Group theory for Feynman diagrams in non-Abelian gauge theories*

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A simple and systematic method for the calculation of group-theoretic weights associated with Feynman diagrams in non-Abelian gauge theories is presented. Both classical and exceptional groups are discussed.

I. INTRODUCTION

The increased interest in non-Abelian gauge theories has in recent years led to the computation of many higher-order Feynman diagrams.[1-11] Asymptotic form-factor and scattering amplitude calculations are of special interest, because they suggest that it might be possible to sum up diagrams with arbitrary numbers of soft gluons just as one can sum up soft-photon processes in QED. In such a program the analysis of the momentum integrals proceeds by the traditional techniques developed for QED calculations. The new aspect, characteristic of non-Abelian gauge theories, is the emergence of a group-theoretic weight (or weight,[12] for short) associated with each Feynman diagram. The dramatic cancellations among various diagrams occur through interplay of their group-theoretic weights and their momentum-space integrals. So the study of weights becomes of interest, as it might suggest cancellation patterns needed for summations of diagrams.

In this paper we give a general method for computing group-theoretic weights, and give explicit rules for SU(n), SO(n), Sp(n), G_2, E_6, F_4, and E_7 symmetry groups. We restrict ourselves to the models with quarks in the defining (lowest dimensional) representation, but the method can be extended to higher representations. As only global symmetry is assumed, we can compute weights not only in symmetric gauge theories, but also in those spontaneously broken gauge theories where all particles within a multiplet have the same mass.

The evaluation procedure is very simple. We think of the weight itself as a Feynman integral (over a discrete lattice), and introduce Feynman diagrammatic notation to replace the unwieldy algebraic expressions. Then we give two relations; the first eliminates all three-gluon vertices, and the second eliminates all internal gluon lines. The result is a sum over a unique set of irreducible group-theoretic tensors which form a natural basis for all Lie algebras. All this is accomplished without recourse to any explicit representation of the group generators and structure constants. As a by-product, we learn how to count quickly the number of invariant couplings for arbitrary numbers of quarks and gluons, thus avoiding involved reductions of outer products of representations by Young tableaux.

In most calculations, one looks for properties which arise solely from gauge invariance, and there the explicit numerical values of weights should really not be necessary. While in some such calculations[1-4,13] it is appealing to express simple diagrams in terms of quadratic Casimir operators (so that the form of the expression is independent of the particular gauge group and the particular representation), for higher-order diagrams there is no simple way of relating weights to generalized Casimir operators,[14,15] and such an approach becomes very cumbersome. Then the explicit expressions for weights might be both suggestive and useful as checks for the cancellations among various diagrams. Another application of explicit weight expressions is $1/n$ expansions[16] for which the above evaluation method gives simple and direct estimates.[17]

Possibly, a novel aspect of this paper is its treatment of exceptional groups. It is known[18-22] that exceptional groups arise from invariance of norms defined on octonion spaces, but the demonstration is rather difficult (it involves Jordan algebras over octonionic matrices). We skirt the complexities of this underlying structure by giving a formulation of exceptional groups purely in terms of the geometrical properties of their defining representations. Intuition so developed might be of use to quark-model builders. We give the following example: Because SU(3) has a cubic invariant $\epsilon^{abc}q_a q_b q_c$, it is possible to build a three-quark color singlet with desirable phenomenological properties.[23] Are there any other groups that could accommodate three-quark color singlets? It turns out that the defining representations of G_2, F_4, and E_6 are among groups with such invariants. A systematic discussion of such invariants shall be given elsewhere.[24]

In the past, most weight calculations have involved SU(n) and, even more specifically, SU(3). This has led to the development of methods specific to SU(n).[25-35] For the sake of completeness and comparison, we pursue this traditional line for a while and find ourselves at an impasse.

The organization of the paper is as follows. In Sec. II, we state the evaluation rules. In Sec. III, we introduce diagrammatic notation and derive various relationships true for all Lie groups, while particular groups are defined in Sec. IV. An example of weight evaluation is given in Sec. V. In Sec. VI, we discuss group-theoretic tensor bases and relations between basis tensors for specific representations, while higher representations are touched upon in Sec. VII. Full Feynman rules are stated in Appendix A. Appendix B is a long discussion of an older method of weight evaluation, specific to SU(n). For readers interested only in models with classical symmetry groups, Figs. 1–3 summarize all that is needed for weight computation.

II. RULES FOR THE EVALUATION OF GROUP-THEORETIC WEIGHTS

For our model we take a Yang-Mills theory with massive quarks of n colors and N massless gluons, defined by the classical Lagrangian density

$$\mathcal{L} = -\tfrac{1}{4}F_i^{\mu\nu}F_{i\mu\nu} + \bar{q}(i\not{D} - m)q\,,$$

$$F_i^{\mu\nu} = \partial^\mu A_i^\nu - \partial^\nu A_i^\mu + C_{ijk}A_j^\mu A_k^\nu\,, \tag{2.1}$$

$$D_a^{\mu b} = \delta_a^b\partial^\mu - iA_i^\mu (T_i)_a^b\,,$$

$$a,b=1,2,\ldots,n,\quad i,j=1,2,\ldots,N$$

where the n complex quark fields q_a transform as the defining (lowest-dimensional cogredient) representation of a compact simple N-dimensional Lie group $\mathcal{G}$, and the N Hermitian gluon fields A_i transform as its adjoint (regular) representation. In Yang-Mills theory the coupling constant e of the usual QED is generalized to quark-gluon coupling matrices $(T_i)_a^b$. They are generators of $\mathcal{G}$, close a Lie algebra

$$[T_i, T_j] = iC_{ijk}T_k\,, \tag{2.2}$$

$$\mathrm{Tr}\,T_i = 0\,, \tag{2.3}$$

and can be chosen to satisfy a normalization condition[36]

$$\mathrm{Tr}(T_iT_j) = a\delta_{ij}\,. \tag{2.4}$$

For example, for SU(n), the conventional choice[37] is $T_i = \frac{1}{2}g\lambda_i$ and $a = \frac{1}{2}g^2$. In this paper a shall remain arbitrary throughout. $\sqrt{a}$ can be thought of as the overall coupling constant for a simple group $\mathcal{G}$, and powers of $\sqrt{a}$ count the number of vertices in a diagram. That the gluon self-couplings $-iC_{ijk}$ also scale as $\sqrt{a}$ is evident from (2.2).

The Lagrangian (2.1) generates the usual Feynman diagrams. There is no mixing between the spacetime and the gauge group $\mathcal{G}$, and the Feynman amplitude associated with a diagram G factorizes into $W_G M_G$, where W_G is the group-theoretic weight consisting of various $(T_i)_a^b$ and C_{ijk}, and M_G arises from the integrals over internal momenta and is similar to QED Feynman amplitudes. Even though M_G will not concern us in this paper, we give the rules for its computation in Appendix A. We note that while in momentum space there are four-gluon vertices, for W_G there exist only three-gluon couplings, because the group-theoretic factors in a four-gluon vertex have the form $C_{ijk}C_{klm}$.

The group-theoretic weight W_G is a product of the following factors (all repeated indices are summed over):

(a) for each internal quark line, a factor δ_a^b, $a,b = 1,2,\ldots,n$,

(b) for each internal gluon or ghost line, a factor δ_{ij}, $i,j = 1,2,\ldots,N$,

(c) for each quark-quark-gluon vertex, a factor $(T_i)_a^b$,

(d) for each three-gluon or ghost-ghost-gluon vertex, a factor $-iC_{ijk}$,

(e) for the four-gluon vertex, the factors
$-(C_{imj}C_{lmk} + C_{iml}C_{jmk})$ (multiplying $g_{\lambda\nu}g_{\mu\xi}$),
$-(C_{imk}C_{jml} + C_{imj}C_{kml})$ (multiplying $g_{\lambda\xi}g_{\mu\nu}$),
$-(C_{iml}C_{kmj} + C_{imk}C_{lmj})$ (multiplying $g_{\lambda\mu}g_{\xi\nu}$),
where gluon group and Lorentz indices are paired as (i,λ), (j,μ), (k,ν), and (l,ξ) (see also Fig. 24).

The weight W_G for an arbitrary Feynman amplitude G is evaluated in two steps:

(1) Reexpress all three-gluon vertices $-iC_{ijk}$ in terms of the defining representation:

(a) If $\mathcal{G}$ is SU(n) or E_6

$$iC_{ijk} = \frac{1}{a}\mathrm{Tr}(T_iT_jT_k - T_kT_jT_i)\,; \tag{2.5}$$

(b) if $\mathcal{G}$ is SO(n), Sp(n), G_2, F_4, or E_7

$$iC_{ijk} = \frac{2}{a}\mathrm{Tr}(T_iT_jT_k)\,. \tag{2.6}$$

(2) Eliminate all internal gluon lines $\cdots(T_i)_d^c\cdots(T_i)_b^a\cdots$ by replacing them with gluon projection operators:

$$\frac{1}{a}(T_i)_b^a(T_i)_d^c = \begin{cases} \delta_d^a\delta_b^c - \frac{1}{n}\delta_b^a\delta_d^c & \text{for SU}(n), \quad (2.7)\\ \frac{1}{2}(\delta_d^a\delta_b^c - \delta^{ac}\delta_{bd}) & \text{for SO}(n), \quad (2.8)\\ \frac{1}{2}(\delta_d^a\delta_b^c + f^{ac}f_{bd}) & \text{for Sp}(n), \quad (2.9)\end{cases}$$

with n even, $f^{ac} = -f^{ca}$, $f^{ac}f_{cb} = \delta_b^a$.

$$\frac{1}{a}(T_i)^a_b(T_i)^c_d=\begin{cases}\frac{1}{2}(\delta^a_d\delta^c_b-\delta^{ac}\delta_{bd})-\frac{1}{\alpha}f^a{}_{be}f^{ec}{}_d \text{ for } G_2, & (2.10)\\ \frac{1}{6}\delta^a_d\delta^c_b+\frac{1}{18}\delta^a_b\delta^c_d-\frac{5}{3\alpha}d^{ace}d_{ebd} \text{ for } E_6, & (2.11)\\ \frac{1}{9}(\delta^a_d\delta^c_b-\delta^{ac}\delta_{bd}) -\frac{7}{9\alpha}(d^{ace}d_{ebd}-d^a{}_{de}d^{ec}{}_b) \text{ for } F_4, & (2.12)\\ \frac{1}{24}\left(\delta^a_d\delta^c_b+f^{ac}f_{bd}-\frac{2}{\alpha}d^{aceg}f_{eb}f_{gd}\right) \text{ for } E_7. & (2.13)\end{cases}$$

(We do not know how to evaluate E_8.)

The rules for the exceptional groups are supplemented by the identities of Sec. IV which define the associated invariants. Graphically, the above rules are summarized in Fig. 1. Section V gives an example of how the rules are used in a typical computation.

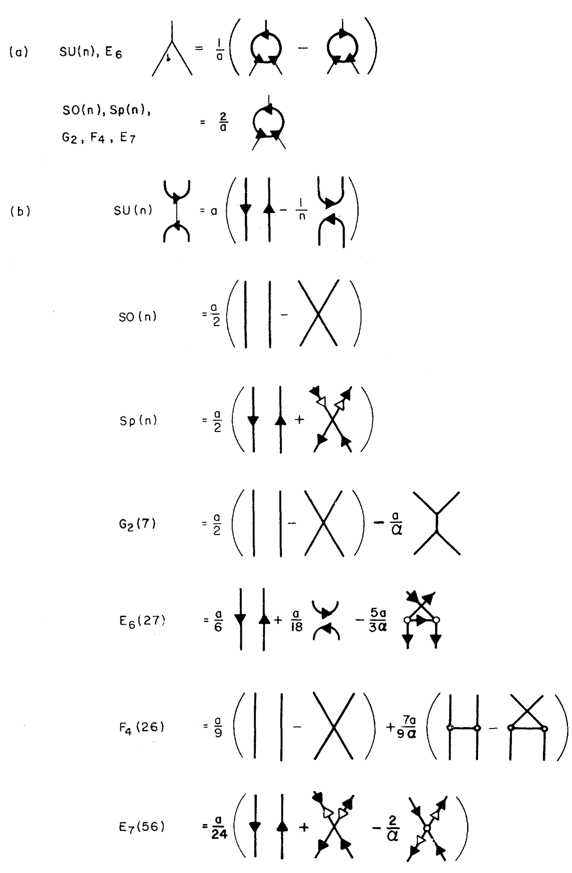

FIG. 1. Weight evaluation rules for the defining representations of all simple groups except E_8. (a) Elimination of a three-gluon coupling $-iC_{ijk}$, (b) elimination of an internal gluon line. Further rules for exceptional groups are given in Sec. IV.

III. LIE ALGEBRA IN DIAGRAMMATIC NOTATION

A group-theoretic weight W_G can be visualized as a Feynman diagram in which the internal lines represent sums over all colors of the associated particles, and vertices represent their couplings. There is never any need to label the lines and vertices; the equivalent points on the paper represent the same index in all terms of a diagrammatic equation.[38] Besides automatically keeping track of indices, diagrams make it easier to recognize the symmetries of more complicated expressions.

In this section algebraic relations shall be transcribed into diagrammatic equations which apply to any semisimple Lie algebra with quarks in any representation. The diagrammatic Feynman rules are given in Fig. 2. Figure 3 summarizes the basic relations of a semisimple Lie algebra. Note that Fig. 3(a) fixes the sign convention for $-iC_{ijk}$; indices circle the vertex in anticlockwise direction. If the direction of the quark line were reversed, the right-hand side would change sign.

Figures 3(e) and 3(f) count the numbers of quarks and gluons, respectively: $\delta^a_a=n$, $\delta^i_i=N$. The above

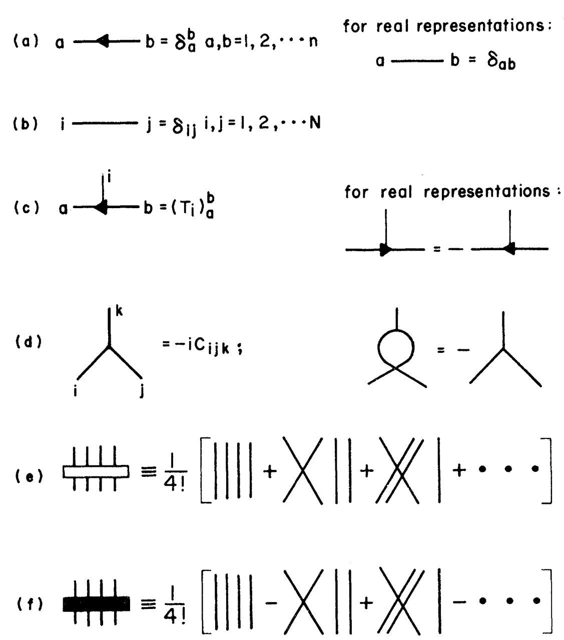

FIG. 2. (a) Quark propagator, (b) gluon or ghost propagator, (c) quark-quark-gluon vertex. The arrow denotes the direction of multiplication of T_i matrices. Whenever omitted, it is assumed to be pointing to the left for quarks going through the diagram, or counterclockwise for closed quark loops, (d) three-gluon or ghost-ghost-gluon vertex. Indices circle the vertex counterclockwise, (e) symmetrization symbol, (f) antisymmetrization symbol (a generalized Kronecker δ).

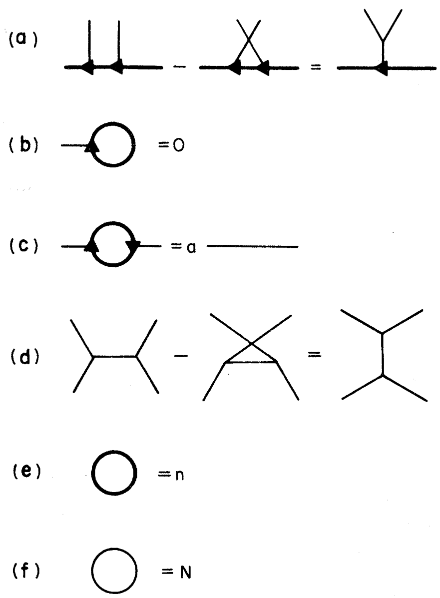

FIG. 3. (a) Lie commutator for the quark representation, (b) tracelessness condition ("color conservation"), (c) normalization convention, (d) Jacobi identity (or Lie commutator) for the adjoint representation, (e) quark number, (f) gluon number.

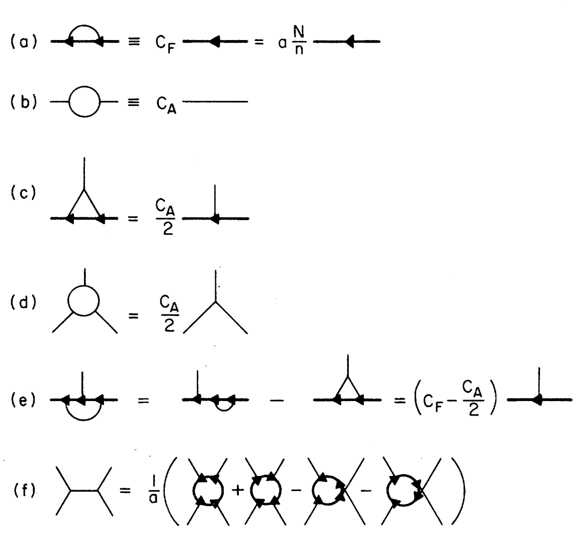

FIG. 4. (a) Quadratic Casimir operator for the defining representation, (b) quadratic Casimir operator for the adjoint representation. The remaining figures are examples of the reduction of (c) a quark-quark-gluon vertex, (d) a three-gluon vertex, and (e) another quark-quark-gluon vertex.

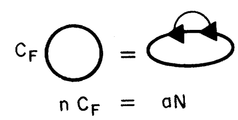

FIG. 5. A diagrammatic computation of the quadratic Casimir operator for the fundamental representation.

definitions already enable us to perform some simple calculations. For example, to calculate the quadratic Casimir operator for the quark representation, Fig. 4(a), we form a trace (join the external quark lines) and use Figs. 3(c), 3(e), and 3(f), as outlined in Fig. 5, to obtain

$$C_F = a\frac{N}{n}. \tag{3.1}$$

In other words, if we know the gluon projection operators [as those listed in (2.7) through (2.13)], we can *compute* the dimension of the algebra by tracing the normalization relation (2.4):

$$N = \frac{1}{a}\mathrm{Tr}(T_i T_i). \tag{3.2}$$

Existence of the gluon Casimir operator C_A [see Fig. 4(b)] is a necessary and sufficient condition that the algebra is semisimple. For compact groups $C_A > 0$. If the group is simple,[42,43]

$$\mathrm{Tr}(T_i T_j) = l\,\mathrm{Tr}(C_i C_j) \tag{3.3}$$

[where l is called the index of the representation, and the adjoint (or regular) representation of $\mathcal{G}$ is constructed from matrices $(C_i)_{jk} = -iC_{ijk}$], Figs. 3(c) and 4(b) are compatible. For a semisimple group, this is generally not true. Joining gluon indices in commutators Figs. 3(a) and 3(d) leads to relations in Figs. 4(c) and 4(d). Similarly, the relation Fig. 4(e) follows from the commutation relation Fig. 3(a).

The antisymmetry of C_{ijk} leads to vanishing of nonplanar diagrams of Fig. 6 as well as all diagrams that contain these as subdiagrams. This follows from the commutation relations of Fig. 3, but it is easily seen as a consequence of the skewness of C_{ijk}, Fig. 2(d). For example, interchange of vertices $1 \leftrightarrow 2$ in Fig. 6(a), and $1 \leftrightarrow 2$, $3 \leftrightarrow 4$, and $5 \leftrightarrow 6$ in Fig. 6(d) gives a factor $(-1)^3$ from skewness of C_{ijk}, while the diagrams are mapped into themselves. The obscure diagram of Fig. 6(d) is related to the Peterson graph[39] in graph theory, while Fig. 6(a) is related to the nonplanar Kuratowski graph.[39,44]

One quickly runs out of relations achievable by Lie algebra manipulations. For example, at this

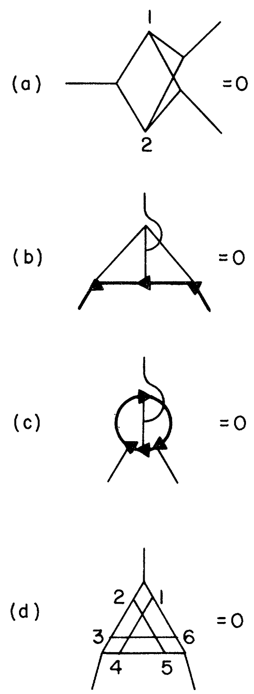

FIG. 6. Some diagrams that vanish because of the skew symmetry of C_{ijk}.

point we have no clue to the evaluation of the gluon Casimir operator C_A of Fig. 4(b), let alone any more complicated diagrams, such as the one of Fig. 7 (related to a quartic Casimir operator). For that, it is necessary to concentrate on specific groups, as we shall do in the next section. However, it should be emphasized that for the study of general properties of gauge theories, the techniques of this section are all that is needed. For vertex exponentiation this is evident from published calculations.[1,2] For quark-quark and gluon-gluon scattering[5-9] the difficulty lies in finding a spontaneous-symmetry-breaking mechanism which maintains the global symmetry (i.e., equal gluon masses). When such a scheme is found (as for example in the Bardakci-Halpern model[45]), one finds that nothing specific to the group enters into cancellations between relevant weights.[46]

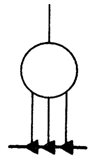

FIG. 7. A sixth-order quark-quark-gluon vertex graph.

IV. WEIGHT EVALUATION

Our objective is to express the group-theoretic weight of an arbitrary diagram as

$$W=\sum_{m=1}^{\beta} C_m T^{(m)}, \tag{4.1}$$

where $T^{(m)}$ are some basis tensors which carry the external particles' indices, and C_m are real coefficients. If $T^{(m)}$ are independent, C_m can be computed by solving a set of linear equations

$$W^n = C_m t^{mn}, \quad m,n=1,2,\ldots,\beta \tag{4.2}$$

where $t^{mn}\equiv T^{\dagger(m)}\cdot T^{(n)}$, $W^m = T^{\dagger(m)}\cdot W$. $T^{\dagger(m)}$ is obtained from $T^{(m)}$ by a reversal of all quark lines, and the product is formed by a contraction of all pairs of corresponding indices. For example, any gluon self-energy weight can be expressed in terms of a single basis $T^{(1)}=\delta_{ij}$, $W_{ij}=C_1\delta_{ij}$ (in this case $W^1=W_{ii}$ and $t^{11}=N$).

W^n and t^{mn} are weights of diagrams with no external legs, which we shall refer to as vacuum weights. From (4.2) it is clear that vacuum weights carry all the information needed for weight evaluation. They also have a direct combinatoric significance. We have already noted that single-loop vacuum weights count the number of ways in which a loop can be colored [Figs. 3(e) and 3(f)]. For arbitrary weights a hint is given by SO(3), where the weight of a gluon vacuum diagram is simply the number of ways of coloring the lines of the diagram with the three colors meeting at each vertex.[44] In general, a vacuum weight is a combinatoric number generated by some more complicated "graph coloring rule."

How is this "coloring rule" built into vacuum weights? If we eliminate gluon self-couplings by (2.5), we note that the remaining couplings $(T_i)^a_b$ always appear in the combination $(T_i)^a_b(T_i)^c_d$. It is this combination that must implement the "coloring rule." What is its significance? As $(T_i)^a_b q^b q_a$ transforms as the adjoint representation (see Behrends *et al.*,[52] Sec. V A for a demonstration), $(T_i)^a_b(T_i)^c_d q^d q_c$ picks out the part of the quark-antiquark product that transforms as a gluon. Repeated applications of $(1/a)(T_i)^a_b(T_i)^c_d$ reduce to a single application through the normalization convention (2.4); hence, we shall refer to $(1/a)(T_i)^a_b(T_i)^c_d$ as the *gluon projection operator*. The problem of weight evaluation is solved once such projection operators are known.

A gluon projection operator is also a weight (diagrammatically, a Born term in quark-quark scattering), and, according to (4.1), we can express it in terms of quark-quark scattering basis tensors. To construct a complete set of these, we need to know all invariants of the particular quark representation. There is no simple way to enumerate the invariants of an arbitrary representation; let us instead concentrate on models with quarks in the defining representation (the lowest-dimensional cogredient representation[47,48]). All higher representations can be constructed from the defining representation; in particular, the adjoint representation emerges as the $(T_i)_d^c q^d q_c$ term in the Clebsch-Gordan series $\bar{q}\otimes q = A\oplus\cdots$. Furthermore, in the defining representation a classical group has a simple geometrical interpretation [such as length preservation for SO(n)]. The main thrust of this section will be to use such invariance properties to characterize the exceptional groups as well.

A. Invariants of the defining representation

Motivated by the existence of invariants such as $\delta^{ab}q_a q_b$ for SO(n), we study unitary transformations G_b^a which preserve an arbitrary polynomial

$$P(Gq) = P(q),\qquad (4.3)$$
$$P(q) \equiv g^{ab\cdots f} q_a q_b \cdots q_f .$$

Infinitesimal parametrization $G = 1 + i\epsilon_i T_i$ gives us a differential statement of $P(g)$ invariance,

$$\frac{\partial P(Gq)}{\partial \epsilon_i} = 0,$$

so that the generators (if a nontrivial group exists) must satisfy[51]

$$(T_i)_c^a g^{cb\cdots f} + (T_i)_c^b g^{ac\cdots f} + (T_i)_c^f g^{ab\cdots c} = 0. \qquad (4.4)$$

Contracting this with $(1/a)(T_i)_e^d$ we obtain an *invariance condition* for gluon projection operators.

Suppose g^{abc} is an invariant tensor. Then $g^{abc}g_{cde}$, $g^{abc}g^{def}$, and so forth automatically satisfy (4.4) and give us no new constraints on T_i. Let us therefore concentrate on primitive invariant tensors (primitives). They are defined by the requirement that any invariant tensor can be expressed in terms of chains of their contractions (which, diagrammatically, can be disconnected or connected, but cannot contain loops). We assume that the number of primitives is finite [hence, the number of bases in (4.1) is also finite]. Any weight is expressible in terms of primitives; in particular, the gluon projection operator will be of the form

$$\frac{1}{a}(T_i)_b^a (T_i)_d^c = C_1 \delta_b^a \delta_d^c + C_2 \delta_d^a \delta_b^c + C_3 g^{ace} g_{ebd} + \cdots . \qquad (4.5)$$

Substituting this into invariance conditions (4.4), we obtain conditions on C_m and $g^{ab\cdots f}$, which, as will be shown, suffice to determine the gluon projection operator up to the overall normalization a. In case of exceptional groups, the invariance conditions are so constraining that they can be realized only in certain dimensions[24] (dimensional constraints already appear in classical groups; the symplectic invariant can be realized only in even dimensions).

Our intention is merely to demonstrate that if we know the invariants of the defining representation, we can construct the gluon projection operators and evaluate any weight. Hence, we shall simply state the primitive invariants for each defining representation and show the conditions they must satisfy. Again, as we are computing vacuum weights, we shall find that no explicit realizations of $g^{ab\cdots c}$ are needed, only some identities which implement the "coloring rules."

All simple Lie algebras are generated by a small set of primitives which are either fully symmetric ($d^{ab\cdots c}$) or fully antisymmetric ($f^{ab\cdots c}$). All defining representations preserve δ_b^a and the Levi-Civita tensor in n dimensions, $\epsilon^{ab\cdots f}$. Their further primitive invariant tensors are

SU(n): $\cdots$,

SO(n): δ_{ab},

Sp(n): f^{ab}, n even

G_2: δ_{ab}, f_{abc},

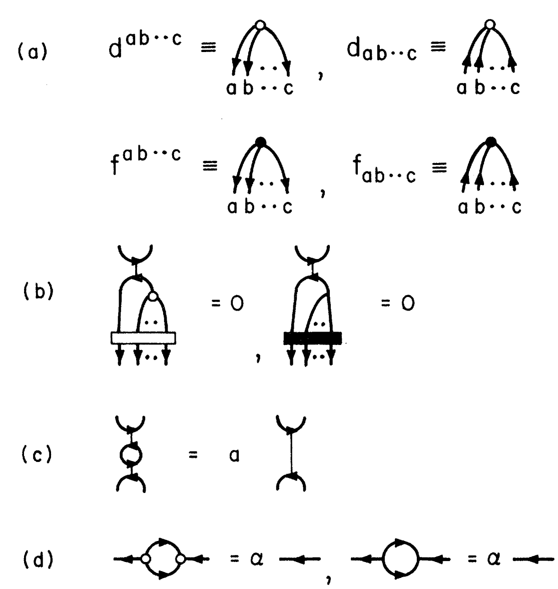

FIG. 8. (a) Diagrammatic notation for fully symmetric tensors $d^{ab\cdots c}$, $d_{ab\cdots c}$ and fully antisymmetric tensors $f^{ab\cdots c}$, $f_{ab\cdots c}$, (b) invariance conditions for gluon projection operators, (c) normalization convention for gluon projection operators, (d) normalization convention for cubic quark self-couplings.

E_6: d^{abc},

F_4: δ_{ab}, d_{abc},

E_7: f^{ab}, d^{abcd},

E_8: δ_{ij}, C_{ijk}, unknown.

Before we proceed with the discussion of individual groups, let us make a few observations that will apply to all cases. Owing to the full (anti) symmetry of ($f^{ab\cdots}$) $d^{ab\cdots}$ tensors, the invariance conditions can be stated very compactly (Fig. 8). $f^{ab\cdots}$ and $d^{ab\cdots}$ can be interpreted as quark self-couplings. Unlike quark-gluon couplings $(T_i)^a_b$, whose scale is fixed relative to C_{ijk} by (2.2), they have no *a priori* relation to gauge couplings, and to characterize their scale we introduce an arbitrary normalization α. For cubic couplings we can define α by

$$d_{abc}d^{cbd} = \alpha\delta^d_a, \tag{4.6}$$

$$f_{abc}f^{cbd} = \alpha\delta^d_a. \tag{4.7}$$

α for different groups need not be the same.

B. Special unitary groups SU(n)

The defining representation of SU(n) is a set of all unitary ($U^\dagger U = 1$) and unimodular ($\det U = 1$) $[n\times n]$ matrix transformations acting on an n-dimensional complex vector space (n quarks). The infinitesimal transformations can be parametrized by $N = n^2 - 1$ traceless Hermitian matrices $(T_i)^a_b$ which close a Lie algebra (2.2). The invariants are the Hermitian (sesquilinear[47]) metric δ^a_b (which imposes the unitarity condition; $\bar{q}q \equiv q^b\delta^a_b q_a$ is preserved) and the Levi-Civita tensor in n dimensions, $\epsilon^{ab\cdots f}$. The contragredient Levi-Civita tensor acts as an inverse to the cogredient one in the sense that a direct product of the two can be expressed as a generalized Kronecker δ function [see also (6.4)]

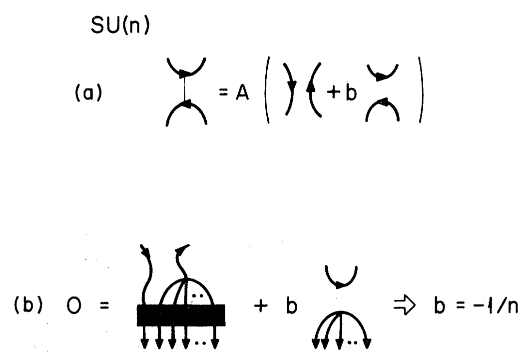

FIG. 9. (a) The most general form of the gluon projection operator for SU(n), (b) the Levi-Civita tensor invariance condition.

$$\epsilon^{ab\cdots f}\epsilon_{pq\cdots u} = \delta^{ab\cdots f}_{pq\cdots u}. \tag{4.8}$$

Gluon projection operator expansion (4.5) is of the form

$$\frac{1}{a}(T_i)^a_b(T_i)^c_d = A(\delta^a_d\delta^c_b + b\delta^a_b\delta^c_d), \tag{4.9}$$

which we give diagrammatically in Fig. 9(a). [Any possible $\epsilon^{ab\cdots c}$ terms reduce to the above two by (4.8)]. Substituting this expression into $\epsilon^{abc\cdots}$ invariance condition Fig. 8(b), we obtain the equation Fig. 9(b), which, when contracted with δ^a_b (in the only way possible, the incoming line with any outgoing line) yields $b = -1/n$. We now see how a projection operator[53] works; $\delta^a_b\delta^c_d$ removes the singlet from a quark-antiquark state, leaving $N = n\times n - 1$ gluons. Tracelessness of T_i ensures that the gluon does not connect to the vacuum (i.e., that the group is semisimple). From the normalization convention Fig. 8(c) $A = 1$, and we can verify that the number of gluons is indeed $N = n^2 - 1$ by evaluating (3.2).

C. Special orthogonal groups SO(n)

The defining representation of SO(n) is a set of all orthogonal ($R^TR = 1$) and unimodular ($\det R = 1$) $[n\times n]$ matrix transformations acting on an n-dimensional complex vector space (n quarks). The defining invariant is a symmetric tensor $d^{ab} = d^{ba}$ (and its inverse $d_{ab} = d_{ba}$) introduced diagrammatically in Fig. 10(a). The remainder of Fig. 10 derives the gluon projection operator from the in-

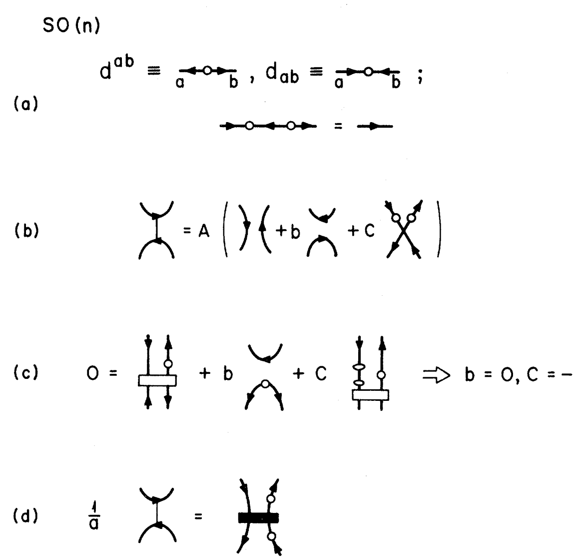

FIG. 10. (a) Diagrammatic notation for SO(n)-invariant tensor d_{ab}, (b) the most general form of the gluon projection operator for SO(n), (c) d_{ab} invariance condition, (d) gluon projection operator for SO(n).

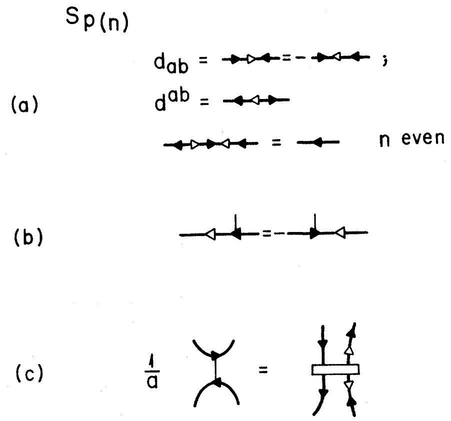

FIG. 11. (a) Diagrammatic notation for Sp(n)-invariant tensor f_{ab}, (b) f_{ab} invariance condition, (c) gluon projection operator for Sp(n).

variance of d^{ab} [Fig. 8(b)]. By diagonalizing d^{ab} and rescaling q^a fields, we can always find a representation where $d_{ab} = \delta_{ab}$. There is no distinction between upper and lower indices (quark = antiquark, the representation is real), and in diagrams we can omit all d^{ab} tensors and all line arrows, and note that because of (4.4) the generators are antisymmetric: $(T_i)_{ab} = -(T_i)_{ba}$. They are clearly traceless, and it is easily verified that the Levi-Civita tensor $\epsilon^{ab\cdots f}$ in n dimensions is preserved as well.

In the conventional choice of SO(n) generators[47,54] with only two nonzero elements ± 1, the normalization is fixed by $a = 2g^2$.

D. Symplectic groups Sp(n)

The invariant preserved by the defining representation of Sp(n) is a skew-symmetric metric[47,54] $f^{ab} = -f^{ba}$ (and its inverse $f_{ab} = -f_{ba}$). An inverse exists only if f^{ab} is nonsingular, $\det(f) \neq 0$. The skew-symmetry of f^{ab} allows that only for even-dimensional

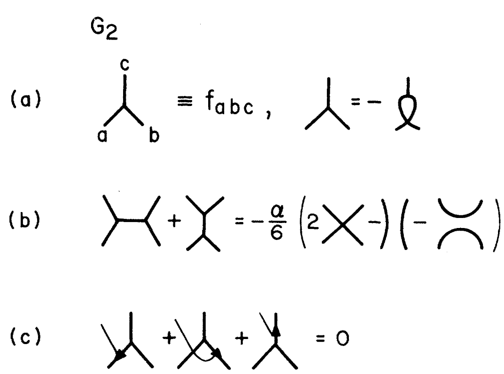

FIG. 12. (a) Diagrammatic notation for the tensor f_{abc} for the exceptional group $G_2(7)$, (b) the "alternativity" relation which relates contractions of pairs of f_{abc}, (c) the invariance condition for f_{abc}.

defining representations.[55] Construction of the gluon projection operator (Fig. 11) proceeds as in the SO(n) case.

E. Exceptional group G_2 (Ref. 56)

The defining representation of G_2 ($n = 7$) preserves a symmetric invariant δ_{ab} [G_2 is a subgroup of SO(7)], and a fully antisymmetric cubic invariant f^{abc}. It is possible to show that G_2 is the only nontrivial simple group that possesses such invariants,[24] and that f_{abc} must satisfy the alternativity relations[18,52] given in Fig. 12(b). By these relations two out of three tensors $f_{abe}f_{ecd}$, $f_{ace}f_{ebd}$, and $f_{ade}f_{ebc}$ can always be eliminated in favor of the third and some combination of δ^a_b's. As in the SO(n) case, δ_{ab} invariance makes generators T_i antisymmetric, and the gluon projection operator (4.5) has the form given in Fig. 13(a). From the identity Fig. 13(b), we derive relation Fig. 13(c), which determines the gluon projection operator through invariance of f_{abc} [Fig. 8(b)]. Actually, Fig. 13(c) (through a few more applications of the alternativity relations) leads to a very strong statement[52] that any chain of three f_{abc} can be reduced to a sum of terms linear in f_{abc} by the equation of Fig. 13(d). This guarantees that even though the projection operator (2.10) replaces internal gluon lines by internal quark lines, the resulting weights can always be reduced to the bases (4.1). The gluon number, evaluated by (3.2), is indeed $N = 14$. Further relations are given in Fig. 14.

An explicit realization of tensors f_{abc} is given by octonions.[57,58] In this framework G_2 is the auto-

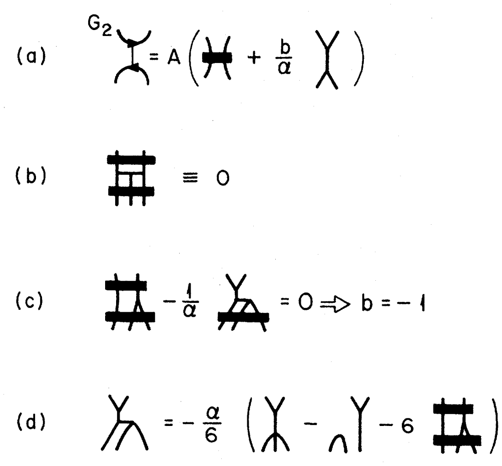

FIG. 13. (a) The most general form of the gluon projection operator for $G_2(7)$ (δ_{ab} invariance has already been imposed), (b) an identity between contractions of three f_{abc} which arises from the skew-symmetry of f_{abc}, and leads to (c) the invariance condition for the gluon projection operator, (d) identity that reduces any chain of contractions of more than two f_{abc}.

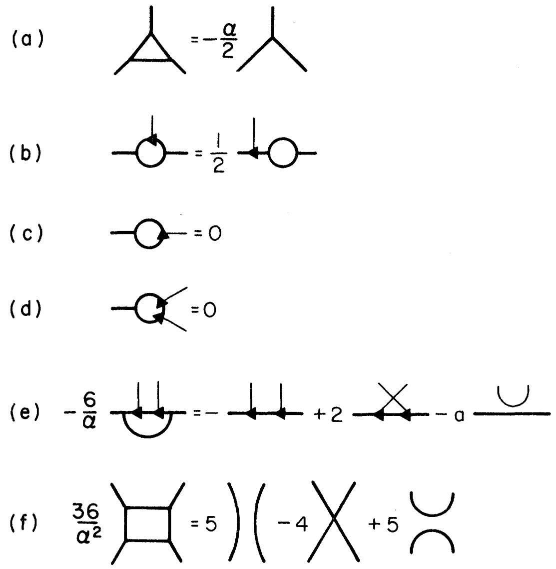

FIG. 14. Some derived relations between f_{abc} tensors useful in the computations of weights for G_2.

morphism group of octonions, i.e., it is a set of all $[7\times 7]$ real matrices G_{ab} such that the transformation

$$e'_a = G_{ab}e_b,\quad a,b=1,2,\ldots,7$$

preserves the octonionic multiplication rule

$$e_a e_b = -\delta_{ab} + f_{abc}e_c, \qquad (4.10)$$

where f_{abc} are given explicitly in Ref. 58; for our purposes, it is sufficient to note that octonions satisfy the alternativity condition if

$$[xyz] \equiv (xy)z - x(yz),$$

$$[xyz] = [zxy] = [yzs] = -[yxz],$$

where x,y,z are arbitrary octonions. The alternativity relation Fig. 12(b) follows from the multiplication rule (4.10) and the alternativity condition.[59] Equation (4.10) also fixes the normalization (4.7) $\alpha=-6$. Then $-\alpha$ is simply the number of distinct colorings of diagram Fig. 8(d) allowed by the octonion multiplication rule.

F. Exceptional group E_6

The defining representation of E_6 ($n=27$) preserves a fully symmetric cubic invariant d_{abc} (and its inverse d^{abc}).[19-21,60-62] No condition relating $d_{abc}d^{cde}$ type tensors exists and the only nontrivial relation[24] on d^{abc} tensors is a trilinear Springer relation[60] [Fig. 15(b)] which arises from the requirement of d^{abc} invariance [Fig. 8(b)]. This relation enables us to compute the gluon projection operator [whose general form is given by Fig. 15(a)]. Evaluation of (3.2) yields the dimension of the algebra of E_6, $N=78$.

Springer's relation arises from the characteristic equation for $[3\times 3]$ Hermitian octonion matrices. The gluon projection operator (2.11) was actually first constructed by Freudenthal[61] in a very different notation (as a derivation of a Jordan algebra). His normalization convention is $\alpha=\frac{5}{2}$.

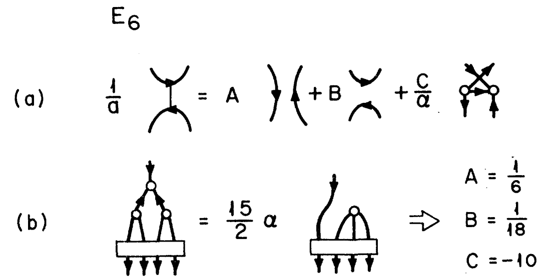

FIG. 15. (a) The most general form of the gluon projection operator for $E_6(27)$. (b) Springer's relation. Together with the invariance condition for the gluon projection operator, it fixes the constants in (a).

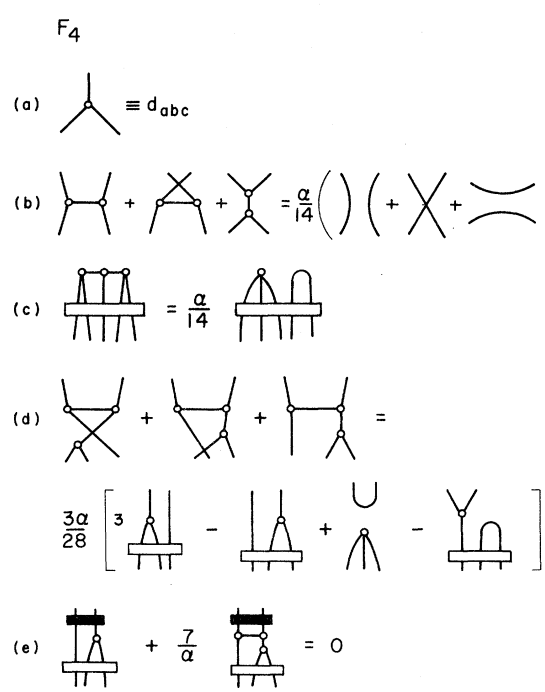

FIG. 16. Diagrammatic notation for the tensor d_{abc} for the exceptional group $F_4(26)$, (b) "characteristic" relation which relates contractions of pairs of d_{abc}, (c) expansion of this identity [which follows from (b)] leads to (d) a relation between contractions of three d_{abc}. Antisymmetrization in top legs and symmetrization in bottom legs yields (e) the Jordan identity which together with the invariance condition for d_{abc} fixes the gluon projection operator for $F_4(26)$.

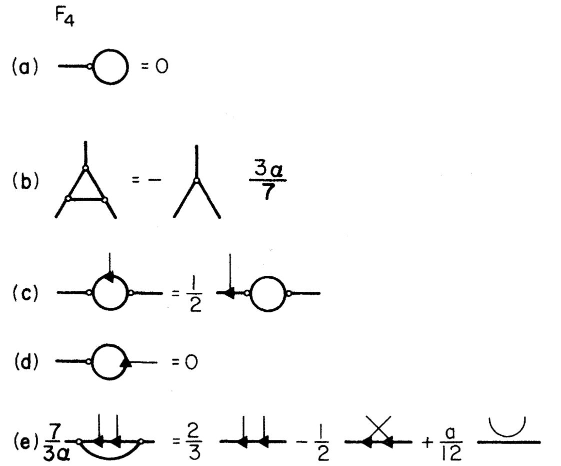

FIG. 17. Some derived relations between d_{abc} tensors useful in computations of F_4 weights.

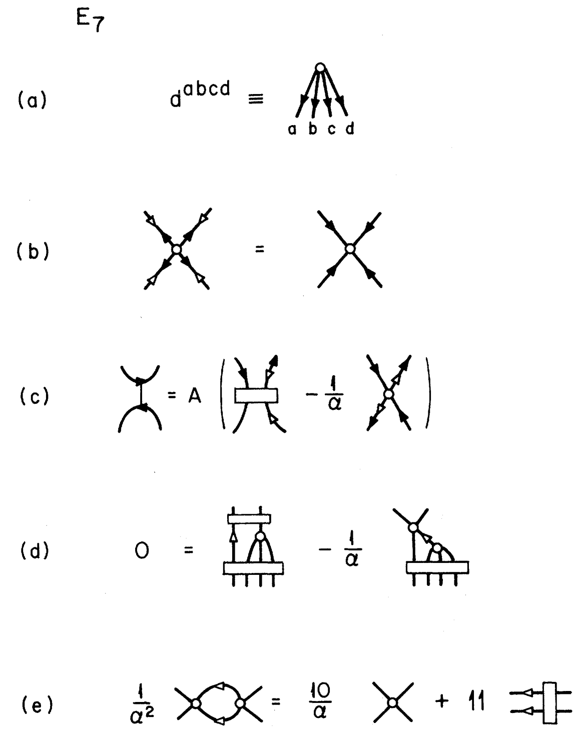

FIG. 18. (a) Diagrammatic notation for the tensor d_{abcd} for the exceptional group $E_7(56)$, (b) symplectic invariant tensor f^{ab} relates d^{abcd} and d_{abcd}, (c) the most general form of the gluon projection operator (f^{ab} invariance has already been imposed), (d) Brown relation which relates contractions of pairs of d_{abcd}, (e) reduction of a one-loop diagram.

G. Exceptional group F_4

The defining representation of F_4 ($n=26$) preserves[19-21,63] both d_{abc} and δ_{ab}. To derive F_4, it is necessary to assume that a relation between bilinear combinations $d_{abe}d_{ecd}$ exists. The only nontrivial relation[24] of such type is the characteristic relation[60] of Fig. 16(b). The gluon projection operator is constructed the way it was constructed for G_2. The identity of Fig. 16(c) leads us to the Jordan identity of Fig. 16(e), which together with the d_{abc} invariance [Fig. 8(b)] fixes the projection operator up to an overall normalization. The normalization convention [Fig. 8(c)] then yields the gluon projection operator given in (2.12). There are $N=52$ gluons. Further relations are given in Fig. 17.

An explicit realization of tensors d_{abc} is given by octonion matrices. In this framework[20] F_4 is the isomorphism group of the exceptional simple Jordan algebra of traceless Hermitian $[3\times 3]$ matrices x with octonion matrix elements. The nonassociative multiplication rule for elements x can be written as

$$\begin{aligned}
&x \equiv x_a e_a, \quad a=1,2,\ldots,26\\
&\mathrm{Tr}\, e_a = 0, \quad e_a \text{ is a } [3\times 3] \text{ basis matrix},\\
&e_a e_b = e_b e_a = \frac{\delta_{ab}}{3}\underline{1} + d_{abc} e_c,\\
&\mathrm{Tr}\,\underline{1} = 3, \quad \underline{1} \text{ is a } [3\times 3] \text{ unit matrix.}
\end{aligned} \tag{4.11}$$

Transformations of F_4 preserve the quadratic form $\mathrm{Tr}(x^2)$ [the length in 26-dimensional space, so that F_4 is a subgroup of SO(26)], as well as a fully symmetric cubic form

$$\mathrm{Tr}(xyz) = \mathrm{Tr}(yxz) = \mathrm{Tr}(yzx) = d_{abc}x_a y_b z_c .$$

The characteristic equation for traceless $[3\times 3]$ matrices

$$x^3 - \tfrac{1}{2}\mathrm{Tr}(x^2)x - \tfrac{1}{3}\mathrm{Tr}(x^3)\underline{1} = 0$$

gives a relationship between contractions of pairs of d_{abc}, drawn in Fig. 16(b). (Characteristic equations are discussed in Sec. VI.) The Jordan identity $(xy)x^2 = x(yx^2)$ is automatically satisfied; it is just the relation of Fig. 16(e). Normalization is fixed by (4.11), $\alpha = \frac{7}{3}$.

H. Exceptional group E_7

The defining representation of E_7 ($n=56$) preserves a skew-symmetric tensor f^{ab} [E_7 is a subgroup of Sp(56)] and a fully symmetric quartic invariant,[61,64,65] d^{abcd} [Fig. 18(a)]. f_{ab}, f^{ab} raise and lower indices [Fig. 18(b)]. The gluon projection operator can have the general form of Fig. 18(c). The invariance of d_{abcd} gives the Brown relation[64] [Fig. 18(d)], which enables us to compute Fig. 18(e), impose the normalization condition Fig. 8(c), and derive (2.13). The evaluation of the gluon number gives $N=133$.

In the explicit realization of tensors d_{abcd} by octonion matrices,[64] the conventional normalization is $\alpha = \frac{1}{3}$.

I. Exceptional group E_8

The defining representation of E_8 ($n = N = 248$) is also the adjoint representation, so our method of reducing everything to the lowest-dimensional representation is of no help. Still, if the invariants of the defining representation of E_8 were known, we would be able to reduce higher-order weight diagrams to a basic set just as for all other simple groups. Known invariants are δ_{ab} and C_{abc}, and other invariants are certainly higher than quartic. The Tits construction,[18-21] which relates $SU(n) \to E_6$, $SO(n) \to F_4$, and $Sp(n) \to E_7$, suggests (extrapolating octonions $\to E_8$) that the E_8 invariant is a fully symmetric octet $d_{abcdefgh}$. We do not know whether this is true and we hope we shall never need to know.

We should also point out that we have not proved that our identities for F_4, E_6, and E_7 suffice to evaluate any weight. We have only verified this for all vacuum weights up to 4 loops (F_4 and E_6) and 3 loops (E_7).

V. ILLUSTRATIVE EXAMPLES

Evaluation of any W_G is now almost trivial, especially for classical groups. We just proceed applying systematically the rules of Fig. 1, first eliminating all three-gluon vertices, and then removing all internal gluon lines. Removal of each gluon line reduces W_G into a sum of weights of lower order. Eventually we end up with a set of irreducible tensor bases, each multiplied by some polynomial in n (n is the number of quark colors).

As an example, we evaluate the SO(n) quadratic

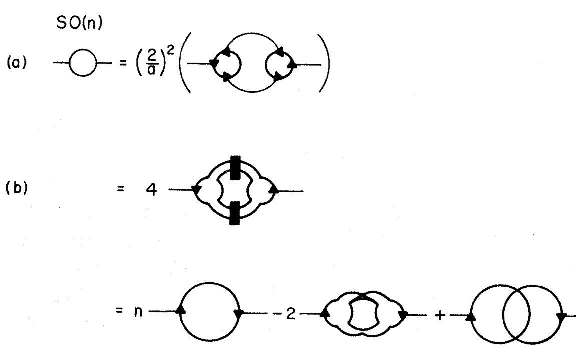

FIG. 19. A sample diagrammatic computation: quadratic Casimir operator for the adjoint representation of SO(n). (a) C_{ijk} are replaced by the defining representation, (b) internal gluons are replaced by gluon projection operators, and (c) the expression is expanded and evaluated.

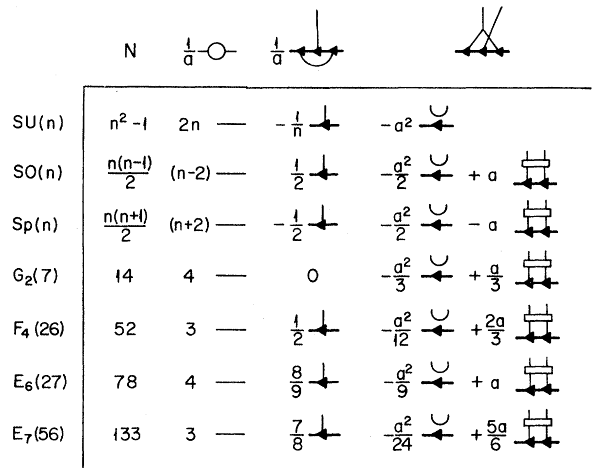

FIG. 20. A tabulation of some simple weight evaluations.

Casimir operator for the adjoint representation (gluons) in Fig. 19. We find that

$$C_A = a(n-2). \tag{5.1}$$

Other such results are tabulated in Fig. 20. Of course, dimensions and Casimir operators (or representation indices) are all tabulated in the literature[42,43] and our algorithm is unnecessary for their evaluation. However, we can now calculate the weight of any diagram. A typical example would be computation of all the weights that appear in the SU(n) quark-quark scattering calculation,[7] or the order of the first nonleading term in $1/n$ expansion for various groups.[17]

VI. RELATIONS BETWEEN BASIS TENSORS

The procedure outlined in Secs. I–V always leads us to a unique set of tensors: $(T_i)_a^b$ and traces over T_i matrices. In other words, we are expressing all W_G in terms of the defining representation. Let us illustrate this by writing all irreducible bases $T^{(m)}$ for quark-quark scattering weights [see (4.1)]:

$$\mathrm{SU}(n):\ \delta_d^a\delta_b^c,\ \delta_b^a\delta_d^c, \qquad (\beta = 2)$$

$$\mathrm{SO}(n):\ \delta_d^a\delta_b^c,\ \delta_b^a\delta_d^c,\ \delta^{ac}\delta_{bd}, \qquad (\beta = 3)$$

$$\mathrm{Sp}(n):\ \delta_d^a\delta_b^c,\ \delta_b^a\delta_d^c,\ f^{ac}f_{bd}, \qquad (\beta = 3)$$

$$\mathrm{G}_2(7):\ \delta_d^a\delta_b^c,\ \delta_b^a\delta_d^c,\ \delta^{ac}\delta_{bd},\ f^a{}_{be}f^{ec}{}_d, (\beta = 4)$$

and so forth. These bases appear naturally in our approach, but they are by no means the only possible choice. For example, we can replace the "color exchange" base $\delta_d^a\delta_b^c$ by the "color flip" base[5-8] $(T_i)_b^a(T_i)_d^c$ using relations (2.7)–(2.13). As another example, we write down all irreducible tensor invariants for a process with r external gluons and no external quarks, the set of all dis-

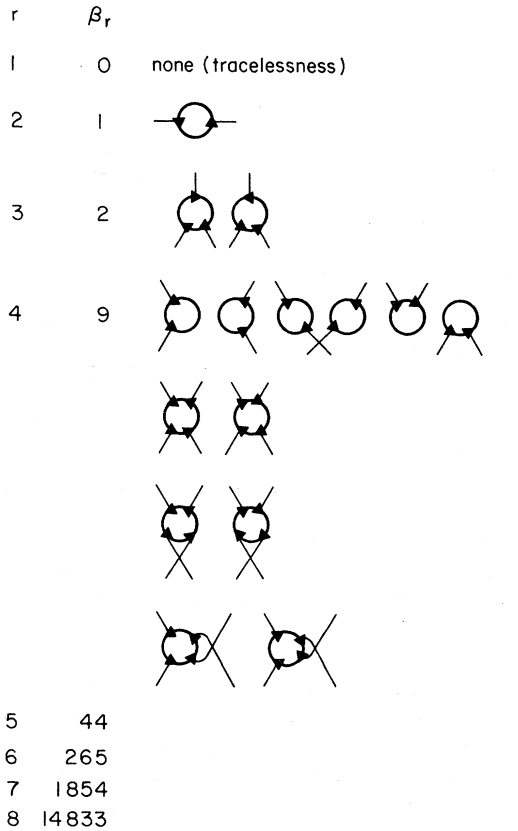

FIG. 21. Tensor bases for processes with $r = 2, 3, \ldots$ external gluons and no external quarks. These are also the complete and independent bases for SU(n) tensors as long as $n \geq r$.

tinct traces over r T_i matrices (Fig. 21).

β_r, the number of all distinct tensors of rank r, is the number of ways in which r T_i matrices can be grouped into traces over their products, with the restriction that $\mathrm{Tr}(T_i)=0$. β_r can be calculated in a number of arduous ways, such as by Young tableaux,[28,66] or by the method of Appendix B. However, it turns out that β_r had already been calculated in 1708,[67,68] and is known as a number of derangements, or subfactorial

$$\beta_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots (-1)^n \frac{1}{n!}\right). \tag{6.1}$$

Not all tensor bases thus enumerated are necessarily independent, because they might be related through the invariants of the defining representation. β_r was calculated from a single condition, tracelessness. Thus, traces over T_i form natural bases for all simple Lie groups, SU(n) in particular. For SO(n), Sp(n), G_2, F_4, and E_7, the clockwise and anticlockwise directions of loops in Fig. 21 are related by δ_{ab}, f_{ab} invariance, and the number of independent bases is reduced:

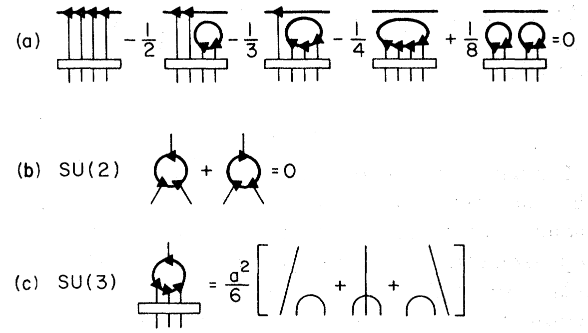

FIG. 22. (a) A characteristic equation for [4×4] matrices, (b) characteristic equation for SU(2) [there are no d_{ijk} coefficients; see Fig. 25(d)], (c) Macfarlane *et al.* relation for SU(3).

$$\mathrm{SO}(n): \beta_3 = 1,\ \beta_4 = 6,\ \text{etc.} \tag{6.2}$$

Further relations, dependent on the dimensionality of the defining representation, arise from the characteristic equations for $[n \times n]$ matrices (i.e., from the invariance of the Levi-Civita tensor). The characteristic polynomial[69] is defined as

$$P(x) = \det|A - Ix| = \sum_{k=0}^{n} (-x)^{n-k} \frac{1}{k!} \delta^{a_1 a_2 \cdots a_k}_{b_1 b_2 \cdots b_k} A^{b_1}_{a_1} \cdots A^{b_k}_{a_k}, \tag{6.3}$$

where

$$\delta^{ab\cdots f}_{pq\cdots u} = \det \begin{vmatrix} \delta_{ap}\delta_{bp} \cdots \delta_{fp} \\ \delta_{aq}\delta_{bq} \\ \vdots \\ \delta_{au} \cdots \quad \delta_{fu} \end{vmatrix} \tag{6.4}$$

is the generalized Kronecker δ. Identity $P(A)=0$ yields the characteristic equation for A:

$$0 = \sum_{k=0}^{n} A^{n-k} \frac{(-1)^k}{k!} \delta^{a_1 a_2 \cdots a_k}_{b_1 b_2 \cdots b_k} \times A^{b_1}_{a_1} A^{b_2}_{a_2} \cdots A^{b_k}_{a_k}.$$

Now if we substitute $A = a_i T_i$, where T_i are generators of the group $\mathcal{G}$, for each n we obtain various relations between tensor invariants. As an example, we work out the $n=4$ case diagrammatically in Fig. 22(a). The indices are symmetrized because the whole expression is multiplied by a symmetric factor $a_i a_j a_k a_l$, summed over all i, j, k, and l. More familar relationships are worked out explicitly for SU(2) and SU(3) in Figs. 22(b) and 22(c). The SU(3) relationship can be rewritten in terms of d_{ijk} tensors, the form of which has been originally derived by Macfarlane *et al.*[27] Higher SU(n) relationships have been worked out in Ref. 29. Such relations do not affect the correctness of our general procedure for weight evaluation.

VII. HIGHER REPRESENTATIONS

In Sec. IV we have constructed gluon projection operators from the invariants of the quark representation. This approach is by no means restricted to the defining representation; in Appendix B we shall give an example of a calculation in terms of the invariants of the adjoint representation. That calculation will exemplify the difficulties arising in the study of higher representations; it is not easy to find a complete set of invariants for an arbitrary representation, and even when those are found, the evaluation of weights can still be difficult.

However, we already have a simpler solution for one higher representation; we know how to compute weights of diagrams with all particles in the adjoint representation. We evaluate them by rewriting them in terms of the defining representation. This suggests that we should attempt to express the particular higher representation in terms of the defining representation; once that is accomplished, the weights can be evaluated by the methods of Sec. IV. In principle, we always know how to construct any representation from the defining one by the Young symmetrization procedure.

As an example we construct the antisymmetric second-rank tensor representation of SU(n).[49] The projection operator $\frac{1}{2}(\delta_a^c\delta_b^d - \delta_a^d\delta_b^c)$ picks out the antisymmetric part of a two-quark state $q_c q_d$, and the generator of SU(n) transformations is

$$(t_i)_{ab}^{cd} = \tfrac{1}{2}[(T_i)_a^c\delta_b^d - (T_i)_b^c\delta_a^d + \delta_a^c(T_i)_b^d - \delta_b^c(T_i)_a^d],$$

where $a, b, \ldots = 1, 2, \ldots, n$, and T_i are the generators of the defining representation of SU(n) (Sec. IV B). This is a nice example of how compact the diagrammatic notation is[70] (Fig. 23) compared to

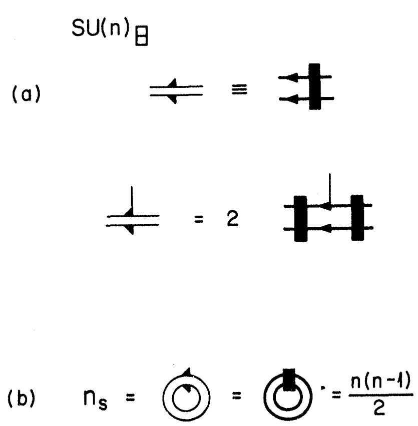

FIG. 23. (a) Diagrammatic notation for the antisymmetric second-rank tensor representation of SU(n), (b) computation of its dimension.

tensor notation. To check this construction we compute the dimension [Fig. 3(f)] and the index (3.3) and verify[42,43,49] that

$$n_S = \frac{n(n-1)}{2},$$

$$\frac{1}{a}\mathrm{Tr}(t_i t_i) = n - 2.$$

Further examples of projection operators for higher representations are given by Behrends *et al.*[52]

We should also mention that there already exist algorithms for computing weights of arbitrary representations. For example, Agrawala and Belinfante[71] have developed a computer program for evaluation of SU(n) invariants.

ACKNOWLEDGMENTS

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APPENDIX A: COMPLETE FEYNMAN RULES FOR $M_G W_G$

With the definition of the group-theoretic weight W_G given in Sec. II, the rules for M_G are easily

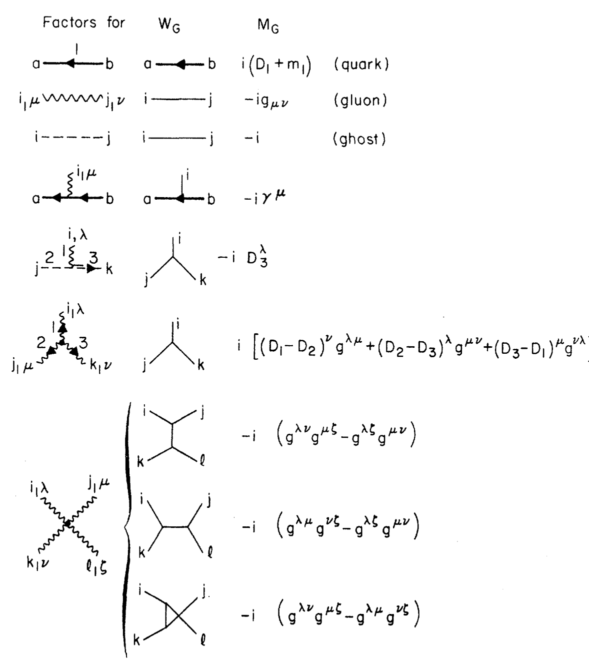

FIG. 24. Factors for the group-theoretic weights W_G and Feynman momentum integrals M_G in the Feynman gauge.

constructed by consulting some standard reference, such as Abers and Lee.[72] In this appendix we state the full rules for unrenormalized Feynman amplitudes in (unbroken) non-Abelian gauge theories as an extension of the rules for constructing Feynman-parametric integrals given previously.[73] Factors of rule 5, of Ref. 73, are now replaced by the factors of Fig. 24. Additionally M_G gets a factor -1 for each quark or ghost loop.

APPENDIX B: EVALUATION OF SU(N) WEIGHTS USING f- AND d-TENSOR BASES

In this appendix we extend the SU(3) method of Dittner[28] to SU(n). The generalized Gell-Mann $[n\times n]$ λ matrices together with I, iI, and $i\lambda$ span all complex matrices,[37] so we can write a multiplication law for λ matrices as

$$\mathrm{SU}(n):\ \lambda_i\lambda_j=(\bar{a}+ib)\delta_{ij}I+(d_{ijk}+if_{ijk})\lambda_k. \qquad \text{(B1)}$$

This relation, which has no obvious analogs for other simple groups, is the departure point for most of the earlier attempts at weight evaluation.[26-34] The tensors δ_{ij}, d_{ijk}, and f_{ijk} are numerically invariant in the sense that they are the same for all equivalent representations $\lambda_i\to u^\dagger\lambda_i u$, $u^\dagger u=1$. They are real by definition. $b=0$ because of the Hermiticity of λ_i, while $\bar{a}$ is related to the arbitrary normalization of Eq. (2.4), $a=(ng^2/4)\bar{a}$.

According to Sec. IV, we can evaluate any weight if we know how to evaluate vacuum weights. There λ_i matrices always appear in traces, $\mathrm{TR}(\lambda_i\lambda_j\cdots\lambda_m)$, and they can be eliminated by the repeated application of the λ-multiplication rule [depicted in Fig. 25(b)]. The problem of weight evaluation for SU(n)

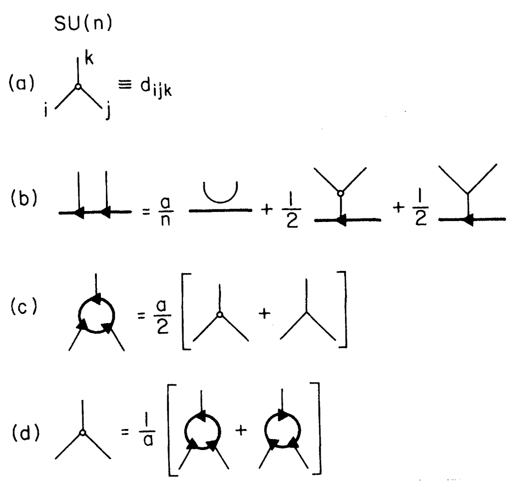

FIG. 25. (a) Notation for the (fully symmetric) numerical tensor d_{ijk}, (b) multiplication rule for SU(n) matrices $T_i\equiv\frac{1}{2}g\lambda_i$, (c) decomposition of three external gluon quark-loop into real and imaginary parts, (d) d_{ijk} as its real part.

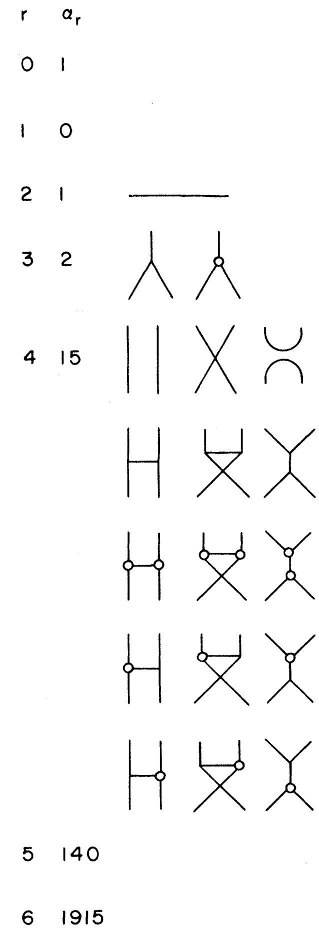

FIG. 26. Construction of all simple d and f tensors with r external gluons.

is then reduced to the problem of evaluation of vacuum weights built solely from the adjoint representation invariant tensors δ_{ij}, f_{ijk}, and d_{ijk}. Dittner solves this by setting up a chain of sets of linear equations of type (4.2), which make it possible (in principle) to compute weights with $l+1$ loops once all vacuum weights with up to l loops

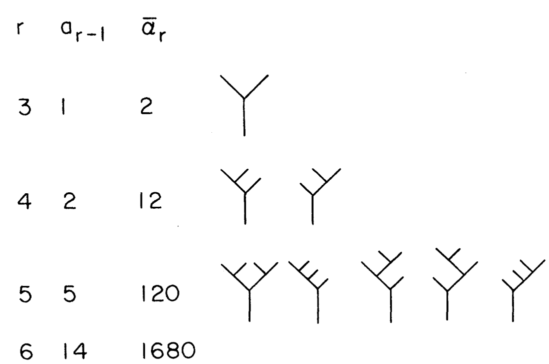

FIG. 27. Catalan's trees.

are known. To achieve this, it is necessary to construct independent bases for processes with $r = 2, 3, \ldots$ external gluon legs.

The simplest set of tensors for each r is easily constructed (see Fig. 26). To enumerate them, we start a systematic construction by drawing all Catalan[68,74] trees in Fig. 27, whose number is Catalan's number (the number of ways in which a product of n numbers can be evaluated)

$$a_{r-1} = \frac{(2r-4)!}{(r-1)!\,(r-2)!}, \quad a_0 \equiv 0. \tag{B2}$$

By $(r-1)!$ permutations of all branches, and factor 2 for each crotch (f or d tensor), we obtain the number of all distinct *connected* tensors

$$\overline{\alpha}_r = 2^{r-2}(2r-5)!!, \quad \overline{\alpha}_1 \equiv 0, \quad \overline{\alpha}_2 \equiv 1 \tag{B3}$$

where $(2n-1)!!$ is the product of the first n-odd integers, $7!! \equiv 7\times5\times3\times1$. To relate $\overline{\alpha}_r$ to the α_r, the number of all distinct tensors (connected and unconnected) we introduce generating functions

$$A(t) \equiv \sum_{r=0}^{\infty} \frac{\alpha_r}{r!} t^r, \tag{B4}$$

$$\overline{A}(t) \equiv \sum_{r=0}^{\infty} \frac{\overline{\alpha}_r}{r!} t^r = \tfrac{1}{12}[-1 + 6t + (1-4t)^{3/2}]. \tag{B5}$$

The numbers of connected and disconnected graphs are related in the usual fashion,

$$A(t) = e^{\overline{A}(t)}. \tag{B6}$$

By differentiation with respect to t, this can be restated as

$$\alpha_r = \sum_{r=0}^{r-1} \binom{r-1}{k} \overline{\alpha}_{r-k}\alpha_k, \tag{B7}$$

which enables us to calculate recursively α_r listed in Fig. 26.

However, tensors so constructed are redundant, and if we attempt to use them to expand an arbitrary tensor with r external gluons, we would not be able to calculate the expansion coefficients, because the determinant of the system of α_r equations vanishes for $r > 3$.

So our next task is to find all the relations between α_r tensors. These stem from the associativity of T_i matrices. For example, $\mathrm{Tr}(T_iT_jT_kT_l)$ can be evaluated in two ways, by pairing matrices either as $\mathrm{Tr}(T_iT_j)(T_kT_l)$ or $\mathrm{Tr}(T_jT_k)(T_lT_i)$, and then using Fig. 25(b). The two evaluations give the relationship of Fig. 28(a). There are $(4-1)! = 6$ distinct connected tensor bases (Fig. 21) with four T_i each, giving us $\overline{\gamma}_4 = 6$ relationships. We cast those in the form familiar from the literature,[26-28] three equations for the real parts [Fig. 28(b)] and three for the imaginary parts [Fig. 28(c)]. Figure 28(c) states that d_{ijk} are invariant

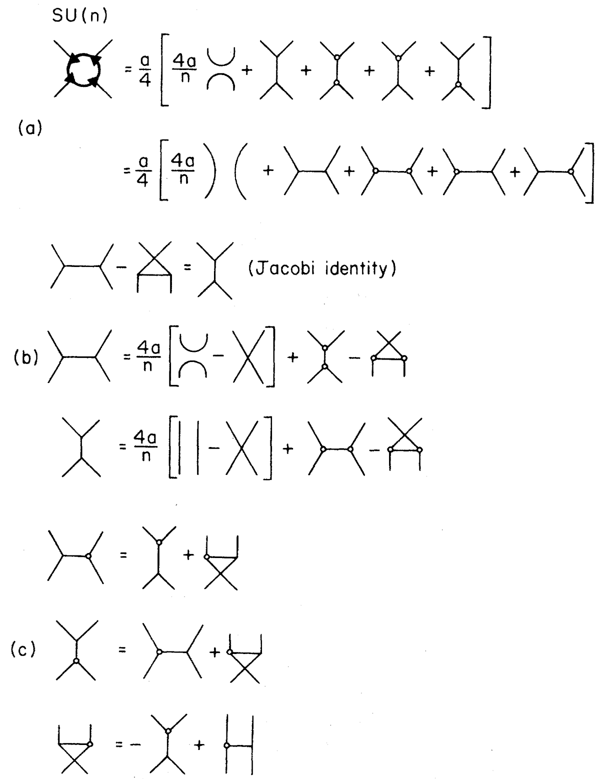

FIG. 28. (a) Associativity of T_i matrices leads to relations between various d and f tensors. All relations between (b) real and (c) imaginary parts of simple tensors with four external gluons.

[see (4.4) and remember that $(T_i)_{jk} = -if_{ijk}$ for the adjoint representation of SU(n)]. The second and third lines of Fig. 28(b) are two versions of the SU(n) generalization[26,27] of the SU(2) relationship

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}. \tag{B8}$$

Glancing back at the gluon projection operator for F_4 [Fig. 1(b)], we realize that this is the gluon projection operator for models with quarks in the adjoint representation of SU(n).

The number of associativity relations for arbitrary r is again related to Catalan's number, which is nothing but the number of associativity patterns

$$\overline{\gamma}_r = (r-1)!\,(a_{r-1} - 1), \quad r \geq 2. \tag{B9}$$

For each r there are

$$\beta_r = \overline{\alpha}_r - \overline{\gamma}_r = (r-1)! \quad r \geq 2 \tag{B10}$$

independent connected tensors. The total number of independent tensors β_r is given by

$$B(t) \equiv \sum_{r=0}^{\infty} \frac{\beta_r}{r!} t^r, \tag{B11}$$

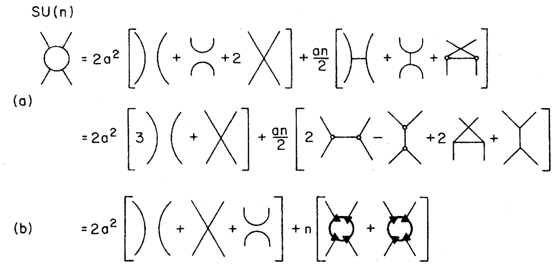

FIG. 29. Gluon "box" diagram evaluated in (a) two different f and d bases and (b) T_i basis.

$$\bar{B}(t)=\sum_{r=2}^{\infty}\frac{\bar{\beta}_r}{r}t^r$$
$$=-t-\ln(1-t), \tag{B12}$$
$$B(t)=e^{\bar{B}(t)}=\frac{e^{-t}}{1-t}. \tag{B13}$$

But $B(t)$ is precisely the generating function for subfactorials, so we have rederived the simple counting of (6.1) in a complicated way.

Once a set of β_r independent tensors has been constructed, the tensor to be simplified is expanded in this basis. By contracting all its indices with each basis tensor, a set of β_r linear equations is obtained. Now it is necessary to solve these equations—for the details, we refer the reader to Dittner's papers.[28] To illustrate the form of the results, we give the reduction of a gluon "box" diagram in two (of many possible) choices of f, d bases [Fig. 29(a)]. For comparison with the method of Sec. VI, we also evaluate the same diagram in T_i bases, Fig. 29(b).

To summarize, for SU(n) the knowledge of the invariants of the adjoint representation leads to a feasible method of weight evaluation. However, compared with the evaluation via the defining representation, it suffers from numerous drawbacks. It introduces a tensor d_{ijk} that does not appear in the original interaction Lagrangian, and leads to arbitrariness in the choice of tensor bases (note that the T_i bases are unique). Finally, it involves solving large sets of linear equations; already for $r=4$ we found it convenient to do the algebra on a computer.[75] By contrast, if we use the defining representation, evaluation never requires solving any equations (for classical groups, at least): It is a systematic procedure of eliminating internal gluons one by one until only irreducible tensors are left. If there are d_{ijk} couplings in the model, they are easily incorporated into our scheme by Fig. 25(d).

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