X}(I/I^2) \to I^n/I^{n+1}$$

induce a surjection of $\mathcal{O}_X$-algebras

$$\mathrm{Sym}_{\mathcal{O}_X}(I/I^2) \to \bigoplus_{n\geq 0} I^n/I^{n+1},$$

which gives a closed embedding of X-schemes

$$C_{X/Y} \hookrightarrow N_{X/Y}.$$

Definition A.5.7 (cf. [16], Définition 0.15.1.7)**.** Let A be a ring. We say that a sequence $(f_j)_{1\leq j\leq d}$ of elements $f_j \in A$ is *regular* if for each n, the image of f_j in

$$\frac{A}{(f_1, \ldots, f_{j-1})}$$

is a not zero or a zero divisor.

Definition A.5.8 (cf. [16], Définition 16.9.2)**.** Let $i : X \to Y$ be a closed embedding of schemes with ideal sheaf I. We say that i is a *regular embedding (of codimension d)* if there exist affine open subsets $U_i = \mathrm{Spec}(A_i)$ of Y with $i(X) \subseteq \bigcup_i U_i$, such that, for each i, the ideal sheaf $I|_{U_i} \subseteq \mathcal{O}_{U_i}$ corresponds to an ideal $(f_1, \ldots, f_d) \subseteq A_i$ generated by a regular sequence $(f_j)_{1\leq j\leq d}$ in A_i.

Remark A.5.9. We can interpret Definition A.5.8 as follows: a closed subscheme X of Y is regularly embedded of codimension d if it can be locally defined by d independent equations.

Proposition A.5.10. *Let $i : X \to Y$ be a closed embedding of schemes with ideal sheaf I. If i is a regular embedding of codimension d then*

(1) I is of finite type, i.e. there is an open cover $\{U_i = \mathrm{Spec}(A_i)\}$ of Y by affine open sets such that $I|_{U_i}$ corresponds to a finitely generated ideal in A_i,

(2) the conormal sheaf I/I^2 is a locally free $\mathcal{O}_X$-module of rank d, and

(3) $C_{X/Y} = N_{X/Y}$.

The converse holds so long as Y is locally Noetherian.

Proof. This follows from [16] Corollaire 16.9.4 and Proposition 16.9.10. □

The following is the analogue of the Hausdorff condition in the category of schemes.

Definition A.5.11. Let $f : X \to S$ be a morphism of schemes. We say that f is *separated* if the diagonal map

$$\Delta : X \to X \times_S X$$

is a closed embedding.

We now introduce the notion of flatness of morphisms of schemes. This homological condition captures the intuitive notion of a map with continuously varying fibres. Recall that a module M over a ring A is called *flat* if the functor

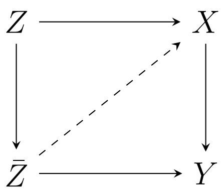
$$M \otimes_A - : A\textbf{-mod} \to A\textbf{-mod}$$

is exact.

Definition A.5.12. Let $f : X \to Y$ be a morphism of schemes. We say that f is *flat* if for every $x \in X$, $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,f(x)}$-module.

Example A.5.13. Let $A \to B$ be a homomorphism of rings. Then the induced map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is flat if and only if B is a flat A-module.

Definition A.5.14 (cf. [16], Définition 17.1.1)**.** Let $f : X \to Y$ be a morphism of schemes. We say that f is *formally smooth* (resp. *formally étale*) if for any commutative diagram of solid arrows



where $\bar{Z}$ is an affine scheme and Z is a closed subscheme of $\bar{Z}$ with nilpotent ideal sheaf, there exists a (resp. there exists a unique) dashed arrow such that the diagram commutes.

Definition A.5.15 (cf. [16], Définition 17.3.1)**.** Let $f : X \to Y$ be a morphism of schemes. We say that f is *smooth* (resp. *étale*) if f is formally smooth (resp. formally étale) and locally of finite presentation.

Proposition A.5.16. *Let $f : X \to Y$ be a smooth (resp. étale) morphism of schemes. Then f is flat and the relative cotangent sheaf $\Omega_{X/Y}$ is a vector bundle (resp. $\Omega_{X/Y} = 0$).*

A.6 Intersection theory

In this section we recall some of the basics of intersection theory on schemes and stacks. In what follows, all schemes are separated and of finite type over a fixed base field k.

The main idea of intersection theory is to associate to a sequence of abelian groups $\{A_p(X)\}_{p\in\mathbb{Z}_{\geq 0}}$ to a scheme X, called *Chow (homology) groups*, such that the elements of $A_p(X)$ are related to the p-dimensional subvarieties of X. These groups are functorial for several classes of well-behaved morphisms in a compatible way. We first need a few preliminaries on integral schemes and dimension.

Definition A.6.1. Let W be a topological space. We say that W is *irreducible* if for any nonempty open sets U and V in W, the intersection $U \cap V$ is nonempty.

Definition A.6.2. Let X be a topological space. The *(combinatorial) dimension* of X is

$$\dim X = \sup \left\{ n \;\middle|\; \begin{array}{c} \emptyset \subsetneq W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_n \\ \text{is a chain of irreducible closed subspaces of } X \end{array} \right\}.$$

If $Y \subseteq X$ is an irreducible closed subset, the *codimension* of Y in X is

$$\operatorname{codim}(Y, X) = \sup \left\{ n \;\middle|\; \begin{array}{c} Y \subsetneq W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_n \\ \text{is a chain of irreducible closed subspaces of } X \end{array} \right\}.$$

Proposition A.6.3 ([14], Proposition I.2.). *Let X be a scheme. Then the map*

$$x \mapsto \overline{\{x\}}$$

is a bijection from points of X to irreducible closed subsets of X.

Definition A.6.4. Let W be a scheme. We say that W is *irreducible* if the underlying topological space $|W|$ is irreducible. We say that W is *reduced* if for every point $w \in W$, the local ring $\mathcal{O}_{W,w}$ has no nilpotent elements. We say that W is *integral* if W is both reduced and irreducible.

Remark A.6.5. Let X be a scheme, and let $|W|$ be an irreducible closed subset of X. Then there is a unique integral closed subscheme W of X with underlying topological space $|W|$. In the case where $X = \mathrm{Spec}(A)$ is affine, the irreducible closed subset $|W|$ corresponds to a point $w \in X$ by Proposition A.6.3, which in turn corresponds to a prime ideal $\mathfrak{p}$ in A, and we set $W = \mathrm{Spec}(A/\mathfrak{p})$. The general case is obtained by gluing the results of this construction along affine open subsets of X.

Definition A.6.6 (cf. [10], Section 1.3). Let k be a field and let X be a scheme of finite type over k. If $p \in \mathbb{Z}_{\geq 0}$, the *group of p-cycles in X* is the free abelian group $Z_p(X)$ generated by the integral closed subschemes of X of dimension p. We write

$$Z_*(X) = \bigoplus_{p \in \mathbb{Z}_{\geq 0}} Z_p(X).$$

The Chow homology group $A_p(X)$ is obtained by taking the quotient of $Z_p(X)$ with respect to a suitable equivalence relation, which we describe as follows.

Let Y be an integral scheme. By Proposition A.6.3, there exists a unique generic point $y \in Y$ such that $Y = \overline{\{y\}}$. It follows that the local ring $\mathcal{O}_{Y,y}$ has a unique prime ideal. Since Y is integral, $\mathcal{O}_{Y,y}$ is an integral domain, so $\{0\} \subseteq \mathcal{O}_{Y,y}$ is the unique prime ideal, and $\mathcal{O}_{Y,y}$ is a field. We denote this field by $R(Y)$ and call it the *field of rational functions on Y*.

Example A.6.7. Let k be a field. Then the scheme $\mathbb{P}^1_k$ is integral. The field of rational functions on $\mathbb{P}^1_k$ is $R(\mathbb{P}^1_k) = k(x)$, the field of rational functions in one variable over k.

Let Y be a scheme, and let W be an integral closed subscheme of Y. Write $\mathcal{O}_{Y,W}$ for the local ring $\mathcal{O}_{Y,w}$ where w is the unique point in Y such that $\overline{\{w\}} = W$. If Y is integral, then

$$R(Y) = \mathrm{Frac}(\mathcal{O}_{Y,W}),$$

where $\mathrm{Frac}(\mathcal{O}_{Y,W})$ is the field of fractions of the integral domain $\mathcal{O}_{Y,W}$. So if $r \in R(Y)^*$ is a nonzero rational function on Y, then there exist elements $f, g \in \mathcal{O}_{Y,W}$ with $r = f/g$. If W is of codimension 1 in Y, the *order of* r *at* W is defined as

$$\mathrm{ord}_W(r) = l_{\mathcal{O}_{Y,W}}\left(\frac{\mathcal{O}_{Y,W}}{(f)}\right) - l_{\mathcal{O}_{Y,W}}\left(\frac{\mathcal{O}_{Y,W}}{(g)}\right),$$

where $l_A(M)$ denotes the length of the A-module M. By the results of [10], Section A.3, this gives a well-defined abelian group homomorphism

$$\mathrm{ord}_W : R(Y)^* \to \mathbb{Z}.$$

If Y is an integral scheme of finite type over k of dimension $p+1$ and $r \in R(Y)^*$, the *divisor of* r is

$$\mathrm{div}(r) = \sum_W \mathrm{ord}_W(r)[W] \in Z_p(Y).$$

Definition A.6.8. Let X be a scheme of finite type over a field k, let $p \in \mathbb{Z}_{\geq 0}$ and let $\alpha, \beta \in Z_p(X)$. We say that α and β are *rationally equivalent* if there exist finitely many integral closed subschemes Y_i of X and rational functions $r_i \in R(Y_i)^*$, such that

$$\beta - \alpha = \sum_i \mathrm{div}(r_i),$$

where we identify $\mathrm{div}(r_i)$ with a cycle in $Z_p(X)$ via the natural inclusion $Z_p(Y_i) \to Z_p(X)$. The pth *Chow homology group* of X is the group $Z_p(X)$ modulo rational equivalence. We write

$$A_*(X) = \bigoplus_{p \in \mathbb{Z}_{\geq 0}} A_p(X).$$

Definition A.6.9. Let X be a scheme of finite type over a field k. The *irreducible components* of X are the maximal irreducible closed subsets of X. The *fundamental cycle* of X is

$$[X] = \sum_i m_i[X_i] \in Z_*(X)$$

where the sum is over irreducible components X_i of X, and for each i,

$$m_i = l_{\mathcal{O}_{X,X_i}}(\mathcal{O}_{X,X_i})$$

is the length of the local ring $\mathcal{O}_{X,X_i}$ as a module over itself. Since X is of finite type over k, the underlying topological space of X is Noetherian, so this is a finite sum by [14], 0.2.2.5. The *fundamental class* of X is the image

$$[X] \in A_*(X).$$

If Y is a closed subscheme of a scheme X, then we can regard the fundamental cycle and fundamental class of Y as elements $[Y] \in Z_*(X)$ and $[Y] \in A_*(X)$.

Let $f : X \to Y$ be a flat morphism of schemes of finite type over k. We say that f has *relative dimension* n if for every irreducible component Y_i of Y, we have

$$\dim f^{-1}(Y_i) = \dim Y_i + n.$$

If $f : X \to Y$ is flat of relative dimension n, then, for all $p \in \mathbb{Z}_{\geq 0}$, we have an induced map

$$f^* : Z_p(Y) \to Z_{p+n}(X),$$

given by

$$f^*([W]) = [f^{-1}(W)]$$

for all integral closed subschemes of Y. By [10], Lemma 1.7.1 and Theorem 1.7, f^* descends to a homomorphism

$$f^* : A_p(Y) \to A_{p+n}(X)$$

such that $f^*[W] = [f^{-1}(W)]$ for all closed subschemes W of Y.

A.7 Algebraic stacks

In this section, we review some aspects of the theory of algebraic stacks.

An algebraic stack is a generalisation of a scheme in which points can have automorphisms. A rich source of examples is the theory of moduli spaces. For a moduli space X, the points of X classify some geometric objects, such as algebraic curves or vector bundles. Very often these objects have automorphisms, so it is natural (and often essential) to keep track of these automorphisms in the space X.

The formal definition of algebraic stack begins as follows. Fix a base scheme S. Endowing $\mathbf{Sch}_{/S}$ with the étale topology (see Example 2.2.5), the Yoneda embedding $j : \mathbf{Sch} \to \mathbf{Sh}(\mathbf{Sch}_{/S})$ identifies the category of schemes over S with a full subcategory of the category of sheaves on $\mathbf{Sch}_{/S}$, consisting of sheaves with a particularly nice local form. To define algebraic stack, we introduce potential automorphisms by replacing sheaves with stacks in groupoids.

We give a minimalist definition of algebraic stacks below. Note that for simplicity we have dropped a number of conditions usually found in the literature.

Definition A.7.1 (cf. [6], Definition 4.6, [1], Definition 5.1 and [29, Tag 026O])**.** Let X be a stack over $\mathbf{Sch}_{/S}$ with the étale topology. We say that X is *algebraic* or *Artin* if X satisfies the following conditions.

(1) The diagonal map $X \to X \times X$ is representable by algebraic spaces.

(2) There exists a scheme U and a smooth surjective morphism of stacks $U \to X$.

We say that X is *Deligne-Mumford* if there exists a scheme U and an étale surjective morphism $U \to X$.

Definition A.7.1 requires a few words of explanation. An *algebraic space* is a mild generalisation of a scheme. Roughly, an algebraic space over S is a well-behaved sheaf on $\mathbf{Sch}_{/S}$ which admits an étale surjective map from a scheme. For a full treatment, see [20]. We say that a morphism $X \to Y$ of stacks on $\mathbf{Sch}_{/S}$ is *representable by algebraic spaces* if for every morphism $U \to Y$ with U (the sheaf represented by) a scheme, the 2-fibre product $U \times_Y X$ is an algebraic space. We say that a morphism $U \to X$ is *smooth surjective* (resp. *étale surjective*) if for every morphism $V \to X$ with V a scheme, the 2-fibre product $U \times_X V \to V$ is smooth surjective (resp. étale surjective).

Remark A.7.2. Some authors use different conventions to ours when defining algebraic stacks. The variations include restricting the site $\mathbf{Sch}_{/S}$ to the subcategory of locally Noetherian schemes, and endowing $\mathbf{Sch}_{/S}$ with a finer topology, such as the fppf or fpqc topologies. The 2-category of algebraic stacks thus obtained is usually independent of these choices.

Remark A.7.3. Let X be a Deligne-Mumford stack. The existence of an étale surjection $U \to X$ with U a scheme implies that the points of X have only finite automorphism groups. If X is merely an Artin stack, then the automorphism groups of X may be finite dimensional smooth algebraic spaces.

Remark A.7.4. Let X be a Deligne-Mumford stack. Since there exists an étale surjective $U \to X$ with U a scheme, we can define the étale site of X as the category $X_{\acute{e}t}$ of schemes together with étale morphisms to X with an appropriate topology. Using $X_{\acute{e}t}$, we can generalise many concepts from scheme theory (such as quasi-coherent sheaves, vector bundles, cotangent sheaves, cotangent complexes, etc.) to Deligne-Mumford stacks. Artin stacks are somewhat trickier to work with, as they do not in general have a well-behaved étale site.

Example A.7.5 (cf. [6], Example 4.8)**.** Let X be an S-scheme and let G be a group scheme over S (i.e. a scheme together with a group structure on the functor of points) acting on X. The *quotient stack* $[X/G]$ is the stack associated to the fibred category $[X/G]_0$ over $\mathbf{Sch}_{/S}$ with

$$\mathrm{Ob}[X/G]_0(U) = X(U)$$

and

$$\mathrm{Hom}(x, y) = \{g \in G(U) \mid gx = y\}$$

for $U \in \mathrm{Ob}\,\mathbf{Sch}$ and $x, y \in X(U)$. If G is a finite group, then $[X/G]$ is a Deligne-Mumford stack. More generally, if G is a smooth group scheme over S, then $[X/G]$ is an Artin stack.

In Section 6.3, it is useful to have an extension of the intersection theory of Section A.6 to algebraic stacks. In [21], A. Kresch constructs Chow homology groups ([21], Definition 2.1.11) $A_p(X)$ for $p \in \mathbb{Z}$ and X an Artin stack of finite type over a base field k, and flat pullbacks ([21], Section 2.2) $f^* : A_p(Y) \to A_{p+n}(X)$ for $f : X \to Y$ a flat morphism of algebraic stacks of relative dimension $n \in \mathbb{Z}$. Here we note that a morphism $f : X \to Y$ of algebraic stacks is *flat of relative dimension* n if there exists a smooth surjective map $g : U \to X$ with U a scheme such that g has relative dimension m and $f \circ g : U \to Y$ is flat of relative dimension $m + n$.

Remark A.7.6. Unlike for schemes, the Chow homology groups $A_p(X)$ of an algebraic stack X do not necessarily vanish for $p < 0$, and we can have morphisms which are flat of negative relative dimension.

Appendix B

The cotangent complex

In this appendix, we review Illusie's theory of cotangent complexes on ringed sites. We recall the definition and main properties, and give explicit descriptions of many of the maps used in this thesis. The main purpose of including this appendix is to see that the cotangent complex has some stronger functoriality properties than those explicitly stated in [19].

B.1 Simplicial algebra

In this section, we recall the theory of simplicial objects in a category $\mathcal{D}$. Of particular interest to us are the case where $\mathcal{D}$ is an abelian category, and the case where $\mathcal{D} = \mathbf{Sh}(\mathcal{C}, \mathbf{CR})$ is the category of sheaves of commutative rings on a site $\mathcal{C}$.

In what follows, let $\mathbf{\Delta}$ be the category with objects the totally ordered sets $[0, n] = \{0, 1, 2, \ldots, n\}$ for $n \in \mathbb{Z}_{\geq 0}$, and morphisms the order-preserving maps.

Definition B.1.1. Let $\mathcal{D}$ be a category. The *category of simplicial objects in* $\mathcal{D}$ is the category $\mathbf{s}D$ of functors

$$X : \mathbf{\Delta}^{op} \to \mathcal{D}$$

and natural transformations between them. If X is a simplicial object in $\mathcal{D}$, we often denote the object $X([0, n])$ by X_n.

Remark B.1.2. Let $\mathcal{D}$ be a category. There is a fully faithful functor $\mathcal{D} \to \mathbf{s}\mathcal{D}$ which takes an object A of $\mathcal{D}$ to the constant functor

$$\begin{aligned} K(A, 0) : \mathbf{\Delta}^{op} &\to \mathcal{D} \\ [0, n] &\mapsto A. \end{aligned}$$

When there is no danger of confusion, we write A in place of $K(A, 0)$.

Definition B.1.3. Let $n \in \mathbb{Z}_{\geq 0}$. For each $i \in [0, n]$, the ith *coface map* is

$$\begin{aligned} d^i : [0, n-1] &\to [0, n] \\ k &\mapsto \begin{cases} k, & \text{if } k < i, \\ k+1, & \text{if } k \geq i, \end{cases} \end{aligned}$$

and the ith *codegeneracy map* is

$$\begin{aligned} s^i : [0, n+1] &\to [0, n] \\ k &\mapsto \begin{cases} k, & \text{if } k \leq i, \\ k-1, & \text{if } k > i. \end{cases} \end{aligned}$$

If X is a simplicial object in a category $\mathcal{D}$, d^i and s^i induce maps

$$d_i = X(d^i) : X_n \to X_{n-1}$$

and

$$s_i = X(s^i) : X_n \to X_{n+1}$$

called the *face* and *degeneracy maps* of X.

It is elementary and well-known that the coface and codegeneracy maps generate the category $\mathbf{\Delta}$. It follows that to define a simplicial object X in $\mathcal{D}$, it suffices to define objects X_n and maps $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ satisfying appropriate relations. If X and Y are simplicial objects in $\mathcal{D}$, a morphism $f : X \to Y$ consists of a collection $\{f_n : X_n \to Y_n\}_{n \in \mathbb{Z}_{\geq 0}}$ of morphisms in $\mathcal{D}$, which commute with the face and degeneracy maps of X and Y.

We now consider the case where $\mathcal{D} = \mathcal{A}$ is an abelian category. In this case, there is an intimate connection between simplicial objects in $\mathcal{A}$ and complexes in $\mathcal{A}$.

Remark B.1.4. The notion of complex introduced in Section 4.1 is sometimes called a *cochain complex.* A *chain complex* in $\mathcal{A}$ is a sequence

$$M = [\cdots \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} M_0 \xrightarrow{\partial_0} M_{-1} \xrightarrow{\partial_{-1}} M_{-2} \xrightarrow{\partial_{-2}} \cdots],$$

where $M_i \in \mathrm{Ob}(\mathcal{A})$ and $\partial_i \circ \partial_{i-1} = 0$ for all $i \in \mathbb{Z}$. We can regard any cochain complex M as a chain complex by setting

$$M_i = M^{-i} \quad \text{and} \quad \partial_i = d^{-i}$$

for all $i \in \mathbb{Z}$. When working with chain complexes instead of cochain complexes, we also write $C_{\leq n}(\mathcal{A})$ and $C_{\geq n}(\mathcal{A})$ in place of $C^{\geq -n}(\mathcal{A})$ and $C^{\leq -n}(\mathcal{A})$ respectively.

Let M be a simplicial object in $\mathcal{A}$. The *normalised chain complex* of M is the chain complex $N(M)$ with nth term

$$N(M)_n = \bigcap_{i=0}^{n-1} \ker d_i$$

and differential

$$\partial_n = (-1)^n d_n : N(M)_n \to N(M)_{n-1}$$

where $d_i : M_n \to M_{n-1}$ is the ith boundary map for M. The relations in $\mathbf{\Delta}$ ensure that $\partial_n \circ \partial_{n+1} = 0$, so $N(M)$ is indeed a chain complex. This construction defines a functor,

$$N : \mathbf{s}\mathcal{A} \to C_{\geq 0}(\mathcal{A}),$$

from simplicial objects in $\mathcal{A}$ to chain complexes in $\mathcal{A}$. The key result is the following.

Theorem B.1.5 (Dold-Kan correspondence)**.** *The functor $N : \mathbf{s}\mathcal{A} \to C_{\geq 0}(\mathcal{A})$ is an equivalence of categories.*

A reference for this result in the case that $\mathcal{A} = \mathbf{Ab}$ is the category of abelian groups is [12], Corollary III.2.3. The proof in the general case is identical.

Example B.1.6. Let M be any object in $\mathcal{A}$. By Remark B.1.2, we can think of M as a constant simplicial object, which we also denote by M. In this case, the normalised chain complex of M is

$$[\cdots \to 0 \to M \to 0 \to \cdots],$$

with M placed in degree 0.

For the remainder of this section, we fix a site $\mathcal{C}$.

The advantage of working with simplicial objects is that they provide us with a reasonable generalisation of complexes that works in non-abelian settings. Of particular interest to us is the category $\mathbf{sSh}(\mathcal{C}, \mathbf{CR})$ of simplicial sheaves of rings over $\mathcal{C}$. For brevity, we will write $\mathbf{sCR}(\mathcal{C})$ in place of $\mathbf{sSh}(\mathcal{C}, \mathbf{CR})$, and we will call its objects *simplicial rings over* $\mathcal{C}$.

Definition B.1.7. Let A be a simplicial ring over $\mathcal{C}$, and let $n \in \mathbb{Z}_{\geq 0}$. The nth *homotopy sheaf* of A is the sheaf of abelian groups

$$\pi_n(A) = H_n(N(A)),$$

where $N(A)$ is the normalised chain complex of the simplicial sheaf of abelian groups underlying A. We say that a morphism $f : A \to B$ of simplicial rings on $\mathcal{C}$ is a *weak equivalence* if the induced map $\pi_n(f) : \pi_n(A) \to \pi_n(B)$ is an isomorphism for all $n \in \mathbb{Z}_{\geq 0}$.

Remark B.1.8. Let A be a simplicial ring over $\mathcal{C}$, and denote by $\partial_n : N(A)_n \to N(A)_{n-1}$ the differential of the normalised chain complex $N(A)$. Then

$$\ker \partial_0 = N(A)_0 = A_0$$

and $\operatorname{im} \partial_1$ is a sheaf of ideals in A_0, so

$$\pi_0(A) = \frac{\ker \partial_0}{\operatorname{im} \partial_1}$$

naturally has the structure of a sheaf of rings. It is immediate that if $f : A \to B$ is a morphism of simplicial rings over $\mathcal{C}$, then $\pi_0(f) : \pi_0(A) \to \pi_0(B)$ is a morphism of sheaves of rings on $\mathcal{C}$.

Definition B.1.9. Let A be a simplicial ring over $\mathcal{C}$. An A-*module* is a simplicial abelian group M over $\mathcal{C}$ together with a morphism $A \times M \to M$ of simplicial sheaves, such that for each $n \in \mathbb{Z}_{\geq 0}$ the induced map $A_n \times M_n \to M_n$ gives M_n the structure of an A_n-module. If M and N are A-modules, a *morphism of* A-*modules* is a morphism $f : M \to N$ of simplicial sheaves of abelian groups such each $f_n : M_n \to N_n$ is a morphism of A_n-modules. We denote the category of A-modules by A-**mod**.

Remark B.1.10. If A is a sheaf of rings on $\mathcal{C}$, then there is a canonical equivalence of categories

$$\mathbf{s}A\textbf{-mod} \simeq K(A,0)\textbf{-mod},$$

where $K(A, 0)$ denotes the constant simplicial ring given by A.

B.2 The cotangent complex

In this section, we recall Illusie's definition of the cotangent complex of a morphism of sheaves of rings on a site $\mathcal{C}$. Roughly, if A is a sheaf of rings, the cotangent complex of an A-algebra B is the left derived functor of the module of Kähler differentials introduced in Section A.4. It is constructed by taking a simplicial resolution of B by free A-algebras and taking Kähler differentials termwise. It will be useful to introduce free simplicial resolutions in the following general setting.

Definition B.2.1. Let $\mathcal{C}$ be a site and let A be a simplicial ring over $\mathcal{C}$. An *A-algebra* is a simplicial ring B over $\mathcal{C}$ together with a morphism $A \to B$. We say that B is *termwise free* if for each $n \in \mathbb{Z}_{\geq 0}$, there exists a sheaf F_n on $\mathcal{C}$ and an isomorphism of A_n-algebras

$$B_n \cong A_n[F_n],$$

where $A_n[F_n]$ is the free A_n-algebra generated by the sheaf F_n. If B is any A-algebra, a *free resolution* of B is weak equivalence $\tilde{B} \to B$, such that $\tilde{B}$ is a termwise free A-algebra.

Proposition B.2.2. *Let $\mathcal{C}$ be a site and let A be a simplicial ring over $\mathcal{C}$. Then every A-algebra has a free resolution.*

Let $\mathcal{C}$ be a site and let $A \to B$ be a morphism of sheaves of rings on $\mathcal{C}$. If $\tilde{B} \to B$ is a free resolution of B over A (considered as a constant simplicial ring) then we can form the $\tilde{B}$-module $\Omega_{\tilde{B}/A}$ of Kähler differentials, given by

$$\left(\Omega_{\tilde{B}/A}\right)_n = \Omega_{B_n/A}.$$

Tensoring with B gives a simplicial B-module

$$\Omega_{\tilde{B}/A} \otimes_{\tilde{B}} B.$$

Definition B.2.3. The *cotangent complex* of B over A is the normalised chain complex

$$L_{B/A} = N(\Omega_{\tilde{B}/A} \otimes_{\tilde{B}} B).$$

The following proposition shows that as an object in $D(B)$, $L_{B/A}$ is independent of the choice of free resolution $\tilde{B}$.

Proposition B.2.4. *Let $A \to B$ be a morphism of sheaves of rings over a site $\mathcal{C}$. If $\tilde{B}_1 \to B$ and $\tilde{B}_2 \to B$ are free resolutions of B, then there is a canonical isomorphism*

$$N(\Omega_{\tilde{B}_1/A} \otimes_{\tilde{B}_1} B) \to N(\Omega_{\tilde{B}_2/A} \otimes_{\tilde{B}_2} B)$$

in $D(B)$.

Remark B.2.5. Let $A \to B$ be a morphism of rings on a site $\mathcal{C}$. It follows from Proposition A.4.9 that there is a canonical isomorphism

$$H^0(L_{B/A}) \to \Omega_{B/A}.$$

Definition B.2.6. Let $\pi : X \to S$ be a morphism of schemes. The *cotangent complex of X over S* is

$$L_{X/S} = L_{\mathcal{O}_X/\pi^{-1}\mathcal{O}_S},$$

where $L_{\mathcal{O}_X/\pi^{-1}\mathcal{O}_S}$ is the cotangent complex of $\mathcal{O}_X$ over $\pi^{-1}\mathcal{O}_S$ on the Zariski site X_{Zar}.

Remark B.2.7. The cotangent complex of schemes is defined using the Zariski site X_{Zar}. If $L_{X_{ét}/S_{ét}}$ denotes the cotangent complex of $\mathcal{O}_{X_{ét}}$ over $\pi^{-1}\mathcal{O}_{S_{ét}}$ on $X_{ét}$, then there is a canonical map $z^*L_{X/S} \to L_{X_{ét}/S_{ét}}$ where $z : (X_{ét}, \mathcal{O}_{X_{ét}}) \to (X_{Zar}, \mathcal{O}_X)$ is the morphism of ringed sites of Example 2.3.6. It follows from [19], Chapitre III, Proposition 3.1.1 that this map is an isomorphism in $D(\mathcal{O}_{X_{ét}})$. We will not usually distinguish between $L_{X/S}$ and $L_{X_{ét}/S_{ét}}$.

B.3 Functoriality and exactness

In this section we study the functoriality and exactness properties of the cotangent complex. For our purposes, we need somewhat stronger coherence properties than are stated in [19], which is the main motivation for including this Appendix.

A full treatment of these coherence properties is beyond the scope of this work. For a detailed account using ∞-categories, see [23], Section 8.3. For our purposes, it will suffice to use the homotopy coherence formalism introduced in Section 4.4.

Let $\mathcal{C}$ be a site, and let K be a small category. As in Section 2.5, define a topology on the product category $\mathcal{C} \times K^{op}$ by declaring a family $\{(f_i, g_i) : (U_i, k_i) \to (U, k)\}_{i \in I}$ to be a covering if $\{f_i : U_i \to U\}_{i \in J}$ is a covering in $\mathcal{C}$, where

$$J = \{i \in I \mid g_i \text{ is an isomorphism}\}.$$

Then for any category $\mathcal{D}$, there is a canonical equivalence of categories

$$\mathbf{Sh}(\mathcal{C}, \mathcal{D})^K \simeq \mathbf{Sh}(\mathcal{C} \times K^{op}, \mathcal{D}).$$

In particular, if $A_\bullet \to B_\bullet$ is a morphism of diagrams of sheaves of rings on $\mathcal{C}$, then we can consider the cotangent complex

$$L_{B_\bullet/A_\bullet} \in \operatorname{Ob} D(B_\bullet).$$

This object is constructed by taking free resolutions of sheaves of rings on $\mathcal{C}$ at each place in the diagram to obtain a commutative diagram of simplicial rings, then taking modules of Kähler differentials at each place. The following proposition is clear from the construction.

Proposition B.3.1. *In the setup above, for each $k \in \operatorname{Ob} K$, the "evaluation at k" functor,*

$$D(B_\bullet) \to D(B_k),$$

takes $L_{B_\bullet/A_\bullet}$ to L_{B_k/A_k}.

Consider the special case where K is the commutative square below.

$$\begin{array}{ccc} 1 & \longrightarrow & 2 \\ \downarrow & & \downarrow \\ 3 & \longrightarrow & 4 \end{array} \tag{B.3.1}$$

Then a morphism $A_\bullet \to B_\bullet$ of diagrams of shape K is a commutative diagram of sheaves of rings as follows.

$$\begin{array}{ccccc} A_1 & & \longrightarrow & & A_2 & & \\ \downarrow & \searrow & & & \downarrow & \searrow & \\ & & A_3 & & \longrightarrow & & A_4 \\ B_1 & & \downarrow & \longrightarrow & B_2 & & \downarrow \\ & \searrow & & & & \searrow & \\ & & B_3 & & \longrightarrow & & B_4 \end{array}$$

A $B_\bullet$-module is a commutative diagram

$$\begin{array}{ccc} M_1 & \longrightarrow & M_2 \\ \downarrow & & \downarrow \\ M_3 & \longrightarrow & M_4, \end{array} \tag{B.3.2}$$

where each M_i is a B_i-module, and each $M_i \to M_j$ is a homomorphism of B_i-modules. There is a right exact functor $B_\bullet$**-mod** $\to B$**-mod**$^\square$ to the category of commutative squares in B**-mod**, which takes a diagram above to the diagram

$$\begin{array}{ccc} M_1 \otimes_{B_1} B_4 & \longrightarrow & M_2 \otimes_{B_2} B_4 \\ \downarrow & & \downarrow \\ M_3 \otimes_{B_3} B_4 & \longrightarrow & M_4. \end{array}$$

This has a left derived functor,

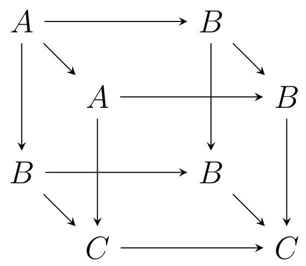
$$D(B_\bullet) \to D(B_4^\square), \tag{B.3.3}$$

defined via flat resolutions. The image of $L_{B_\bullet/A_\bullet}$ under this functor is a homotopy coherent diagram

$$\begin{array}{ccc} L_{B_1/A_1}\overset{\mathrm{L}}{\otimes}_{B_1} B_4 & \longrightarrow & L_{B_2/A_2}\overset{\mathrm{L}}{\otimes}_{B_2} B_4 \\ \downarrow & & \downarrow \\ L_{B_3/A_3}\overset{\mathrm{L}}{\otimes}_{B_3} B_4 & \longrightarrow & L_{B_4/A_4}, \end{array}$$

where $\overset{\mathrm{L}}{\otimes}$ denotes the left derived functor of tensor product.

As a special case, suppose we have morphisms of rings $A \to B \to C$ over $\mathcal{C}$. Then the diagram



gives a homotopy coherent square as follows.

$$\begin{array}{ccc} L_{B/A}\overset{\mathrm{L}}{\otimes}_B C & \longrightarrow & L_{B/B}\overset{\mathrm{L}}{\otimes}_B C \\ \downarrow & & \downarrow \\ L_{C/A} & \longrightarrow & L_{C/B} \end{array}$$

Since we plainly have $L_{B/B} = \Omega_{B/B} = 0$, this gives a coherent triangle

$$L_{B/A}\overset{\mathrm{L}}{\otimes}_B C \to L_{C/A} \to L_{C/B}$$

in $D(C)$.

Proposition B.3.2 (cf. [19], II.2.1.2.1)**.** *The coherent triangle*

$$L_{B/A}\overset{\mathrm{L}}{\otimes}_B C \to L_{C/A} \to L_{C/B}$$

is exact.

Corollary B.3.3. *Let* $X \xrightarrow{f} Y \to S$ *be morphisms of schemes. Then there is a canonical exact triangle,*

$$Lf^* L_{X/S} \to L_{Y/S} \to L_{Y/X},$$

in $D(Y)$*.*

B.4 Application to deformation theory

In this section, we recall the key points in the proof of the following theorem.

Theorem B.4.1 (cf. [19] Théorème III.1.2.3 and [25], Theorem A.7)**.** *Let* $\mathcal{C}$ *be a site, let* $A \to B$ *be a morphism of rings over* $\mathcal{C}$ *and let* J *be a* B*-module. Then there is a natural equivalence*

$$\mathrm{ch}(R\underline{\mathrm{Hom}}^\bullet(L_{B/A}, J[1])) \simeq \underline{\mathbf{Ext}}_A(B, J)$$

of stacks on $\mathcal{C}$*. Here* $\underline{\mathbf{Ext}}_A(B, J)$ *denotes the stack of square-zero extensions of* B *by* J *over* A*.*

Sketch of proof. The proof is in two main steps.

The first step is to construct a natural equivalence

$$\mathrm{ch}(R\underline{\mathrm{Hom}}^\bullet(\Omega, J[1])) \simeq \underline{\mathbf{Ext}}(\Omega, J)$$

for Ω a complex of B-modules and J a B-module. Here $\underline{\mathbf{Ext}}(\Omega, J)$ is the stack of extensions

$$0 \to J \to \tilde{\Omega} \to \Omega \to 0$$

of Ω by J. This equivalence is constructed in two stages. First, we construct a natural map

$$\mathrm{pch}(\underline{\mathrm{Hom}}^\bullet(\Omega, I[1])) \to \underline{\mathbf{Ext}}(\Omega, I)$$

for any B-module I. This takes a morphism of complexes $\Omega \to I[1]$ (a section of $\mathrm{pch}(\underline{\mathrm{Hom}}^\bullet(\Omega, I[1]))$) to the extension with

$$\tilde{\Omega} = \Omega \times_{I[1]} [I \to I].$$

A morphism of sections f and g of $\mathrm{pch}(\underline{\mathrm{Hom}}^\bullet(\Omega, I[1]))$ can be identified with a morphism of complexes $\Omega \to [I \to I]$, which determines a morphism of extensions

$$\Omega \times_{I[1],f} [I \to I] \to \Omega \times_{I[1],g} [I \to I].$$

The induced morphism of stacks,

$$\mathrm{ch}(\underline{\mathrm{Hom}}^\bullet(\Omega, I[1])) \to \underline{\mathbf{Ext}}(\Omega, I),$$

is fully faithful in general, and an equivalence for I injective. Now choose an injective resolution $\tilde{J}$ for J and write

$$[I^{-1} \to I^0] = \tau^{\leq 0}(\tilde{J}[1]).$$

Then

$$\mathrm{ch}(R\underline{\mathrm{Hom}}^\bullet(\Omega, J[1])) = \mathrm{ch}(\underline{\mathrm{Hom}}^\bullet(\Omega, [I^{-1} \to I^0])).$$

We have a 2-commutative square

$$\begin{array}{ccc} \mathrm{ch}(\underline{\mathrm{Hom}}^\bullet(\Omega, I^{-1}[1])) & \longrightarrow & \mathrm{ch}(\underline{\mathrm{Hom}}^\bullet(\Omega, I^0[1])) \\ \downarrow & & \downarrow \\ \underline{\mathbf{Ext}}(\Omega, I^{-1}) & \longrightarrow & \underline{\mathbf{Ext}}(\Omega, I^0) \end{array}$$

where the left and right vertical arrows are an equivalence and fully faithful respectively. Hence, we get an equivalence from the fibre of the top row (over the zero section of $\underline{\mathrm{Hom}}^\bullet(\Omega, I^{-1})$) to the fibre of the bottom row (over the trivial extension). But the fibre of the bottom row is simply

$$\underline{\mathbf{Ext}}(\Omega, J)$$

and, from the exact triangle

$$\underline{\mathrm{Hom}}^\bullet(\Omega, [I^{-1} \to I^0]) \to \underline{\mathrm{Hom}}^\bullet(\Omega, I^{-1}[1]) \to \underline{\mathrm{Hom}}^\bullet(\Omega, I^0[1]),$$

the fibre of the top row is

$$\mathrm{ch}(\underline{\mathrm{Hom}}^\bullet(\Omega, [I^{-1} \to I^0])) = \mathrm{ch}(R\underline{\mathrm{Hom}}^\bullet(\Omega, J[1]))$$

by Corollary 4.5.3. Thus we have an equivalence

$$\mathrm{ch}(R\underline{\mathrm{Hom}}^\bullet(\Omega, J[1])) \simeq \underline{\mathbf{Ext}}(\Omega, J)$$

as claimed.

The second step of the proof is to construct a natural equivalence

$$\underline{\mathbf{Ext}}_A(B, J) \to \underline{\mathbf{Ext}}(L_{B/A}, J).$$

Fix a free resolution $\tilde{B} \to B$ of B over A. Given a square-zero extension

$$0 \to J \to C \to B \to 0$$

of B, we get a square-zero extension

$$0 \to J \to C \times_B \tilde{B} \to \tilde{B} \to 0,$$

of $\tilde{B}$, where we regard J as a constant simplicial B-module. Since $\tilde{B}$ is termwise free over A, by Proposition A.4.9, we get an exact sequence

$$0 \to J \to \Omega_{C\times_B \tilde{B}/A} \otimes \tilde{B} \to \Omega_{\tilde{B}/A} \to 0.$$

Tensoring with B gives an exact sequence

$$0 \to J \to \Omega_{C\times_B \tilde{B}/A} \otimes B \to \Omega_{\tilde{B}/A} \otimes_{\tilde{B}} B \to 0$$

since $\Omega_{\tilde{B}/A}$ is a (termwise) flat $\tilde{B}$-module. Taking normalised chain complexes gives an extension

$$0 \to J \to N(\Omega_{C\times_B \tilde{B}/A} \otimes B) \to L_{B/A} \to 0,$$

which gives a section of $\underline{\mathbf{Ext}}(L_{B/A}, J)$.

To construct the inverse equivalence, take an extension

$$0 \to J \to \tilde{\Omega}_0 \to \Omega_{\tilde{B}/A} \otimes_{\tilde{B}} B \to 0$$

corresponding via the Dold-Kan correspondence (Theorem B.1.5) to an extension of $L_{B/A}$ by J. Pulling back under the map $\Omega_{\tilde{B}/A} \to \Omega_{\tilde{B}/A} \otimes_{\tilde{B}} B$ gives an extension

$$0 \to J \to \tilde{\Omega} \to \Omega_{\tilde{B}/A} \to 0$$

and hence a square-zero extension

$$0 \to J \to \tilde{B} \oplus \tilde{\Omega} \to \tilde{B} \oplus \Omega_{\tilde{B}/A} \to 0$$

of simplicial A-algebras. Pulling back under the homomorphism

$$\tilde{B} \xrightarrow{(\mathrm{id},d)} \tilde{B} \oplus \Omega_{\tilde{B}/A},$$

where d denotes the universal derivation, we get a square-zero extension

$$0 \to J \to \tilde{C} \to \Omega_{\tilde{B}/A} \to \tilde{B} \to 0,$$

where

$$\tilde{C} = \tilde{B} \oplus \tilde{\Omega} \times_{\tilde{B}\oplus\Omega_{\tilde{B}/A}} \tilde{B}.$$

Taking π_0, this gives a square-zero extension

$$0 \to J \to C \to B \to 0$$

with $C = \pi_0(\tilde{C})$.

Thus we have the desired natural equivalence

$$\mathrm{ch}(R\underline{\mathrm{Hom}}^\bullet(L_{B/A}, J[1])) \simeq \underline{\mathbf{Ext}}(L_{B/A}, J) \simeq \underline{\mathbf{Ext}}_A(B, J).$$

□

Remark B.4.2. The equivalence of Theorem B.4.1 is natural in A, B and J. We can make precise the naturality in A as follows. Given a fixed ring B on $\mathcal{C}$, we have an object

$$L_{B/A_\bullet} \in D(B^K)$$

for any diagram $A_\bullet$ of rings on $\mathcal{C}$ of shape K and any morphism $A_\bullet \to B$. This gives a 2-commutative diagram of stacks

$$\mathrm{ch}(R\underline{\mathrm{Hom}}^\bullet(L_{B/A_\bullet}, J[1])) \in \mathrm{Ob}\,\mathrm{Ho}(\mathbf{St}(\mathcal{C})^{K^{op}})$$

for any B-module J. Following through the proof of Theorem B.4.1, we get an equivalence

$$\mathrm{ch}(R\underline{\mathrm{Hom}}^\bullet(L_{B/A_\bullet}, J[1])) \simeq \underline{\mathbf{Ext}}_{A_\bullet}(B, J),$$

where $\underline{\mathbf{Ext}}_{A_\bullet}(B, J)$ is the strictly commutative diagram of stacks obtained from the forgetful maps. The naturality in B and J is similar.

B.5 Truncated cotangent complexes of schemes

In this section, we describe a computation of the truncated cotangent complex $\tau^{\geq -1} L_{X/S}$ of a morphism of schemes $X \to S$.

The main algebraic result is the following.

Theorem B.5.1 (cf. [19], Corollaire III.1.2.9.1)**.** *Let $\mathcal{C}$ be a site and let $A \to B \to C$ be morphisms of sheaves of rings on $\mathcal{C}$. If $B \to C$ is surjective with kernel $I \subseteq B$, then there is a canonical morphism*

$$L_{C/A} \to [I/I^2 \to \Omega_{B/A} \otimes_B C].$$

Moreover, if the natural map $L_{B/A} \to \Omega_{B/A}$ is an isomorphism and $\Omega_{B/A}$ is a flat B-module, then the above map induces an isomorphism

$$\tau^{\geq -1} L_{C/A} \to [I/I^2 \to \Omega_{B/A} \otimes_B C]$$

in $D(C)$.

Sketch of proof. We recall the construction of the morphism

$$L_{C/A} \to [I/I^2 \to \Omega_{B/A} \otimes_B C].$$

Form the square-zero extension

$$0 \to I/I^2 \to B/I^2 \to C \to 0$$

of C by I/I^2. Using the proof of Theorem B.4.1, this induces an exact sequence

$$0 \to I/I^2 \to N(\Omega_{D/A} \otimes_D C) \to L_{C/A} \to 0$$

of complexes of C-modules, where D is the simplicial ring

$$D = B/I^2 \times_C \tilde{C}$$

for some free resolution $\tilde{C} \to C$ of C. Thus, we have a quasi-isomorphism

$$\mathrm{Cone}(I/I^2 \to N(\Omega_{D/A} \otimes_D C)) \to L_{C/A}.$$

The commutative diagram

$$\begin{array}{ccccc} I/I^2 & \longrightarrow & D & & \\ \downarrow & & \downarrow & \searrow & \\ & & & & C \\ & & & \nearrow & \\ I/I^2 & \longrightarrow & B & & \end{array}$$

induces a commutative diagram of complexes

$$\begin{array}{ccc} I/I^2 & \longrightarrow & N(\Omega_{D/A} \otimes_D C) \\ \downarrow & & \downarrow \\ I/I^2 & \longrightarrow & \Omega_{B/A} \otimes_B C. \end{array}$$

Thus, we get a morphism

$$\mathrm{Cone}(I/I^2 \to N(\Omega_{D/A} \otimes_D C)) \to \mathrm{Cone}(I/I^2 \to \Omega_{B/A} \otimes_B C),$$

which gives the desired morphism

$$L_{C/A} \to [I/I^2 \to \Omega_{B/A} \otimes_B C].$$

□

Proposition B.5.2 (cf. [19], Corollaire III.3.1.3 and Corollaire III.3.2.7)**.** *Let S be a scheme and let $i : X \to Y$ be a closed embedding of S-schemes with ideal sheaf $I \subseteq i^{-1}\mathcal{O}_Y$. If Y is smooth over S, then there is an isomorphism*

$$\tau^{\geq -1} L_{X/S} \simeq [I/I^2 \to i^*\Omega_{Y/S}]$$

in $D(X)$. If i is a regular embedding, then this gives an isomorphism

$$L_{X/S} \simeq [I/I^2 \to i^*\Omega_{Y/S}].$$

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