Love, betrayal and commitment.*

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Abstract

We present a theoretical framework that captures a possible role for love and mutual esteem in supporting efficient outcomes in dynamic decisions within the household.

1. Introduction.

In many contexts there are problems with limited commitment in intertemporal interactions; examples include worker training and workers and firms (see Malcomson (1997)); sharing risk between villagers (see, for example, Ligon, Thomas and Worrall (2002)) and central bakers and consumers. The problem of limited commitment also arises within families; see, for example: Basu (2006), Dercon and Goldstein (2000), Dubois and Ligon (2003),Duflo and Udry (2003), Goldstein (2002), Hess (2004), Ligon (2002), Lundberg and Pollak (2003), Mazzocco (2007) and Wahhaj (2007). Although the general analysis of limited commitment

*I thank Yoram Weiss for many stimulating conversations. I am grateful to the audience at the 2008 Lechene family workshop at IFS for their scepticism on love; this lead directly to this paper.

provides some guidelines for the analysis of families there are distinct differences between the contexts. In particular, family relationships often involve love which is conspicuously absent from the worker-firm relationship or in interactions between village members and is unthinkable for central bankers. In this paper we propose a theory of love and betrayal that sustains many Pareto efficient outcomes for families that would not otherwise be available to rational forward looking members of a partnership. The model is developed in the specific context of the location decision model of Mincer (1978) and Lundberg and Pollak (2003) but has wider application. In this model a couple, a and b, are presented with an opportunity to increase their joint income if they move to another location. The problem arises that the move shifts power within the household toward one partner (partner b, say) and the other partner (a) will veto the move if he or she is worse off after the move. Promises by partner b are not incentive compatible since a does not have any credible punishment threat. We suggest that if one partner exercises too aggressively their bargaining power then the other partner loses some regard (or love) for them. The important element is that this loss of love (by a in our case) is out of the control of the affected partner; in this sense, this is betrayal. In a model with mutual love, this 'punishment' is often sufficient to deter a partner from exercising their full bargaining power.

2. Caring and Pareto weights.

Consider a married couple a ('her') and b ('him'). Income, normalised to unity, is divided between them so that a receives x for private consumption and b receives $1 - x$. Each person has the same strictly increasing, strictly concave felicity function, so that:

$$\begin{aligned} u_a &= u(x) \\ u_b &= u(1-x) \end{aligned} \tag{2.1}$$

We normalise $u(0) = 0$ so that $u(x) > 0$ for $x > 0$. Each person also cares for the other with individual utility functions given by:

$$\begin{aligned} W_a(x) &= u_a + l_a u_b \\ &= u(x) + l_a u(1-x) \end{aligned} \tag{2.2}$$

$$\begin{aligned} W_b(x) &= l_b u_a + u_b \\ &= l_b u(x) + u(1-x) \end{aligned} \tag{2.3}$$

where $l_s \geq 0$ is person s's caring for the other person. We assume that the values of the caring parameters are outside the control of either partner. Note that with the non-negative normalisation on u, each partners welfare is increasing in the caring they feel ($\partial W_a / \partial l_a \geq 0$ and similarly for b). This assumption has a substantive impact on the analysis below; we regard it as reasonable to impose a normalisation that ensures that if one partner's love for the other increases (and allocations are held fixed) then his or her welfare increases.

To rule out pathological cases with 'excess caring' we restrict $l_a l_b$ to be strictly less than unity[1]; in this case the two partners will not agree on the distribution of income and we have to specify some mechanism to choose a value for x. Rather than choosing an explicit game form, we shall simply assume that there is some (collective) procedure that leads the household to behave as though it maximises the function:

$$\hat{W} = W_a + \mu W_b \tag{2.4}$$

The *objective Pareto weight*, μ, captures all of the external factors that might affect the power of b within the household. These could include: tradition and societal norms; relative incomes or wages; outside options in the event of divorce (including relative attractiveness); the use of violence and idiosyncratic propensities to dominate or to accommodate the other's wishes.

[1] An alternative formulation to weights on the felicity functions is to have each partner weighting the other person's utility function function. This gives an equivalent formulation, given our restrictions $l_a l_b < 1$. See Bergstrom (1989) and Bernheim and Stark (1988).

The household welfare function is:

$$\begin{aligned}
\hat{W} &= u(x) + l_a u(1-x) + \mu(l_b u(x) + u(1-x)) \\
&= (1+\mu l_b) u(x) + (l_a+\mu) u(1-x) && (2.5) \\
&\sim u(x) + \frac{(l_a+\mu)}{(1+\mu l_b)} u(1-x) && (2.6) \\
&= u(x) + \lambda u(1-x) && (2.7)
\end{aligned}$$

(where the $\sim$ denotes that these two expressions represent the same ordering). We denote λ the *subjective Pareto weight* to distinguish it from the objective value that applies if there is no caring ($=\mu$). As can be seen, b's relative weight in the allocation of consumption, λ, is strictly decreasing in his love for her (l_b); strictly increasing in her love for him (l_a) and strictly increasing in b's objective Pareto weight μ (since $l_a l_b < 1$). In this framework, an increase in love looks like a loss of power, when we consider allocations.

The level of consumption for a is given as:

$$\hat{x}(l_a, l_b, \mu) = \arg\max_x \{u(x) + \lambda u(1-x)\} \tag{2.8}$$

The following gives the properties for a's optimal consumption:

$$\frac{\partial \hat{x}}{\partial l_a} < 0, \frac{\partial \hat{x}}{\partial l_b} > 0, \frac{\partial \hat{x}}{\partial \mu} < 0 \tag{2.9}$$

The individual indirect utilities are given by:

$$\begin{aligned}
\hat{V}_a(l_a, l_b, \mu) &= u(\hat{x}) + l_a u(1-\hat{x}) \\
\hat{V}_b(l_a, l_b, \mu) &= l_b u(\hat{x}) + u(1-\hat{x}) && (2.10)
\end{aligned}$$

We have the following properties for the realised indirect utilities:

$$\begin{aligned}
&\frac{\partial \hat{V}_a}{\partial l_a} \gtrless 0, \frac{\partial \hat{V}_a}{\partial l_b} > 0, \frac{\partial \hat{V}_a}{\partial \mu} < 0 \\
&\frac{\partial \hat{V}_b}{\partial l_a} > 0, \frac{\partial \hat{V}_b}{\partial l_b} \gtrless 0, \frac{\partial \hat{V}_a}{\partial \mu} > 0 && (2.11)
\end{aligned}$$

The ambiguous 'own love' response arises since:

$$\frac{\partial \hat{V}_a}{\partial l_a} = \left(\frac{(1-l_a l_b)\mu}{1+\mu l_b}\right) u'(1-\hat{x}) \frac{\partial \hat{x}}{\partial l_a} + u(1-\hat{x}) \tag{2.12}$$

where the first term on the rhs is negative and the second is positive. Thus a's realised welfare may be decreasing in her love for him if it leads to a fall in consumption that outweighs the gain from her extra weighting of his utility.

3. Commitment and betrayal.

Now we introduce a decision that the two people can make that leads to a potential Pareto improvement. To fix ideas, suppose that there is a potential move to another location that would increase household income but would also increase b's objective bargaining power (as in Lundberg and Pollak (2003)). Either partner has a veto on the move. Partner b will always favour the move. Person a will veto the move if she is worse off after the move; that is, if her share of the bigger pie is less than she receives of the smaller pie. Thus we can model the moving decision as being under a's control. In the event that a vetoes the move, b would be willing to forgo his extra power to realise the potential gain for both. The question is how to do this if there is no formal commitment mechanism. The mechanism we have in mind is that b promises not to use his new found power if they move. If, however, they do move and then b chooses to exercise his new found power, then a loses some caring for him. This is out of her control (she feels 'betrayed') and acts as a punishment device for b. The rest of this paper explores the implications of such a model.[2]

Suppose there is a (moving) decision, d_a, that can be made by a that costlessly increases household income from unity to $y > 1$. If this is the only effect then, of course, a would always choose $d_a = 1$. However, we also assume that the decision increases b's objective Pareto weight to $\mu(1+m)$ where $m \geq 0$. If a chooses $d_a = 1$ then b can choose whether or not to exercise the new found gain in the Pareto weight; denote this decision by $d_b = 1$ if b chooses to exercise his increased power. Critically we shall assume that this decision has an impact on a's caring for b. We assume that a choice of $d_b = 1$ represents a betrayal to a and she finds herself caring less for b. This is the novel mechanism that operates between married partners that is missing in non-family relationships. The fall in her caring for him is taken to be exogenous so that a has an automatic and hence credible punishment for b choosing to take advantage of his improved position. We assume that a's new caring parameter for b is $l_a(1 - smd_b)$ where $s \in [0, m^{-1}]$.[3] Thus a's loss of caring for b if he chooses $d_b = 1$ depends on both

[2] We neglect the option in which they divorce and the husband moves to the new location.

[3] We could have taken the new level of caring to be $l_a(1 - sd_b)$ with $s \in [0,1]$, but the

an extrinsic factor, s, and the amount by which b gains from choosing to exercise his increased power.

We have to consider three cases: $(d_a, d_b) = (0, 0)$ (no moving, case I), $(1, 0)$ (moving with b not exercising his new power, case II) and $(1, 1)$ (moving with b exercising his new power, case III). Case I is the situation described in the previous subsection with:

$$\hat{W} = (1 + \mu l_b) \, u(x) + (l_a + \mu) \, u(1 - x) \tag{3.1}$$

For case II, the household utility function used to choose x is given by:

$$\tilde{W} = (1 + \mu l_b) \, u(x) + (l_a + \mu) \, u(y - x) \tag{3.2}$$

For the case III we have:

$$\begin{aligned} \breve{W} &= (1 + \mu (1 + m) \, l_b) \, u(x) + (l_a (1 - sm) + \mu (1 + m)) \, u(y - x) \\ &= ((1 + \mu l_b) + \mu m l_b) \, u(x) + ((l_a + \mu) + m (\mu - sm)) \, u(y - x) \end{aligned} \tag{3.3}$$

Let the choices associated with the three cases be denoted $\hat{x}$, $\tilde{x}$ and $\breve{x}$ respectively. We have the following comparative statics conditions:

$$\frac{\partial \tilde{x}}{\partial y} > 0, \frac{\partial \breve{x}}{\partial y} > 0, \frac{\partial \breve{x}}{\partial m} < 0, \frac{\partial \breve{x}}{\partial s} > 0 \tag{3.4}$$

For example, for the final result we use the first order condition:

$$(1 + \mu (1 + m) \, l_b) \, u'(\breve{x}) = (l_a (1 - sm) + \mu (1 + m)) \, u'(y - \breve{x}) \tag{3.5}$$

Taking the partial with respect to s we have:

$$\begin{aligned} &[(1 + \mu (1 + m) \, l_b) \, u''(\breve{x}) + (l_a (1 - sm) + \mu (1 + m)) \, u''(y - \breve{x})] \frac{\partial \breve{x}}{\partial s} \\ &+ l_a s u'(y - \breve{x}) = 0 \end{aligned} \tag{3.6}$$

The term in square brackets is negative and the final term is positive, hence $\frac{\partial \breve{x}}{\partial s} > 0$.

formulation used leads to easier derivations.

The pairs of associated indirect utility functions are given by:

$$\hat{V}_a = u(\hat{x}) + l_a u(1-\hat{x}), \quad \hat{V}_b = l_b u(\hat{x}) + u(1-\hat{x}) \tag{3.7}$$

$$\tilde{V}_a = u(\tilde{x}) + l_a u(y-\tilde{x}), \quad \tilde{V}_b = l_b u(\tilde{x}) + u(y-\tilde{x}) \tag{3.8}$$

$$\breve{V}_a = u(\breve{x}) + l_a(1-sm)u(y-\breve{x}), \quad \breve{V}_b = l_b u(\breve{x}) + u(y-\breve{x}) \tag{3.9}$$

Note that a's utility function changes in case III as she puts less weight on b's consumption; this is a loss of utility that both must take into account when considering the different cases. We have some immediate results.

Lemma 3.1. *(Local selfishness). In all cases, both a and b would prefer more expenditure for themselves than the household maximisation problem yields.*

Lemma 3.2. *Case II Pareto dominates case I: $\tilde{V}_a > \hat{V}_a$ and $\tilde{V}_b > \hat{V}_b$.*

Lemma 3.3. *There is a threshold loss of caring, s^*, such that b weakly prefers II to III if and only if:*

$$s \geq s^* = \frac{\mu(1-l_a l_b)}{(1+\mu l_b) l_a} \tag{3.10}$$

Note that the threshold loss of caring, s^*, does not depend on the increase in income, y, or the change in the Pareto weight, m. Unlike the cases I and II we cannot give any results for the preferences for cases II relative to III. Indeed, the following table shows that any combination of preferences is possible. (uses the utility function $u(c) = \ln(1+c)$). Remark particularly on a prefers III and b prefers II.

4. Decision making.

4.1. Decisions with no loss of caring.

It is convenient to first consider the case in which there is there is no penalty for b choosing $d_b = 1$; that is, $s = 0$. In this case b will always set $d_b = 1$ so that case II is irrelevant and the comparison for the two partners is between cases I and III. In case III, b has a larger share of a bigger income so he always prefers III to I. For person a there is a straight trade-off between a higher y and a higher Pareto weight for b. Indeed, if the gain in income is large enough, a will always prefer III to I so long as b has some caring for a. Formally:

Lemma 4.1. *For any values of (l_a, l_b, μ) with $l_b > 0$ there exists a $\bar{y}$ such that if $y > \bar{y}$, a always prefers III to I.*

If the new income is below $\bar{y}$ then there is some threshold value for m, which we denote $m_0(y)$, such that a value of m below $m_0(y)$ would lead to a agreeing to move. The properties of this function are given in the next proposition.

Proposition 4.2. *$m_0(y)$ is a mapping $[1, \bar{y}) \rightarrow [0, \infty)$ that is strictly increasing with $m_0(1) = 0$.*

If $y \in [1, \bar{y})$ and there are no sanctions for b ($s = 0$) then a will veto the potential Pareto improvement if it entails too large an increase in b's Pareto weight; that is, if $m > m_0(y)$.

4.2. Decisions with a penalty for behaving badly: $s > 0$.

4.2.1. b deciding between II and III.

For the comparison of cases II and III, 3.3 gives that b prefers case II to case III if and only if $s > s^*$ where:

$$s^* = \frac{\mu(1 - l_a l_b)}{(1 + \mu l_b) l_a} \tag{4.1}$$

Thus person b will choose to exercise his extra power (set $d_b = 1$) if s is low. We have:

$$\begin{aligned}
\frac{\partial s^*}{\partial \mu} &= \frac{(1 - l_a l_b)}{(1 + \mu l_b)^2 l_a} > 0 \\
\frac{\partial s^*}{\partial l_a} &= -\frac{\mu}{(1 + \mu l_b)(l_a)^2} < 0 \\
\frac{\partial s^*}{\partial l_b} &= -\frac{\mu(\mu + l_a)}{(1 + \mu l_b) l_a} < 0
\end{aligned} \tag{4.2}$$

Thus b is more likely to exercise his option to set $d_b = 1$ if he has *more* power initially. The intuition here is that if he is not very powerful initially, then he relies on a's caring for him for transfers; choosing to behave badly undermines this. He is also more likely to choose $d_b = 1$ if her love for him (l_a) is weaker and if his love for her (l_b) is weaker.

4.2.2. a deciding between I and III.

When considering a we need to compare cases I and III . Once again we can define a threshold level of income in case III which is high enough to induce a

to prefer III to I. In this case, the threshold value is a function of m and s, $y^*(m,s)$, defined by:

$$u\left(\breve{x}\left(y^*\left(m,s\right),m,s\right)\right)+l_a\left(1-sm\right)u\left(y-\breve{x}\left(y^*\left(m,s\right),m,s\right)\right)=u\left(\hat{x}\right)+l_a u\left(1-\hat{x}\right) \tag{4.3}$$

Analogously to the case considered in the last subsection, we have $y^*(0,s)=1$ for all s. To derive the other properties of y^* we take derivatives with respect to m and s. Considering the first we have:

$$\left(u'\left(\breve{x}\right)-l_a\left(1-sm\right)u'\left(y-\breve{x}\right)\right)\left[\frac{\partial\breve{x}}{\partial y}\frac{\partial y^*}{\partial m}+\frac{\partial\breve{x}}{\partial m}\right]=sl_a u\left(y-\breve{x}\right) \tag{4.4}$$

The right hand side is positive, the first term on the left hand side is positive, $\frac{\partial\breve{x}}{\partial y}>0$ and $\frac{\partial\breve{x}}{\partial m}<0$; together these imply that $\frac{\partial y^*}{\partial m}>0$. For s we have:

$$\left(u'\left(\breve{x}\right)-l_a\left(1-sm\right)u'\left(y-\breve{x}\right)\right)\left[\frac{\partial\breve{x}}{\partial y}\frac{\partial y^*}{\partial s}+\frac{\partial\breve{x}}{\partial s}\right]=ml_a u\left(y-\breve{x}\right) \tag{4.5}$$

Since $\frac{\partial\breve{x}}{\partial s}>0$ this does not give a sign for $\frac{\partial y^*}{\partial s}$.

4.3. Outcomes.

We need only consider values of $m\in\left[0,(s^*)^{-1}\right]$. Figure 4.1below shows the potential outcomes for $y<\bar{y}$.[4] Considering first the axis with $s=0$ (see subsection 4.1), we have that the outcome will be case III if and only if $m\le m_0$.[5] Suppose now that $m>m_0$ and we introduce a small s; this takes us into the south-east (SE) segment marked 'I to I'. In this case there is no change in the moving decision since b will betray and a knows it. However, as s becomes larger the advantage to b from exercising his additional power diminishes until, at the point $s=s^*$, b is indifferent between cases II and III. For higher values of s he will not renege on his promise and a can choose to allow the move. Indeed, as can be seen, for any value of $s>s^*$ we have that the final decision is to move and the husband not exercising his additional power; see Lemma **??**. This is the original intuition that motivated this paper. The final region we consider is the case when $s<s^*$ and $m<m_0$. If both s and m are small, then the decisions in favour of case III are unchanged; see the region marked 'III to III'. The most

[4] The utility function used to generate this figure is $u(x)=\ln(1+x)$. The parameter values are $l_a=0.2$, $l_b=0.5$, $\mu=1$ and $y=1.05$. The range of m is $[0.1,0.15]$.

[5] Here we assume that if a is indifferent between I and III then she chooses $d_a=1$ (which leads to case III).

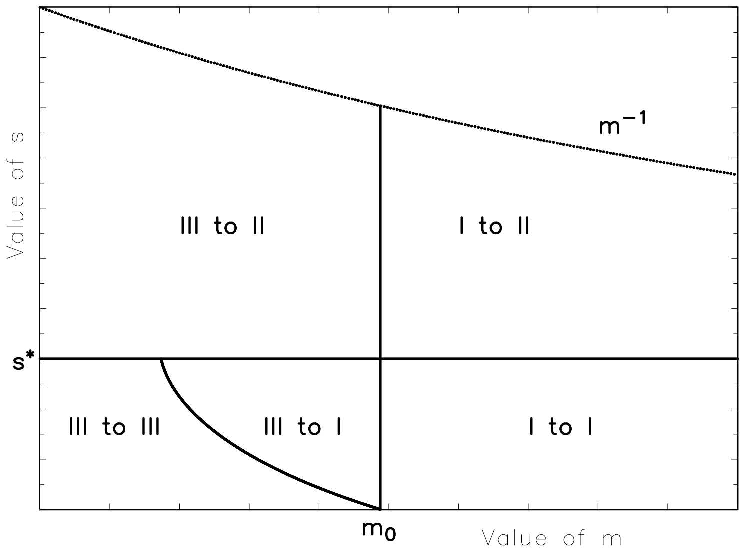

Figure 4.1: Decisions as a function of s and m.

interesting region, from a theoretical point of view, is the region marked 'III to I'. With this configuration of s and m, partner a will reverse their decision in favour of moving since the consequent loss due to her fall in caring for b will outweigh the benefit of a smaller share of a larger income.

[For 'III to I' region, show that if $0 < s < s^*$ and $m = m_0 - \varepsilon$ then a prefers I to III.]

5. Conclusions.

The main contribution of this paper is to show how caring (or love) in a two person interaction might lead to Pareto improvements that would not otherwise be realised without binding commitments. The context is very specific - it is not suggested that other interactions (such as firms and workers or central banks and consumers) would have access to this device.

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Proof of Lemma 3.3. The implicit weights for b in (3.2) and (3.3) are given by:

$$\tilde{\omega}_b = \frac{l_a + \mu}{1 + \mu l_b} \tag{.1}$$
$$\breve{\omega}_b = \frac{l_a (1 - sm) + \mu (1 + m)}{1 + \mu (1 + m) l_b} \tag{.2}$$

We have $\tilde{\omega}_b \geq \breve{\omega}_b$ if and only:

$$s \geq \frac{\mu (1 - l_a l_b)}{(1 + \mu l_b) l_a} = s^* \tag{.3}$$

$\tilde{\omega}_b \geq \breve{\omega}_b$ if $\tilde{x} \leq \breve{x}$. Since the indirect utility function for b is $l_b u (x) + u (y - x)$ in both cases II and III, the ordering of $\tilde{x}$ and $\breve{x}$ establishes the ordering of the two cases.■

Proof of Lemma ?? If $s \geq s^*$ then b prefers case II. a always prefers case II to case III.■

Proof of Lemma ??. ■

Proof Lemma 4.1. [USE THAT MAXIMUM UTILITY FOR A FOR CASE I BOUNDED ABOVE BY UNIT INCOME]For any $y < \bar{y}$ partner a will only prefer III to I if m is small enough. To show this formally, define a threshold income $m_0 (y)$ by:

$$u (\breve{x} (m_0 (y) , y)) + l_a u (y - \breve{x} (m_0 (y) , y)) \equiv u (\hat{x}) + l_a u (1 - \hat{x}) \tag{.4}$$

That is, $m_0 (y)$ is the value of the increase in b's Pareto weight that leaves a indifferent between I and III, given that the new income is y. The following establishes the properties of this function.■

Proof 4.2. If $y < \bar{y}$ there are values of m that satisfy (.4). First we show that for any y there is a unique value for m_0. Take the implicit weight $\omega_a (m)$ defined in the previous proof (**??**). We have:

$$\omega_a (0) = \frac{1 + \mu l_b}{l_a + \mu} > l_b = \lim_{m \to \infty} \omega_a (m) \tag{.5}$$

The inequality holds since $l_a l_b < 1$. The implicit weight for partner a for the first case, $\hat{W}$, (see (3.3)), is given by:

$$\hat{\omega}_a = \frac{1 + \mu l_b}{l_a + \mu} > l_b = \lim_{m \to \infty} \omega_a (m) \tag{.6}$$

Define:

$$\hat{x} = \arg\max_x \hat{\omega}_a u(x) + u(1-x) \tag{.7}$$

$$\breve{x}(m) = \arg\max_x \omega_a(m) u(x) + u(y-x) \tag{.8}$$

The function $\breve{x}(m)$ is strictly decreasing with $\breve{x}(0) > \hat{x}$ and

$$\lim_{m \to \infty} \breve{x}(m) < \hat{x} \tag{.9}$$

since $y < \bar{y}$. Thus $\exists$ a unique m_0 such that $\breve{x}(m_0) = \hat{x}$.$\blacksquare$