ANNALES SCIENTIFIQUES DE L'É.N.S.

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Annales scientifiques de l'É.N.S. 4e série, tome 33, nº 2 (2000), p. 151-179

<http://www.numdam.org/item?id=ASENS_2000_4_33_2_151_0>

Ann. Scient. Éc. Norm. Sup.,
4e série, t. 33, 2000, p. 151 à 179.

HODGE DECOMPOSITION FOR HIGHER ORDER HOCHSCHILD HOMOLOGY

TEIMURAZ PIRASHVILI

ABSTRACT. – Let Γ be the category of finite pointed sets and F be a functor from Γ to the category of vector spaces over a characteristic zero field. Loday proved that one has the natural decomposition $\pi_n F(S^1) \cong \bigoplus_{i=0}^n H_n^{(i)}(F)$, $n \geqslant 0$. We show that for any $d \geqslant 1$, there exists a similar decomposition for $\pi_n F(S^d)$. Here S^d is a simplicial model of the d-dimensional sphere. The striking point is, that the knowledge of the decomposition for $\pi_n F(S^1)$ (respectively $\pi_n F(S^2)$) completely determines the decomposition of $\pi_n F(S^d)$ for any odd (respectively even) d. These results can be applied to the cohomology of the mapping space X^{S^d}, where X is a d-connected space. Thus Hodge decomposition of $H^*(X^{S^1})$ and $H^*(X^{S^2})$ determines all groups $H^*(X^{S^d})$, $d \geqslant 1$. © 2000 Éditions scientifiques et médicales Elsevier SAS

RÉSUMÉ. – Soient Γ la catégorie des ensembles finis pointés et F un foncteur de la catégorie Γ vers la catégorie des espaces vectoriels sur un corps de caractéristique zéro. Loday montre dans (Loday, 1998) que l'on a une décomposition naturelle $\pi_n F(S^1) \cong \bigoplus_{i=0}^n H_n^{(i)}(F)$, $n \geqslant 0$. On démontre dans cet article qu'il existe une décomposition naturelle pour $\pi_n F(S^d)$, où S^d est un modèle simplicial pour les sphères de dimension d. Le fait important ici est que la décomposition pour $d = 1$ (resp. $d = 2$) détermine complètement la décomposition pour tout d impair (resp. pair). Ce résultat peut être appliqué à la cohomologie des espaces fonctionnels X^{S^d}. Donc les décompositions de $H^*(X^{S^1})$ et $H^*(X^{S^2})$ déterminent complètement tous les groupes $H^*(X^{S^d})$, $d \geqslant 1$. © 2000 Éditions scientifiques et médicales Elsevier SAS

0. Introduction

Let A be a commutative algebra and M be an A-module. Let

$$\mathcal{L}(A, M) : \Gamma \to \mathit{Vect}$$

be the functor from the category Γ of finite pointed sets to the category of vector spaces given by

$$\{0, 1, \ldots, n\} \mapsto M \otimes A^{\otimes n},$$

see also Section 1.7. There is a standard way to prolong the functor $\mathcal{L}(A, M) : \Gamma \to \mathit{Vect}$ to a functor from the category Sets_* of all pointed sets to the category of vector spaces by direct limits. By abuse of notation we will still denote this functor by $\mathcal{L}(A, M)$. For any pointed simplicial set $Y : \Delta^{op} \to \mathit{Sets}_*$ we let $H_*^Y(A, M)$ denote the homology of the chain complex associated to the simplicial vector space

$$\Delta^{op} \xrightarrow{Y} Sets_* \xrightarrow{\mathcal{L}(A,M)} Vect.$$

For $Y = S^d$, the sphere of dimension $d \geqslant 1$, one uses the notation $H_*^{[d]}(A,M)$ instead of $H_*^{S^d}(A,M)$. For $Y = S^1$ one recovers the usual Hochschild homology of A with coefficients in M (see [17]). Because of this fact, we call the groups $H_*^{[d]}(A,M)$ *Hochschild homology of order* d *of* A *with coefficient in* M. Higher order Hochschild homology was implicitly defined in [1]. These groups are related to the cohomology of the mapping spaces X^{S^d} in the same way as usual Hochschild homology is related to the cohomology of the free loop space X^{S^1} (see Section 5). The main goal of this paper is to show that the higher order Hochschild homology in the characteristic zero case has a natural decomposition, which, for $d = 1$, is isomorphic to the classical Hodge decomposition [17]. Our methods are new even for $d = 1$ and are based on homological properties of Γ-modules. As a consequence we show that the knowlege of the Hodge decomposition of the cohomology $H^*(X^{S^1})$ of the free loop space determines the cohomology $H^*(X^{S^d})$ of the mapping spaces for all odd d and, similarly, the knowlege of the Hodge decomposition of the cohomology $H^*(X^{S^2})$ determines the cohomology $H^*(X^{S^d})$ for all even d, provided X is a d-connected space (see Section 5).

Let us recall that the Hodge decomposition for Hochschild homology of commutative algebras in the characteristic zero case first was obtained by Quillen [23]. Using sophisticated combinatorics Gerstenhaber and Schack constructed the so-called Eulerian idempotents and proved that they yield a decomposition of Hochschild homology (see [12,16]). It turns out that both decompositions are isomorphic to each other (see [25]). Moreover, Loday made in [16] the important observation that such decompositions have a more general nature. Namely he proved the following result: Let $F:\Gamma \to Vect$ be a functor from finite pointed sets to the category of vector spaces over a characteristic zero field. Let $S^1:\Delta^{op} \to Sets_*$ be the standard simplicial model for the circle, which has only two nondegenerate simplices. Clearly the values of S^1 are finite pointed sets, so one can take the composition of these two functors $F(S^1):\Delta^{op} \to Vect$. In this way one gets the simplicial vector space $F(S^1)$ and one can take the homotopy groups of it. Among other things, Loday proved that there exists a natural decomposition

$$\pi_n F(S^1) \cong \bigoplus_{i=0}^{n} H_n^{(i)}(F), \quad n \geqslant 0,$$

where the groups $H_n^{(i)}(F)$ are defined using Eulerian idempotents. If A is a commutative algebra and M is a symmetric bimodule, then for $F = \mathcal{L}(A,M)$ Loday's result gives exactly the Hodge decomposition for Hochschild homology. Loday obtained in [16] also a similar decomposition for cyclic homology (see Section 3 for more details). Later McCarthy remarked that Loday's decomposition can be obtained using the rotations of the circle [19]. We refer to [30] for relationship between Hodge decomposition of cyclic homology and Hodge filtration in algebraic geometry, and [6] and [29] for related results.

We give an alternative approach of this subject. We do not only give a new purely homological proof of Loday's results, but we obtain essentially more. Using homological algebra of Γ-modules we construct a spectral sequence, whose abutment is $\pi_* F(Y)$. Here Y is any pointed simplicial set. We show that for $Y = S^d$, the spectral sequence collapses at E^2 level and gives a natural decomposition for $\pi_n F(Y)$ in characteristic zero. We give a simple axiomatic characterization of the decomposition of $\pi_* F(S^1)$, from which we deduce that our decomposition and Loday's decomposition are isomorphic. The striking fact is, that in the decomposition of

$\pi_n F(S^d)$ for d odd, the same groups $H_k^{(i)}(F)$ appear but in a different way:

$$\pi_n F(S^d) \cong \bigoplus_{i+dj=n} H_{i+j}^{(j)}(F), \quad n \geqslant 0,\ d = 2k+1 \geqslant 1.$$

We define the Hochschild homology of order d of a commutative algebra A with coefficients in an A-module M to be $\pi_n F(S^d)$, where $F = \mathcal{L}(A, M)$. As an example we compute higher order Hochschild homology for smooth algebras (for all d) and for truncated polynomial algebras (for d odd). We prove that for all d the Hochschild homology of order d of the de Rham complex of a d-connected manifold is isomorphic to the homology of the mapping space X^{S^d}.

It is worth to mentioning that any functor $F : \Gamma \to$ *Vect* gives rise to an abelian spectrum, thanks to the famous result of Segal (see [27], notice that our Γ is the opposite of the original Γ of Segal). Let $\pi_n^{st} F$ be the homotopy groups of the corresponding spectrum. It turns out that the groups $H_n^{(1)}(F)$ (which are known as Harrison homology [16]) and $\pi_n^{st} F$ are isomorphic in the characteristic zero case. Therefore one can think of the groups $\pi_n^{st}\mathcal{L}(A, M)$ as a modification of André–Quillen homology for characteristic $p > 0$. In [22] it is proved that it is isomorphic to a "brave new algebra" version of André–Quillen homology constructed by Robinson and Whitehouse (see [31]).

The paper is organized as follows: In Section 1 we describe the basic properties of the category of functors from pointed finite sets to vector spaces. In the next Section we prove the decomposition properties for $\pi_* F(S^d)$. In Section 3 we reprove Loday's decomposition theorem for cyclic homology. We use the observation that the forgetful functor from Γ to the category of nonempty finite sets is flat in some sense. In Section 4 we prove a version of the Hochschild–Kostant–Rosenberg theorem for smooth Γ-modules, which is the main technical tool for the calculation of higher order Hochschild homology for smooth algebras in the last section. In the same section we also prove that the higher order Hochschild homology of the Sullivan cochain algebra of a d-connected space X is isomorphic to the cohomology of the mapping space X^{S^d}.

This work was written during my visit at the Sonderforschungsbereich der Universität Bielefeld. I would like to thank Friedhelm Waldhausen for the invitation to Bielefeld. The author wishes to thank the referee for many useful critical comments, helpful suggestions and pointing out several mistakes. Proposition 1.6(iii), the last assertion in Corollary 2.5 and the first assertion in Theorem 4.6 are due to the referee. He pointed out also the correct proofs of Proposition 1.12 and Theorem 5.6. After his suggestion now Section 3.2 looks much better than it was in the previous version. The author was partially supported by the grant INTAS-93-2618-Ext and by the TMR network K-theory and algebraic groups, ERB FMRX CT-97-0107.

1. Properties of Γ-modules

1.1. Γ-modules

Let K be a field. In what follows all vector spaces are defined over K. Moreover $\otimes_K$ and Hom_K are denoted by $\otimes$ and Hom respectively. Let *Vect* be the category of vector spaces.

Let Γ be the category of finite pointed sets. For any $n \geqslant 0$, we let $[n]$ be the set $\{0, 1, \dots, n\}$ with basepoint 0. We assume that the objects of Γ are the sets $[n]$. Let Γ-*mod* be the category of all covariant functors from Γ to *Vect*. Similarly *mod*-Γ denotes the category of contravariant functors from Γ to the category of vector spaces. The objects of Γ-*mod* (respectively *mod*-Γ) are called left (respectively right) Γ-modules. We will use the term Γ-module if we do not want to distinguish between left and right Γ-modules.

The categories Γ*-mod* and *mod-*Γ are abelian categories with sufficiently many projective and injective objects. For any $n \geqslant 0$ one defines

$$\Gamma^n := K\big[\mathrm{Hom}_\Gamma([n], -)\big] \quad \text{and} \quad \Gamma_n := K\big[\mathrm{Hom}_\Gamma(-, [n])\big].$$

Here $K[S]$ denotes the free vector space generated by a set S. It is a consequence of the Yoneda lemma that for any left Γ-module F and any right Γ-module T one has natural isomorphisms

(1.1.1) $$\mathrm{Hom}_{\Gamma\text{-}mod}(\Gamma^n, F) \cong F([n])$$

and

(1.1.2) $$\mathrm{Hom}_{mod\text{-}\Gamma}(\Gamma_n, T) \cong T([n]).$$

Therefore Γ^n (respectively Γ_n), $n \geqslant 0$, are projective generators of the category Γ*-mod* (respectively *mod-*Γ). Clearly Γ_0 and Γ^0 are constant functors with the value K. Any pointed map $[m] \to [n]$ yields morphisms of Γ-modules $\Gamma_m \to \Gamma_n$ and $\Gamma^n \to \Gamma^m$.

We shall use the following simple and well known fact very extensively (see Proposition II.1.3 of [11]).

YONEDA PRINCIPLE. – *Let* $G_1, G_2 : \Gamma\text{-}mod \to Vect$ *be right exact functors, commuting with sums. Let* $\widetilde{G}_i$ *be the composition* $\Gamma^{op} \xrightarrow{\Gamma^?} \Gamma\text{-}mod \xrightarrow{G_i} Vect$, $i = 1, 2$, *where the first functor assigns* Γ^n *to* $[n]$. *If* $\widetilde{G}_1 \cong \widetilde{G}_2$, *then* $G_1 \cong G_2$.

1.2. Duality and functors of finite type

For any vector space V one denotes the dual vector space by V^*. One can extend this notion for Γ-modules. For a Γ-module F one defines F^* by

$$F^*([n]) := \big(F([n])\big)^*.$$

Then F^* is a right Γ-module if F is a left Γ-module and vice-versa. A functor is called *of finite type* if it has values in finite dimensional vector spaces. Clearly $F^{**} \cong F$ if F is a functor of finite type. Let F be a left Γ-module and T be a right Γ-module. Then one has a natural isomorphism

$$\mathrm{Hom}_{\Gamma\text{-}mod}(F, T^*) \cong \mathrm{Hom}_{mod\text{-}\Gamma}(T, F^*).$$

Therefore for any projective Γ-module F, the dual module F^* is injective. Moreover the functors Γ^{n*} and Γ_n^* are injective cogenerators of the category Γ*-mod* and *mod-*Γ respectively and one has natural isomorphisms

$$\mathrm{Hom}_{\Gamma\text{-}mod}(F, \Gamma^{n*}) \cong \big(F([n])\big)^* \quad \text{and} \quad \mathrm{Hom}_{mod\text{-}\Gamma}(T, \Gamma_n^*) \cong \big(T([n])\big)^*.$$

Let us notice that the functors Γ^{n*} and Γ_n^* are of finite type. We call them *standard injective objects*.

1.3. Pointwise tensor product

For any left Γ-modules F and T we put

$$(F \otimes T)([n]) := F([n]) \otimes T([n]).$$

Clearly $F \otimes T$ is still a left Γ-module. In the same way one defines the same kind of operation for right Γ-modules. It follows from the definition that

$$\Gamma^n \otimes \Gamma^m \cong \Gamma^{n+m} \quad \text{and} \quad \Gamma_n \otimes \Gamma_m \cong \Gamma_{nm+n+m}, \tag{1.3.1}$$

because $[n] \vee [m] \cong [n+m]$ and $[n] \times [m] \cong [nm+n+m]$. As a consequence we see that *if F and T are projective, then $F \otimes T$ is also projective.* Moreover $\Gamma^n \cong (\Gamma^1)^{\otimes n}$ for any $n \geqslant 0$.

We claim that *if F and T are injective, then $F \otimes T$ is also injective* provided F or T is of finite type. Indeed, any injective object is a retract of a product of standard injectives. One observes that the functor $(-) \otimes T$ commutes with products provided T is of finite type. Therefore it is enough to consider the case when both F and T are standard injectives. In this case the claim is obvious.

1.4. The right Γ-modules t and θ^n

Let t be the right Γ-module given by

$$t([n]) := \mathrm{Hom}_{Sets_*}([n], K).$$

Here the field K is considered as a pointed set with basepoint 0. For an element $i \in [n]$ we let χ_i be the characteristic function of i, i.e., $\chi_i(i) = 1$ and $\chi_i(j) = 0$ if $j \neq i$. Let us remark that χ_i is a pointed map if $i \neq 0$. One has an exact sequence

$$\Gamma_2 \xrightarrow{\alpha} \Gamma_1 \xrightarrow{\beta} t \to 0. \tag{1.4.1}$$

Here α and β are given by

$$\alpha\big([n] \xrightarrow{f} [2]\big) = p_1 \circ f + p_2 \circ f - p \circ f$$

and

$$\beta\big([n] \xrightarrow{g} [1]\big) = \sum_{g(i)=1} \chi_i,$$

where p and p_i are defined as follows: For any nonzero element $i \in [n]$ we let $p_i : [n] \to [1]$ be the pointed map given by $p_i(i) = 1$ and $p_i(j) = 0$ if $j \neq i$. The pointed map $p : [2] \to [1]$ is given by $p(1) = p(2) = 1$. It is quite easy to check that the sequence (1.4.1) is indeed exact. We also refer to [20], where a projective resolution SQ_* of t was constructed, whose beginning is just the above exact sequence. Using this projective resolution one can describe the stable homotopy groups of a functor (see Proposition 2.2) via the Eilenberg–MacLane cubical construction (see also E.13.2.2 of [17]). One observes that

$$t^* \cong Coker(\Gamma^0 \to \Gamma^1). \tag{1.4.2}$$

Indeed, the values of both functors on $[n]$ are the same, namely the vector space generated by $[n]$, modulo the vector space generated by $0 \in [n]$. Here $\Gamma^0 \to \Gamma^1$ is induced by the unique map $[1] \to [0]$, which has a section. Hence $\Gamma^1 \cong \Gamma^0 \oplus t^*$. Therefore t^* is projective and t is injective. Moreover the left Γ-modules $t^{\otimes n *}$ are projective generators, while the right Γ-modules $t^{\otimes n}$ are injective cogenerators.

Let $Sym^* : Vect \to Vect$ (respectively $\Lambda^* : Vect \to Vect$) be the functor which assigns the underlying vector space of the symmetric algebra $Sym^*(V)$ (respectively exterior algebra $\Lambda^*(V)$)

to V. If K has characteristic zero, then Λ^n and Sym^n are natural direct summands of $\otimes^n$. Therefore $\Lambda^n \circ t$, $Sym^n \circ t$ and $\Lambda^n \circ t \otimes Sym^n \circ t$ are injective right Γ-modules, too.

The functor t played already an important role in Mac Lane homology theory (see [20,17,24]) and we will see that it is also very important in the present paper. We will need also the following right modules $\theta^n, n \geqslant 0$. Let $\mathcal{B}$ be the left Γ-module, whose value on $[m]$ is the free vector space generated by the subsets of $[m]$, modulo the vector space generated by the subsets containing 0. If $f:[m] \to [k]$, then the action of f on $\mathcal{B}([m])$ is induced by the direct image of f. Let $\mathcal{B}_n$ be the subfunctor of $\mathcal{B}$ generated by the subsets of cardinality $\leqslant n$. We define the functor θ^n as the dual of the quotient $\mathcal{B}_n/\mathcal{B}_{n-1}$. Clearly $t \cong \theta^1$. One observes that $(\mathcal{B}_n/\mathcal{B}_{n-1})([m])$ is isomorphic to the vector space spanned by the n-element subsets of $\{1,\dots,m\}$. Hence, as a vector space, we have $\theta^n([m]) \cong \Lambda^n(t([m]))$. But, as functors, θ^n and $\Lambda^n \circ t$ are different as soon as $n > 1$.

1.5. The bifunctor $- \otimes_\Gamma -$

For a right Γ-module N and a left Γ-module M we let $N \otimes_\Gamma M$ be the abelian group generated by all elements $x \otimes y$, where $x \in N([n])$, $y \in M([n])$ and $n \geqslant 0$, modulo the relations

$$(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y, \qquad x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2,$$
$$(ax) \otimes y = a(x \otimes y) = x \otimes (ay), \qquad \alpha^*(x') \otimes y = x' \otimes \alpha_*(y).$$

Here $\alpha:[n] \to [m]$ is a morphism in Γ, $x_1, x_2 \in N([n]), y_1, y_2 \in M([n]), x' \in N([m])$ and $a \in K$. In other words $N \otimes_\Gamma M$ is the quotient of $\sum_n N([n]) \otimes M([n])$ by the relation $\alpha^*(x') \otimes y = x' \otimes \alpha_*(y)$. It is well known (see Section 16.7 of [26]) that the bifunctor

$$- \otimes_\Gamma - : \textit{mod-}\Gamma \times \Gamma\textit{-mod} \to \textit{Vect}$$

is right exact with respect to each variable and preserves sums. Moreover, for any left Γ-module F, any right Γ-module T and any $n \geqslant 0$, there exist natural isomorphisms

$$T \otimes_\Gamma \Gamma^n \cong T([n]), \qquad \Gamma_n \otimes_\Gamma F \cong F([n]).$$

Clearly $- \otimes_\Gamma -$ is a left balanced bifunctor in the sense of Cartan and Eilenberg ([7]). Therefore the derived functors of $- \otimes_\Gamma -$ with respect to each variable are isomorphic and we will denote the common value by $\operatorname{Tor}_*^\Gamma(-,-)$. This notion has the standard extension for chain complexes of Γ-modules. Moreover, we will consider a hyperhomology spectral sequence for such Tor-groups.

1.6. PROPOSITION. – *Let F be a left Γ-module and let C_* be a nonnegative chain complex, whose components are projective right Γ-modules.*

(i) *Then there exist a first quadrant spectral sequence*

$$E^2_{pq} = \operatorname{Tor}_p^\Gamma\big(H_q(C_*), F\big) \Rightarrow H_{p+q}(C_* \otimes_\Gamma F).$$

(ii) *Suppose*

(1.6.1) $$\operatorname{Ext}^{m-n+1}_{\textit{mod-}\Gamma}\big(H_n(C_*), H_m(C_*)\big) = 0, \quad \textit{for } n < m,$$

then the spectral sequence is degenerate at E^2 and one has the decomposition

(1.6.2) $$H_n(C_* \otimes_\Gamma F) \cong \bigoplus_{p+q=n} \operatorname{Tor}_p^\Gamma\big(H_q(C_*), F\big)$$

which is natural in F.

(iii) *Suppose that not only the condition* (1.6.1) *holds, but also*

$$\mathrm{Ext}^{m-n}_{\mathit{mod}\text{-}\Gamma}\big(H_n(C_*), H_m(C_*)\big) = 0, \quad \textit{for } n < m, \tag{1.6.3}$$

then the decomposition (1.6.2) *is natural with respect of the action of endomorphisms of* C_*.

Furthermore the statements are still true if F is a chain complex of left Γ-modules.

Proof. – (i) is a standard tool of homological algebra (see, for example, Section 2.4 of [13] and especially Remarque 3 on p. 148 of *loc. cit.*)

(ii) is implicitly in [8]. We give the argument here. A chain complex C_* is called of *type* (A,n) if $H_iC_* = 0$ for $i \neq n$ and $H_nC_* \cong A$. Let us denote such a complex by (A,n). A chain complex C_* is called *formal* if it has the same weak homotopy type as $\bigoplus_{n\geqslant 0}(H_nC_*, n)$. Dold in [8] constructed the Postnikov decomposition of C_* as a kind of tower $\cdots \to C_*^{n+1} \to C_*^n \to \cdots$. Moreover he defined the ith invariant as an element in $H^{i+1}(C_*^{i-1}, H_iC_*)$ and proved that C_* is formal iff all invariants are zero. Clearly it is enough to show that in our case C_* is formal. We will show by induction that all invariants are zero. Thanks to Satz 5.2 [8] the first invariant lies in $\mathrm{Ext}^2_{\mathit{mod}\text{-}\Gamma}(H_0C_*, H_1C_*)$ which is 0 by Assumption (1.6.1) and thus this invariant is zero. Assume, now that all i-invariants are zero for $i < n$. Then C_*^{n-1} is formal. Thus it has the same weak homotopy type as $\bigoplus_{i=0}^{n-1}(H_iC_*, i)$. We can still use Satz 5.2 [8] to get

$$H^{n+1}\big(C_*^{n-1}, H_nC_*\big) \cong \bigoplus_{i=0}^{n-1} \mathrm{Ext}^{n+1-i}_{\mathit{mod}\text{-}\Gamma}(H_iC_*, H_nC_*) = 0.$$

Therefore the nth invariant vanishes and (ii) is proved.

(iii) The homology yields the well-defined homomorphism from the set of homotopy clases of endomorphisms of C_*

$$[C_*, C_*] \to \prod_n \mathrm{Hom}_{\mathit{mod}\text{-}\Gamma}(H_nC, H_nC). \tag{1.6.4}$$

Since C_* is formal the homomorphism (1.6.4) is an epimorphism. It is also monomorphism, because

$$[C_*, C_*] \cong \prod_n \Big[(H_n, n), \bigoplus_m (H_m, m)\Big] \subset \prod_{n,m} \big[(H_n, n), (H_m, m)\big]$$

and therefore

$$[C_*, C_*] \subset \prod_{n,m} \mathrm{Ext}^{m-n}_{\mathit{mod}\text{-}\Gamma}\big(H_n(C_*), H_m(C_*)\big) = \prod_n \mathrm{Hom}_{\mathit{mod}\text{-}\Gamma}(H_nC, H_nC).$$

Hence (1.6.4) is an isomorphism. This shows that any endomorphism of C_* is compatible with the decomposition $\bigoplus_{i=0}(H_iC_*, i)$ up to homotopy and the result follows. □

1.7. (Co)algebras and Γ-modules

Let A be a commutative K-algebra with unit and let M be an A-module, considered as a symmetric A-A-bimodule. Following Loday [16,17] we let $\mathcal{L}(A, M)$ be the left Γ-module given by

$$[n] \mapsto M \otimes A^{\otimes n}.$$

For a pointed map $f:[n]\to[m]$, the action of f on $\mathcal{L}(A,M)$ is given by

$$f_*(a_0\otimes\cdots\otimes a_n):=b_0\otimes\cdots\otimes b_m, \tag{1.7.1}$$

where

$$b_j=\prod_{f(i)=j}a_i,\quad j=0,\dots,m.$$

If A is an augmented algebra then $\mathcal{L}(A,K)$ is denoted by $\mathcal{L}(A)$ for brevity. Dually, if C is a cocommutative K-coalgebra, and N is an A-comodule, one obtains a right Γ-module $\mathcal{J}(C,N)$, which assigns $N\otimes C^{\otimes n}$ to $[n]$. Clearly, for a finite dimensional coalgebra C and a finite dimensional comodule N, one has

$$\mathcal{J}(C,N)^*\cong\mathcal{L}(C^*,N^*).$$

In the coaugmented case we write $\mathcal{J}(C)$ instead of $\mathcal{J}(C,K)$.

We also need to extend the definition of the functor $\mathcal{L}(A,M)$ to commutative graded differential algebras. Let A be a commutative graded differential K-algebra with unit and let M be a graded differential A-module. We let $\mathcal{L}(A,M)$ be the chain complex of left Γ-modules given by

$$[n]\mapsto M\otimes A^{\otimes n}.$$

For a pointed map $f:[n]\to[m]$, the action of f on $\mathcal{L}(A,M)$ is given by

$$f_*(a_0\otimes\cdots\otimes a_n):=(-1)^{\varepsilon(f,a)}b_0\otimes\cdots\otimes b_m,$$

where $b_j=\prod_{f(i)=j}a_i$, $j=0,\dots,m$. Of course now the product is the ordered one and

$$\varepsilon(f,a)=\sum_{j=1}^{n-1}|a_j|\Bigl(\sum_{k\in I_j}|a_k|\Bigr). \tag{1.7.2}$$

Here $I_j=\{k>j\mid 0\leqslant f(k)\leqslant f(j)\}$ (see [29]).

Let us note that for graded (co)algebras one obtains functors from Γ to the category of graded vector spaces. Therefore for any integer i, the ith component of $\mathcal{L}(A)$ and $\mathcal{J}(C)$ defines left and right Γ-modules, which are denoted by $\mathcal{L}_i(A)$ and $\mathcal{J}_i(C)$ respectively.

1.8. *Example.* – Let $A=K[x]/(x^2)$ be the commutative graded algebra where the generator x has degree $d>0$. We claim that there exist the following isomorphisms of left Γ-modules:

$$\mathcal{L}_i\bigl(K[x]/(x^2)\bigr)\cong\Lambda^j\circ t^*,\quad i=jd \text{ and } d \text{ is odd},$$

$$\mathcal{L}_i\bigl(K[x]/(x^2)\bigr)\cong(\theta^j)^*,\quad i=jd \text{ and } d \text{ is even},$$

$$\mathcal{L}_i\bigl(K[x]/(x^2)\bigr)=0,\quad i\neq jd.$$

Indeed

$$\mathcal{L}\bigl(K[x]/(x^2)\bigr)[n]=(K+Kx)^{\otimes n}=\bigoplus_{0\leqslant j\leqslant n}\bigl(Kx^{\otimes j}\bigr)^{\binom{n}{j}},$$

where $x^{\otimes(j+1)} = x^{\otimes j} \otimes x$, $j \geqslant 0$. Therefore the ith dimensional part is zero if $i \neq dj$ and it is isomorphic to $\Lambda^j(K^n)$ if $i = dj$. Moreover the elements $x_J = x_1 \otimes \cdots \otimes x_n$ form a basis for $\mathcal{L}_i(K[x]/(x^2))[n]$. Here J runs through the set of subsets of $\{1,\dots,n\}$ with j elements and $x_i = x$ if $i \in J$ and $x_i = 1$ if $i \notin J$. Since $x^2 = 0$ one observes that the action of $f:[n] \to [m]$ on x_J is zero if $\mathit{Card}\, f(J) < j$ and is equal up to sign to $x_{f(J)}$ if $\mathit{Card}\, f(J) = j$. Based on (1.7.2) one easily shows that the sign is always $+$ for even d and corresponds to the sign in the exterior power for odd d, and the result is proved.

1.9. Category of surjections

Let Ω be the small category of all finite sets and surjections. We will assume that the objects of Ω are the sets

$$\underline{n} := \{1,\dots,n\}, \quad n \geqslant 0,$$

where $\underline{0}$ denotes the empty set. A covariant (respectively contravariant) functor $\Omega \to \mathit{Vect}$ is called a left (respectively right) Ω-module. Clearly the functors

$$\Omega^n = K\left[\mathrm{Hom}_\Omega(\underline{n}, -)\right] \quad \text{and} \quad \Omega_n = K\left[\mathrm{Hom}_\Omega(-, \underline{n})\right]$$

are projective generators in Ω-*mod* and *mod*-Ω respectively. We have a bifunctor $- \otimes_\Omega - :$ *mod*-$\Omega \times \Omega$-*mod* $\to$ *Vect* similar to 1.5 with the same kind of properties.

Let M be a left (respectively right) $K[\Sigma_n]$-module. One denotes by $\Theta(M)$ (respectively $\Theta^{op}(M)$) the unique left (respectively right) Ω-module, which assigns M to $\underline{n}$ and 0 to $\underline{m}$, where $m \neq n$ and for which the action of $\Sigma_n = \mathrm{Hom}_\Omega(\underline{n}, \underline{n})$ on M coincides with the given one. Clearly if M is a simple $K[\Sigma_n]$-module, then $\Theta(M)$ and $\Theta^{op}(M)$ are simple Ω-modules as well. We claim that in this way one obtains all simple Ω-modules up to isomorphism. Indeed, let T be a left (respectively right) Ω-module and n be the minimal number, for which $T(\underline{n}) \neq 0$. Since $K[\Sigma_n] = K[\mathrm{Hom}_\Omega(\underline{n}, \underline{n})]$, we see that $M = T(\underline{n})$ is a representation of Σ_n. Therefore $\Theta(M)$ (respectively $\Theta^{op}(M)$) is well defined. Now one can show that there exists a unique morphism of Ω-modules $\Theta(M) \to T$ (respectively $T \to \Theta^{op}(M)$) which is the identity on $\underline{n}$. From this observation one easily deduces the claim.

For a left Ω-module T we let $T^\natural(\underline{n})$ be the cokernel $\bigoplus T(\underline{n+1}) \to T(\underline{n})$, where the sum is taken over all morphisms $\underline{n+1} \to \underline{n}$. Let X be a right $K[\Sigma_n]$-module. We claim that there is a natural isomorphism

$$\Theta^{op}(X) \otimes_\Omega T \cong X \otimes_{\Sigma_n} T^\natural(\underline{n}). \tag{1.9.1}$$

Indeed, it suffices to observe that $\bigoplus \Theta^{op}(X)(\underline{m}) \otimes T(\underline{m})$ has only one nontrivial component, corresponding to $m = n$.

1.10. Dold–Kan type theorem for Γ-modules

For any $n \geqslant 1$ and any i such that $1 \leqslant i \leqslant n$, one defines the pointed maps

$$r_i : [n] \to [n-1]$$

by $r_i(i) = 0$, $r_i(j) = j$ if $j < i$ and $r_i(j) = j - 1$, if $j > i$. Thanks to (1.4.2) we see that $(\Gamma^0 \to \Gamma^1)^{\otimes n}$ is a resolution of $t^{*\otimes n}$. According to (1.3.1) the very beginning part of this resolution looks as follows:

$$\bigoplus_{i=1}^{n} \Gamma^{n-1} \to \Gamma^n \to t^{*\otimes n} \to 0.$$

Here the first morphism is induced by $(r_1,\dots,r_n)$. Therefore for each left Γ-module T, one has an isomorphism

$$\mathrm{Hom}_{\Gamma\text{-}mod}\big(t^{*\otimes n},T\big)\cong\bigcap_{i=1}^{n}\ker\big(r_{i*}:T([n])\to T([n-1])\big). \tag{1.10.1}$$

Here r_{i*} is the homomorphism induced by r_i.

We claim that there exist an isomorphism

$$\mathrm{Hom}_{\Gamma\text{-}mod}\big(t^{*\otimes n},t^{*\otimes m}\big)\cong K\big[\mathrm{Hom}_{\Omega}(\underline{m},\underline{n})\big]. \tag{1.10.2}$$

Indeed, one observes that $t^{*\otimes m}([n])$ can be identified with the vector space spanned by all maps $\underline{m}\to\underline{n}$. Then the action of $f:[n]\to[k]$ on $\alpha:\underline{m}\to\underline{n}$ is $f\circ\alpha$ if $(f\circ\alpha)^{-1}(0)\neq\emptyset$ and is 0 otherwise. Now it suffices to note that

$$\ker\big(r_{i*}:t^{*\otimes m}([n])\to t^{*\otimes m}([n-1])\big)$$

is spanned by maps $\underline{m}\to\underline{n}$ whose image contains i, $1\leqslant i\leqslant n$. For a left Γ-module T one defines the functor

$$cr(T):\Omega\to Vect$$

as follows. On objects one puts

$$cr(T)(\underline{n}):=\bigcap_{i=1}^{n}\ker\big(r_{i*}:T([n])\to T([n-1])\big).$$

For a surjection $f:\underline{n}\to\underline{m}$ one denotes by $f_0:[n]\to[m]$ the unique pointed map which extends f. Then one easily shows that the image of $cr(T)(\underline{n})\subset T([n])$ under the homomorphism $(f_0)_*:T([n])\to T([m])$ lies in $cr(T)(\underline{m})$ and therefore $cr(T)$ is well defined. According to (1.10.1) and (1.10.2) we have $cr(t^{*\otimes n})\cong\Omega^n$. Since the functors $t^{*\otimes n}$, $n\geqslant 0$ are small projective generators, it follows from the Morita theory that the functor

$$cr:\Gamma\text{-}mod\to\Omega\text{-}mod$$

is an equivalence of categories (see also [21] for this and more general results). Let us notice that a similar equivalence exists also for right modules

$$cr:mod\text{-}\Gamma\cong mod\text{-}\Omega.$$

Let F be a right Γ-module and let T be a left Γ-module. It is also a consequence of the Morita theory that there is an isomorphism

$$F\otimes_\Gamma T\cong cr(T)\otimes_\Omega cr(T). \tag{1.10.3}$$

We turn now to the functor $\mathcal{L}(A,M)$ defined in Section 1.7. Here A is a commutative K-algebra and M is an A-module. We let $\bar A$ be the quotient $A/(K\cdot 1)$. For each element $a\in A$, we let $\bar a$ be the corresponding element in $\bar A$. We need a homomorphism $s:M\otimes\bar A^{\otimes n}\to M\otimes A^{\otimes n}$ which is a section of the natural projection $M\otimes A^{\otimes n}\to M\otimes\bar A^{\otimes n}$. Take an element $x=(m,a_1,\dots,a_n)\in$

$M \otimes A^{\otimes n}$. For any subset $S = \{j_1, \dots, j_k\} \subset \underline{n}$, we set $x_S = (ma_{j_1} \cdots a_{j_k}, a_1, \dots, 1, \dots, a_n)$, where we put 1 in places labeled by $j_1, \dots, j_k$. Then the map s is given by $s(\bar{x}) = \sum_S (-1)^{|S|} x_S$. Clearly

$$cr\big(\mathcal{L}(A, M)\big)(\underline{n}) \cong M \otimes \bar{A}^{\otimes n}. \tag{1.10.4}$$

For a surjection $f : \underline{n} \to \underline{m}$, the induced homomorphism $f_* : M \otimes \bar{A}^{\otimes n} \to M \otimes \bar{A}^{\otimes m}$ is obtained by applying the formula (1.7.1) to each term of $s(\bar{x})$. For example, if $f : \underline{n} \to \underline{n-1}$ is given by $f(1) = 1$, $f(i) = i - 1$, for $i \geqslant 2$, then

$$f_*\big(m, \bar{a}_1, \dots, \bar{a}_n\big) = \big(m, \bar{a}_1\bar{a}_2, \dots, \bar{a}_n\big) - \big(m\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n\big) - \big(m\bar{a}_2, \bar{a}_1, \dots, \bar{a}_n\big). \tag{1.10.5}$$

1.11. Polynomial modules

For a left (respectively right) $K[\Sigma_n]$-module M one denotes by $\theta(M)$ (respectively $\theta^{op}(M)$) the Γ-modules corresponding to $\Theta(M)$ (respectively $\Theta^{op}(M)$) under the equivalence 1.10. For $M = K$ with trivial action (respectively with sign representation) one has the isomorphism $\theta^{op}(M) \cong \theta^n$ (respectively $\theta^{op}(M) \cong \Lambda^n \circ t$). This follows from the fact that $cr(F)(\underline{m}) = 0$ if $\underline{n} \neq \underline{m}$ and $cr(F)(\underline{n}) = M$ for $F = \Lambda^n \circ t$ or $F = \theta^n$.

We will say that *the degree* of a left Ω-module T is less or equal to n if $T(\underline{k}) = 0$ for $k > n$. In this case we write $\deg(T) \leqslant n$. One writes $\deg(T) = n$ if $\deg(T) \leqslant n$ and $\deg(T) \not\leqslant n - 1$. We take the analogous definition for right modules. Then $\deg(T) \leqslant n$ if and only iff $\deg(T^*) \leqslant n$. We will say that *the degree* of a Γ-module T is n if $\deg(cr(T)) = n$. For example $\deg(t) = 1$, $\deg(\Gamma^n) = n$, $\deg(\theta^n) = n$. Therefore any left (respectively right) Γ-module of degree $\leqslant n$ admits a projective (respectively injective) resolution whose components have degree $\leqslant n$. Moreover, if the Γ-module is of finite type, then one can choose the resolutions of finite type as well. Let A be a connected graded commutative algebra, then

$$\deg \mathcal{L}_k(A) = k.$$

This follows from (1.10.4), because $\bar{A}^{\otimes n}$ is zero in dimensions $< n$. Similarly, if C is a connected graded cocommutative coalgebra, then

$$\deg \mathcal{J}_k(C) = k. \tag{1.11.1}$$

1.12. PROPOSITION. – *If K is a field of characteristic zero, then for any left Γ-module T of degree $\leqslant n$ the projective dimension of T is $\leqslant n - 1$. Thus*

$$\textit{proj. dim}(T) \leqslant \deg(T) - 1, \quad T \in \Gamma\textit{-mod}.$$

Similarly for any right Γ-module T one has

$$\textit{inj. dim}(T) \leqslant \deg(T) - 1, \quad T \in \textit{mod-}\Gamma.$$

Proof. – Thanks to Section 1.10 one can work with Ω-modules. Let $\omega^n : K[\Sigma_n]\textit{-mod} \to \Omega\textit{-mod}$ be a functor given by

$$\omega^n(M) = M \otimes_{K[\Sigma_n]} \Omega^n.$$

The functor ω^n takes projective objects to projective objects. Moreover

$$\deg\big(\omega^n(M)\big) \leqslant n$$

for any $M \in K[\Sigma_n]$-*mod*. Since K has characteristic zero, we see that $\omega^n(T(\underline{n}))$ is projective. Clearly there is a natural map $\omega^n(T(\underline{n})) \to T$, whose kernel and cokernel have degrees $< n$ and the result follows by induction. A similar argument based on the functor $\omega_n : mod\text{-}K[\Sigma_n] \to mod\text{-}\Omega$ proves the result for right modules. Here

$$\omega_n(M) = \mathrm{Hom}_{\Sigma_n}(\Omega_n, M). \quad \square$$

1.13. Koszul complexes for sets

One defines the transformations

$$d : \Lambda^{i+1} \circ t^* \otimes Sym^j \circ t^* \to \Lambda^i \circ t^* \otimes Sym^{j+2} \circ t^*$$

and

$$d : Sym^n \circ t^* \to \mathcal{B}_n / \mathcal{B}_{n-1}$$

by their values on the basis

$$d(x_0 \wedge \cdots \wedge x_i \otimes y_1 \cdots y_j) := \sum_{k=0}^{i} (-1)^k x_0 \wedge \cdots \wedge \hat{x}_k \cdots \wedge x_i \otimes x_k^2 y_1 \cdots y_j$$

and

$$d(z_1 \cdots z_n) := \{z_1, \dots, z_n\} mod\, \mathcal{B}_{n-1}.$$

Here $x_0, \dots, x_i, y_0, \dots, y_j, z_1, \dots, z_n \in [m]$. We claim that for any n the following sequences are exact

$$0 \to \Lambda^n \circ t^* \to \Lambda^{n-1} \circ t^* \otimes Sym^2 \circ t^* \to \Lambda^{n-2} \circ t^* \otimes Sym^4 \circ t^* \to \cdots$$
$$\cdots \to t^* \otimes Sym^{2n-2} \circ t^* \to Sym^{2n} \circ t^* \to \mathcal{B}_{2n}/\mathcal{B}_{2n-1} \to 0$$

and

$$0 \to \Lambda^n \circ t^* \otimes t^* \to \Lambda^{n-1} \circ t^* \otimes Sym^3 \circ t^* \to \cdots$$
$$\cdots \to Sym^{2n+1} \circ t^* \to \mathcal{B}_{2n+1}/\mathcal{B}_{2n} \to 0.$$

These facts are true in any characteristic and can be proved as follows. Let $\mathcal{K}$ be the bigraded object which in bidegree (i, j) is zero if i is odd and is $\Lambda^{i/2} \circ t^* \otimes Sym^j \circ t^*$ if i is even. This object is equipped with the differential of bidegree $(-2, 2)$ as defined above. We also consider the graded object $gr_* \mathcal{B}$, whose component in degree n is $\mathcal{B}_n/\mathcal{B}_{n-1}$. We have to prove that the homology of $\mathcal{K}$ is zero if $i > 0$ and is isomorphic to $gr_* \mathcal{B}$ if $i = 0$. One easily shows that $\mathcal{K}(S \vee T) \cong \mathcal{K}(S) \otimes \mathcal{K}(T)$ as differential graded objects. Since $gr_* \mathcal{B}(S \vee T) \cong gr_* \mathcal{B}(S) \otimes gr_* \mathcal{B}(T)$, everything reduces to the case, when the above sequences are evaluated on $[1]$. In this case the assertion is obvious.

1.14. PROPOSITION. –
(i) *Let F be a right Γ-module of degree d, then* $\mathrm{Hom}_{mod\text{-}\Gamma}(\theta^i, F) = 0$ *as soon as $i > d$.*
(ii) *Let K be a field of characteristic zero. Assume $m > j - i$. Then*

$$\mathrm{Ext}^m_{mod\text{-}\Gamma}(\theta^i, \theta^j) = 0.$$

Proof. – (i) Since $cr(\theta^i)(\underline{m}) \neq 0$ only for $m = i$, we see that

$$\mathrm{Hom}_{\mathit{mod}\text{-}\Omega}\big(cr(\theta^i), cr(F)\big) = 0$$

if $i > d$. Hence the result follows from Section 1.10.

(ii) Since we are in characteristic zero, one can take the dual of the exact sequences of Section 1.13 to obtain

$$0 \to \theta^n \to \mathit{Sym}^n \circ t \to \cdots \to \Lambda^{n/2} \circ t \to 0$$

if n is even, and

$$0 \to \theta^n \to \mathit{Sym}^n \circ t \to \cdots \to t \otimes \Lambda^{(n-1)/2} \circ t \to 0$$

if n is odd. By 1.4 we know that $\Lambda^n \circ t$, $\mathit{Sym}^n \circ t$ and $\Lambda^n \circ t \otimes \mathit{Sym}^n \circ t$ are injective. Therefore $\mathrm{Ext}^m_{\mathit{mod}\text{-}\Gamma}(\theta^i, \theta^j)$ is a subquotient of $\mathrm{Hom}_{\mathit{mod}\text{-}\Gamma}(\theta^i, \Lambda^m \circ t \otimes \mathit{Sym}^{j-2m} \circ t)$. Since

$$\deg\big(\Lambda^m \circ t \otimes \mathit{Sym}^{j-2m} \circ t\big) = j - m$$

the result follows from part (i). □

1.15. PROPOSITION. – *Let A be a commutative algebra and M be an A-module. Then there exist natural isomorphisms*

$$(\Lambda^n \circ t) \otimes_\Gamma \mathcal{L}(A, M) \cong M \otimes_A \Omega^n_A,$$
$$\theta^n \otimes_\Gamma \mathcal{L}(A, M) \cong M \otimes_A \mathit{Sym}^n_A\big(\Omega^1_A\big),$$

where Ω^n_A denotes the Kähler differentials.

Proof. – Since proofs in both cases are quite similar, we prove only the first isomorphism. By (1.10.3) one can pass to the category Ω. Let us recall that $cr(\Lambda^n \circ t)$ is isomorphic to $\Theta^{op}(X)$, where X is the sign representation of Σ_n. Therefore we can use (1.9.1) and (1.10.5) to conclude that the tensor product in question is isomorphic to the quotient of $M \otimes \Lambda^n_K(\bar{A})$ by the relation

$$m \otimes (a_1 a_2) \wedge \cdots \wedge a_n = m a_1 \otimes a_2 \wedge \cdots \wedge a_n + m a_2 \otimes a_1 \wedge \cdots \wedge a_n$$

and the result is proved. □

2. Decomposition of $\pi_* F(S^n)$

2.1. The main idea and the link with homotopy theory

Let Sets_* be the category of all pointed sets. There is a standard way to prolong a left Γ-module F to a functor from the category of simplicial pointed sets $s.\mathit{Sets}_*$ to the category of simplicial vector spaces $s.\mathit{Vect}$. First, one can prolong F by direct limits to a functor $\mathit{Sets}_* \to \mathit{Vect}$, then by the degreewise action one obtains a functor $s.\mathit{Sets}_* \to s.\mathit{Vect}$. By abuse of notation we will still denote this functor by F. Similarly any right Γ-module T can be prolonged to a functor from the category of simplicial pointed sets $s.\mathit{Sets}_*$ to the category of cosimplicial vector spaces $\mathit{cos}.\mathit{Vect}$. It is of great interest to understand the structure of the groups $\pi_* F(L)$ for a pointed simplicial set L. Of course of special interest is the case $L = S^n$. Let us remark that any Γ-set A (that is a functor $\Gamma \to \mathit{Sets}_*$, with property $A([0]) = [0]$) gives rise to a binatural transformation

$$X \wedge A(Y) \to A(X \wedge Y)$$

as follows: For any $x \in X$ define $\hat{x}: Y \to X \wedge Y$ by $y \mapsto (x, y)$. Then apply A to get $A(\hat{x}): A(Y) \to A(X \wedge Y)$. Now one defines $X \wedge A(Y) \to A(X \wedge Y)$, by $(x, z) \mapsto A(\hat{x})(z)$. Using this transformation one obtains a map $S^1 \wedge A(S^n) \to A(S^{n+1})$. Here S^1 is a simplicial model of the circle, which has only two nondegenerate simplices and $S^{n+1} = S^1 \wedge S^n$. Of course this can be done with left Γ-modules as well. Indeed if F is a left Γ-module, then $F = F_0 \oplus A$, where F_0 is a constant Γ-module with value $F([0])$. Clearly $A([0]) = 0$ and one can define

$$\pi_*^{st}(F) := \mathit{colim}\, \pi_{*+n} A\big(S^n\big) \cong \mathit{colim}\, \pi_{*+n} F\big(S^n\big).$$

Similarly for a right Γ-module T one puts

$$\pi_{st}^* F := \lim_n \pi^{n+*} F\big(S^n\big).$$

Here π^* denotes the "cohomotopy groups" of a cosimplicial vector space, meaning the homology of the associated cochain complex.

Mimicking Korollar 6.12 in [9] or use Proposition 5.21 of [18], one can prove that this limit always stabilizes and one has an isomorphism

$$\pi_i^{st}(F) \cong \pi_{i+n} F\big(S^n\big) \quad \text{if } n > i. \tag{2.1.1}$$

Now we give the main idea of our approach. For simplicity we take the classical case $n = 1$. Let us recall that S^1 is $[n]$ in dimension n. Moreover $s_i : [n] \to [n+1]$ is the unique monotone injection, whose image does not contain $i + 1$, while $d_i : [n] \to [n-1]$ is given by $d_i(j) = j$ if $j < i$, $d_i(i) = i$ if $i < n$, $d_n(n) = 0$ and $d_i(j) = j - 1$ if $j > i$. Thus $\pi_* F(S^1)$ is by definition the homology of the following complex

$$F(S^1) = \big(F([0]) \leftarrow F([1]) \leftarrow F([2]) \leftarrow \cdots\big),$$

where the boundary map is the alternating sum of the face homomorphisms. Using the tensor product of functors (see Section 1.5) we can write $F([n]) = \Gamma_n \otimes_\Gamma F$. Therefore $F(S^1) = \Gamma_{S^1} \otimes_\Gamma F$, where Γ_{S^1} is the simplicial right Γ-module, which is Γ_n in dimension n. Now one can use the result from Section 1.6 in order to construct a spectral sequence whose abutment is $\pi_* F(S^1)$. It turns out that in the characteristic zero case, this spectral sequence degenerates and we obtain the expected decomposition. In order to realize this program we start with the translation of "homotopy theory" in the language of homological algebra of Γ-modules.

2.2. PROPOSITION. – *For any left Γ-module F, one has an isomorphism*

$$\mathrm{Tor}_*^\Gamma(t, F) \cong \pi_*^{st} F.$$

Moreover, for any right Γ-module T one has an isomorphism

$$\pi_{st}^i T \cong \mathrm{Ext}_{\mathit{mod}\text{-}\Gamma}^*(t, T).$$

Proof. – We use the well-known axiomatic characterisation of *Tor* functors. In order to show the existence of the isomorphism in dimension 0, one observes that $\pi_0^{st} F \cong \pi_1 F(S^1)$ thanks to (2.1.2). By our choice of the model of S^1 one sees that

$$\pi_1 F(S^1) \cong \mathit{Coker}\big(F([2]) \to F([1])\big)$$

and hence by the exact sequence (1.4.1) we get $\pi_0^{st} F \cong t \otimes_\Gamma F$, because $F([n]) \cong \Gamma^n \otimes_\Gamma F$. Obviously the functors $T \mapsto \pi_*^{st} T$ form an exact connected sequence of functors Γ-*mod* $\to$ *Vect* and it is enough to show that they are zero in positive dimensions for any projective T. Therefore one only needs to consider functors like $T = \Gamma^n$. The chain complex associated to the simplicial abelian group $\Gamma^n(S^m)$ is the same as the chains of the product of n copies of S^m. Thanks to the Künneth theorem the homology of this complex is zero in dimensions $> m$ and $< 2m$, and we are done. A similar argument works for right modules. $\square$

In what follows we give the formulation of results only for left Γ-modules. We leave the task to the interested reader to make the trivial reformulation for the corresponding results for right Γ-modules.

2.3. Fundamental spectral sequence

Let L be a pointed simplicial set. It is well-known that the homology $H_*(L)$ of L with coefficients in K is a coaugmented coalgebra. Therefore one can consider the right Γ-module $\mathcal{J}_i(H_*L)$ for any $i \geqslant 0$ (see Section 1.7).

2.4. THEOREM. – *Let F be a left Γ-module and let L be a pointed simplicial set. Then there exists a spectral sequence*

$$E^2_{pq} = \mathrm{Tor}^\Gamma_p\left(\mathcal{J}_q(H_*L), F\right) \Rightarrow \pi_{p+q}\left(F(L)\right).$$

A simplicial map $L \to L'$ induces an isomorphism $\pi_*(F(L)) \to \pi_*(F(L'))$ as soon as $H_*(L) \to H_*(L')$ is an isomorphism.

Proof. – For a pointed set X, we let Γ_X be the right Γ-module given by

$$\Gamma_X = colim_Y \Gamma_Y.$$

Here Y runs finite pointed subsets of X. Clearly

$$\Gamma_X([n]) = colim_Y \Gamma_Y([n]) = colim_Y K\left[\mathrm{Hom}_\Gamma([n], Y)\right] = K[X^n].$$

Moreover, one has the following isomorphisms

$$\Gamma_X \otimes_\Gamma F \cong colim_Y \Gamma_Y \otimes_\Gamma F = colim_Y F(Y) = F(X).$$

We know that Γ_X is a projective right Γ-module for finite X. We claim that this is still true for arbitrary X. First consider the case when X is a countable. Without loss of generality we can assume that $X = \bigcup_n [n]$. Since for any pointed injective map $Z \to Y$, the induced map $\Gamma_Z \to \Gamma_Y$ has a retraction, we see that $\Gamma_X \cong \bigoplus_n Coker(\Gamma_{n-1} \to \Gamma_n)$. Hence Γ_X is a projective. In general one observes that the functor $Sets_* \xrightarrow{\Gamma_?} mod\text{-}\Gamma$, which assigns Γ_X to X, preserves filtered colimits. Based on this fact the claim can easily proved by transfinite induction. We let Γ_L be the following composition

$$\Delta^{op} \xrightarrow{L} Sets_* \xrightarrow{\Gamma_?} mod\text{-}\Gamma.$$

Therefore Γ_L is a simplicial object in *mod*-Γ, which assigns Γ_{L_n} to $[n]$. By abuse of notation we denote the chain complex associated to this simplicial right Γ-module also by Γ_L. Let us calculate the homology of this chain complex. Since $\Gamma_L([n]) \cong K[L^n]$, it follows by the Eilenberg–Zilber and Künneth theorem that $H_*(\Gamma_L)[n] = H_*(L)^{\otimes n}$. Hence one has an isomorphism of right Γ-modules

$$H_i(\Gamma_L) \cong \mathcal{J}_i\big(H_*(L)\big).$$

The relations from Section 1.5 show that $F(L)([n]) \cong \Gamma_{L_n} \otimes_\Gamma F$. So one can write: $F(L) \cong \Gamma_L \otimes_\Gamma F$. Thus the spectral sequence constructed in Proposition 1.6 for $C_* = \Gamma_L$ gives the expected result. □

2.5. COROLLARY. – *Let F be a left Γ-module. Then there exists a spectral sequence*

$$E^2_{pq} \Rightarrow \pi_{p+q} F\big(S^d\big), \quad d \geqslant 1,$$

with $E^2_{pq} = 0$ if $q \neq dj$ and

$$E^2_{pq} = \mathrm{Tor}_p^\Gamma\big(\Lambda^j \circ t, F\big) \quad \text{if } q = dj \text{ and } d \text{ is odd}$$

$$E^2_{pq} = \mathrm{Tor}_p^\Gamma\big(\theta^j, F\big) \quad \text{if } q = dj \text{ and } d \text{ is even.}$$

Moreover, if K is a field of characteristic zero, then the spectral sequence degenerates:

$$\pi_n F\big(S^d\big) \cong \bigoplus_{p+dj=n} \mathrm{Tor}_p^\Gamma\big(\Lambda^j \circ t, F\big)$$

if d is odd and

$$\pi_n F\big(S^d\big) \cong \bigoplus_{p+dj=n} \mathrm{Tor}_p^\Gamma\big(\theta^j, F\big)$$

if d is even. Furthermore, if $(S^d)^f$ is a fibrant model of S^d and $f:(S^d)^f \to (S^d)^f$ is a simplicial map, whose realization is of degree $N \neq 0$, then jth part of the above decompositions corresponds to the eigenspace of $f_*: \pi_n F(S^d) \to \pi_n F(S^d)$, where f_* acts as N^j.

Proof. – Example 1.8 shows that $\mathcal{J}_i(H_*(S^d)) = 0$ if $i \neq jd$ and for $i = jd$ one has $\mathcal{J}_i(H_*(S^d)) \cong \Lambda^j \circ t$ or $\mathcal{J}_i(H_*(S^d)) \cong \theta^j$ depending on the parity of d. Hence the previous result implies that the spectral sequence is of the expected form. By Section 1.4 we know that $\Lambda^i \circ t$ is injective, so $\mathrm{Ext}^i_{\mathit{mod}\text{-}\Gamma}(-, \Lambda^j \circ t) = 0$ for $i > 0$. By Proposition 1.14(ii) one easily shows that

$$\mathrm{Ext}^{d(j-i)+1}_{\mathit{mod}\text{-}\Gamma}\big(\theta^i, \theta^j\big) = 0 = \mathrm{Ext}^{d(j-i)}_{\mathit{mod}\text{-}\Gamma}\big(\theta^i, \theta^j\big)$$

if $i < j$. Therefore the expected decomposition follows from Proposition 1.6(ii). The fact that the action of f_* on $\pi_n F(S^n)$ have expected form easily folows from Proposition 1.6(iii). □

2.6. Relation with Loday's decomposition

Observe that the summand of the decomposition 2.5 corresponding to $j = 0$ is trivial if $p > 0$, because the functor $\Lambda^0 \circ t$ is the constant functor with value K and thus it is isomorphic to Γ_0 and hence it is projective. Now we consider the case when $d = 1$. For every left Γ-module F one has the following decomposition

$$\pi_n F\big(S^1\big) \cong \bigoplus_{i=0}^{n} H_n^{(i)}(F) \tag{2.6.1}$$

if K has characteristic zero (see [16]). We will show that our decomposition in Corollary 2.5 for $d=1$ essentially coincides with Loday's decomposition. In fact we prove more, namely we give an axiomatic characterization of the decomposition of $\pi_n F(S^1)$.

2.7. THEOREM. – *Let K be a field of characteristic zero. Let*

$$\pi_n F(S^1) \cong \bigoplus_{i=0}^{n} H_n^{(i)}(F)$$

be a natural decomposition with the properties:
(i) $H_n^{(0)}(F)=0$ *if* $n>0$.
(ii) *For any* $i \geqslant 1$, *the sequence of functors* $H_n^{(i)}(-): \Gamma\text{-}mod \to Vect$, $n \geqslant i$, *forms an exact connected sequence of functors.*
(iii) *For any* $n \geqslant 0$, *the functor* $H_n^{(n)}(-)$ *is nonzero.*
Then, for any $i \geqslant 1$ *one has a natural isomorphism*

$$H_n^{(i)}(F) \cong \operatorname{Tor}_{n-i}^{\Gamma}(\Lambda^i \circ t, F), \quad n \geqslant i.$$

Moreover for the decomposition (2.6.1) *the properties* (i)–(iii) *hold and therefore our decomposition is isomorphic to the one given in* [16].

Proof. – The fact that (2.6.1) satisfies (i)–(iii) is very easy to check based on the definition given in [16]. So it is enough to show the first statement. If $n=1$, then the result is clear because there is only one summand in both decompositions of $\pi_1 F(S^1)$. In order to prove the theorem, we still use the axiomatic characterization of Tor-groups. Since the functors $H_n^{(i)}(-): \Gamma\text{-}mod \to Vect, n \geqslant i$ form an exact connected sequence of functors, it is enough to show that for a projective F one has the isomorphisms

$$H_i^{(i)}(F) \cong \pi_i F(S^1) \quad \text{and} \quad H_n^{(i)}(F)=0, \quad \text{for } n>i.$$

Let us recall that the functors $t^{*\otimes n}$, $n \geqslant 0$, are projective generators in the category of left Γ-modules. Hence one only needs to consider the case $F=t^{*\otimes n}$. Let us calculate $\pi_* F(S^1)$, when $F=t^{*\otimes n}$. We recall that $t^*(S)$ is the free vector space generated by S, modulo the relation $*=0$. Thus $\pi_* t^*(S^1)$ is nothing but the reduced homology of S^1. Therefore it is a consequence of Eilenberg–Zilber and Künneth theorems that

$$\pi_k t^{*\otimes m}(S^1) \cong K, \quad \text{if } k=m \quad \text{and} \quad \pi_k t^{*\otimes m}(S^1)=0 \quad \text{if } k \neq m. \tag{2.7.1}$$

It follows from property ii) that the functor $H_i^{(i)}(-)$ is right exact for any $i \geqslant 0$. Now consider the case $n=2$. The isomorphism (2.7.1) shows that $H_2^{(2)}(t^{*\otimes n})=0$ except for $n=2$. If this were also true for $n=2$, then $H_2^{(2)}(F)=0$ for all projective F, and therefore $H_2^{(2)}(-)=0$, because $H_2^{(2)}(-)$ is right exact. This contradicts the fact that all components are nontrivial in general. Thus $H_2^{(2)}(t^{*\otimes 2})=K$. Comparing the Hodge decomposition for $n=2$ with (2.7.1), we can conclude that $H_2^{(1)}(F)=0$ for any projective F. This shows that the result is proved for $n=2$. Now we can finish the proof by induction, based on the same argument. □

2.8. Stable homotopy and Harrison homology

For any left Γ-module F, the vector space $F([n])$ is an $K[\Sigma_n]$-module, because $\Sigma_n = Aut_\Gamma([n])$. Therefore the subspace of shuffles Sh_n of $K[\Sigma_n]$ acts on $F([n])$ (see [16]). Moreover, by [16], the subspaces $Sh_n(F) := Sh_nF([n]) \subset F([n]), n \geqslant 0$, define the subcomplex of the complex associated to the simplicial module $F(S^1)$ (let us recall that in dimension n, the simplicial module $F(S^1)$ is just $F([n])$). By definition *Harrison homology* $Harr_*(F)$ of F is the homology of the corresponding quotient.

2.9. COROLLARY. – *Assume K has characteristic zero. Then one has a natural isomorphism*

$$Harr_{*-1}(F) \cong \pi_*^{st}(F).$$

Proof. – By Theorem 3.7 [16], we know that $H_*^{(1)}(F) \cong Harr_*(F)$. Therefore our statement is a consequence of Proposition 2.2 and Theorem 2.7. □

2.10. Link with Mac Lane homology

Let $P(\mathbf{Z})$ be the category of finitely generated free abelian groups. Let $\mathcal{F}(\mathbf{Z})$ be the category of all functors from $P(\mathbf{Z})$ to the category of all abelian groups. Let $T \in \mathcal{F}(\mathbf{Z})$ be a functor. We refer to [14] (see also [17] Chapter 13) for the definition of Mac Lane (co)homology $HML(\mathbf{Z}, T)$ of $\mathbf{Z}$ with coefficient in T. For any functor $T \in \mathcal{F}(\mathbf{Z})$ one denotes by $\widetilde{T}$ the precomposition of T with the functor $\bar{\mathbf{Z}}: \Gamma \to P(\mathbf{Z})$, given by $S \mapsto \bar{\mathbf{Z}}[S]$. Here S is a pointed set and $\bar{\mathbf{Z}}[S]$ is the free abelian group generated by S modulo the relation $* = 0$.

2.11. PROPOSITION. – *For any $T \in \mathcal{F}(\mathbf{Z})$ one has a natural isomorphism*

$$HML_*(\mathbf{Z}, T) \otimes \mathbf{Q} \cong Harr_{*-1}(\widetilde{T}) \otimes \mathbf{Q}.$$

Proof. – According to Corollary 2.9 one only needs to show that

$$HML_*(\mathbf{Z}, T) \otimes \mathbf{Q} \cong \pi_*^{st}(\widetilde{T}) \otimes \mathbf{Q}.$$

This fact is probably well known to experts, but we give the argument here. Following Breen [4], we consider the chain complex $W_*(X)$ of the simplicial set $\mathbf{Z}[S^n] \otimes X$, where $X \in P(\mathbf{Z})$ and $n > 0$ is a natural number. By the well known properties of Eilenberg–Mac Lane spaces we know that $H_i(W_*(X)) \otimes \mathbf{Q} = 0$ for $i < 2n$ and $i \neq n$. Moreover $H_n(W_*(X)) = X$ and each component of $W_i(X)$ is of the form $\mathbf{Z}[X^m]$ for some m. Varying X one obtains the componentwise projective complex $W((-)^*)$ in the category $\mathcal{F}(\mathbf{Z})$, whose homology up to torsion is the functor I^* in dimension $< n$ and zero in dimensions $< 2n$. Here $I^*(X) = X^* = \mathrm{Hom}(X, \mathbf{Z})$. Therefore W_* can be used to calculate $HML_*(\mathbf{Z}, T) = \mathrm{Tor}_*^{P(\mathbf{Z})}(I^*, T)$ up to torsion and the result follows. □

2.12. The groups $\pi_*F(S^d)$ and playing with Chinese puzzles

Let d be an odd number. Comparing Corollary 2.5 and Theorem 2.7 one sees that in the characteristic zero case the groups

$$\pi_nF(S^d) \cong \bigoplus_{i+dj=n} H_{i+j}^{(j)}(F) \tag{2.12.1}$$

for different d differ only by the way of taking the pieces $H_n^{(i)}(F)$ in the decomposition (2.12.1). The same remark is true also for even d. The knowledge of the decomposition for $d = 2$ completely determines the decomposition for all even dimensional spheres. However in even case, in the decomposition of $\pi_nF(S^d)$, only the group $Harr_{n-d+1}(F) \cong H_{n-d+1}^{(1)}(F)$ belongs to Loday's decomposition; all other groups are new.

2.13. Stable homotopy and homology of small categories

We refer to [3] for definition of the (co)homology of small category with coefficient in a bifunctor. For any left Γ-module F and right Γ-module T, we let D^F and D_T be the bifunctors on Γ given by

$$D^F([m],[n]) = t([m]) \otimes F([n]) \quad \text{and} \quad D_T([m],[n]) = T([m]) \otimes t^*([n]).$$

Using Proposition 2.2 and Corollary 3.11 of [14] we obtain the following isomorphisms

$$\pi^*_{st}T \cong H^*(\Gamma, D_T) \quad \text{and} \quad \pi^{st}_* F \cong H^*(\Gamma, D^F). \tag{2.13.1}$$

2.14. Generalization for chain complex of left Γ-modules

Let (F_*, ∂) be a chain complex of left Γ-modules. We can still apply (F_*, ∂) on S^d to get a simplicial object in the category of chain complexes. We can take the total complex $tot(F_*(S^d))$ of the corresponding bicomplex. One denotes by $\pi_* F_*(S^d)$ the homology of $tot(F_*(S^d))$. We have $F_*(S^d) = \Gamma_{S^d} \otimes F_*$. Since in characteristic zero Γ_{S^d} is homotopy equivalent to $\bigoplus_j (\Lambda^j \circ t, jd)$ or $\bigoplus_j (\theta^j, jd)$ depending on the parity of d, one obtains the natural chain map

$$tot(F_*(S^d)) \to \bigoplus_j (\Lambda^j \circ t) \otimes_\Gamma F_*[dj] \tag{2.14.1}$$

or

$$tot(F_*(S^d)) \to \bigoplus_j (\theta^j) \otimes_\Gamma F_*[dj] \tag{2.14.2}$$

depending on the parity of d. This shows that the proof of the Corollary 2.5 is still valid and we have a similar decomposition of $\pi_n F_*(S^d)$ in this generality.

3. Decomposition for cyclic homology

3.1. More categories

Let $\mathcal{F}$ be the small category of finite nonempty sets. We assume that objects of $\mathcal{F}$ are the sets $[n]$. By forgetting the basepoint one gets a functor $\mu : \Gamma \to \mathcal{F}$. Let ΔC be Connes' cyclic category (see [17]). It is well-known that the simplicial circle S^1 has a natural cyclic structure, meaning that the functor $S^1 : \Delta^{op} \to \Gamma$ fits into the commutative diagram (see Proposition 2.11 of [16]):

$$\begin{tikzcd} \Delta^{op} \arrow[r] \arrow[d, "S^1"'] & \Delta C^{op} \arrow[d, "S^1"] \\ \Gamma \arrow[r] & \mathcal{F} \end{tikzcd}$$

Therefore any functor $T : \mathcal{F} \to \mathit{Vect}$ gives rise to the cyclic module $T(S^1)$. We let $HC_*(T)$ be the cyclic homology of this cyclic module. Let us remark, that for $T([n]) = A^{\otimes n+1}$ one gets the cyclic homology of A. Here A is any commutative algebra. In [16] Loday proved that the groups $HC_*(T)$ have a canonical decomposition:

$$HC_n(T) \cong HC_n^{(1)}(T) \oplus \cdots \oplus HC_n^{(n)}(T), \quad n \geqslant 1, \tag{3.1.1}$$

in the characteristic zero case. Moreover he proved that there is a natural isomorphism

$$H_n^{(1)}(T) \cong HC_n^{(1)}(T) \quad \text{for } n \geqslant 3.$$

We give an alternative proof of these facts based on the same ideas which were used before.

3.2. From Γ-modules to $\mathcal{F}$-modules

A *left* (respectively *right*) $\mathcal{F}$*-module* is a covariant (respectively contravariant) functor from $\mathcal{F}$ to the category *Vect*. The categories of left and right $\mathcal{F}$-modules are abelian categories. It is clear that for $\mathcal{F}$-modules one has duality and pointwise tensor products as for Γ-modules. Moreover the functors

$$\mathcal{F}^n = K\big[\mathrm{Hom}_{\mathcal{F}}([n], -)\big] \quad \text{and} \quad \mathcal{F}_n = K\big[\mathrm{Hom}_{\mathcal{F}}(-, [n])\big]$$

are projective generators in $\mathcal{F}$*-mod* and *mod-*$\mathcal{F}$ respectively, while the dual functors $\mathcal{F}^{n*}$ and $\mathcal{F}_n^*$ are injective cogenerators. Clearly

$$\mathcal{F}^n \otimes \mathcal{F}^m \cong \mathcal{F}^{n+m+1} \quad \text{and} \quad \mathcal{F}_n \otimes \mathcal{F}_m \cong \mathcal{F}_{nm+n+m}.$$

These are consequences of the facts that $[n] \coprod [m] \cong [n+m+1]$ and $[n] \times [m] \cong [nm+n+m]$. Therefore *if F and T are projective $\mathcal{F}$-module, then $F \otimes T$ is also projective.* Similarly, *if F and T are injective $\mathcal{F}$-module, then $F \otimes T$ is also injective* provided F or T has values in finite dimensional vector spaces. We have a bifunctor $- \otimes_{\mathcal{F}} - : \textit{mod-}\mathcal{F} \times \mathcal{F}\textit{-mod} \longrightarrow \textit{Vect}$ similar to 1.5 with the same kind of properties. For example one has an isomorphism: $F([n]) \cong \mathcal{F}_n \otimes_{\mathcal{F}} F$. We also observe that Proposition 1.6 is still valid for $\mathcal{F}$-modules. The precomposition with $\mu : \Gamma \longrightarrow \mathcal{F}$ defines the functor from $\mathcal{F}$-modules to Γ-modules. We let μ^* denote this functor. Thus for any $\mathcal{F}$-module F one has $\mu^*(F)([n]) = F([n])$. Let $\nu : \mathcal{F} \longrightarrow \Gamma$ be the functor which adds a disjoint basepoint. The precomposition with ν defines the functor $\nu^* : \textit{mod-}\Gamma \longrightarrow \textit{mod-}\mathcal{F}$. Thus for any right Γ-module T one has $(\nu^* T)(X) = T(\nu X) = T(X \sqcup *)$. The following isomorphism is a consequence of the fact that ν is left adjoint to the functor $\mu : \Gamma \longrightarrow \mathcal{F}$.

$$\mathrm{Hom}_{\textit{mod-}\mathcal{F}}(\nu^* F, G) \cong \mathrm{Hom}_{\Gamma\textit{-mod}}(F, \mu^* G). \tag{3.2.1}$$

Moreover for a left Γ-module F we let $\sigma F : \mathcal{F} \longrightarrow \textit{Vect}$ be the functor, which assigns the sum $\bigoplus_{x \in X} F(X_x)$ to a finite set X. Here X_x is a set X with basepoint x. The proof of the following isomorphism is immediate

$$\mathrm{Hom}_{\mathcal{F}\textit{-mod}}(\sigma F, G) \cong \mathrm{Hom}_{\Gamma\textit{-mod}}(F, \mu^* G). \tag{3.2.2}$$

We see that $\sigma \Gamma^n \cong \mathcal{F}^n$ and $\nu^* \Gamma_n \cong \mathcal{F}_n$.

3.3. PROPOSITION. –
- (i) *The functor μ^* is exact and sends injective $\mathcal{F}$-modules to injective Γ-modules.*
- (ii) *The functor $\sigma : \Gamma\text{-mod} \longrightarrow \mathcal{F}\text{-mod}$ (respectively $\nu^* : \text{mod-}\Gamma \longrightarrow \text{mod-}\mathcal{F}$) is exact and sends projective Γ-modules to projective $\mathcal{F}$-modules.*
- (iii) *One has isomorphisms*

$$\mathrm{Ext}^*_{\mathcal{F}\textit{-mod}}(\sigma F, G) \cong \mathrm{Ext}^*_{\Gamma\textit{-mod}}(F, \mu^* G),$$
$$\mathrm{Ext}^*_{\textit{mod-}\mathcal{F}}(\nu^* F, G) \cong \mathrm{Ext}^*_{\textit{mod-}\Gamma}(F, \mu^* G)$$

and

$$\mathrm{Tor}_*^{\mathcal{F}}(\nu^* F, G) \cong \mathrm{Tor}_*^{\Gamma}(F, \mu^* G).$$

(iv) *If F and G are right Γ-modules, then $\nu^*(F \otimes G) \cong \nu^* F \otimes \nu^* G$ and $\nu^*(\Lambda^i \circ F) \cong \Lambda^i \circ \nu^*(F)$.*

(v) *One has the following isomorphism*

$$\nu^* t \cong \mathrm{Hom}_{Sets}(-, K).$$

Proof. – The exactness of a left (respectively right) adjoint implies that the right (respectively left) adjoint preserves injectives (respectively projectives). Clearly μ^*, ν^* and σ are exact. This implies (i) and (ii). The assertion (iii) is a consequence of (i), (ii), (3.2.1) and (3.2.2), while (iv) and (v) are obvious. □

3.4. A spectral sequence for $HC_*(F)$

According to [16] the cyclic homology $HC_*(F)$ of a left $\mathcal{F}$-module F is defined as the homology of the bicomplex

$$\begin{tikzcd}
\arrow[d] & \arrow[d] & \arrow[d] \\
F([2]) \arrow[d] & F([1]) \arrow[l] \arrow[d] & F([0]) \arrow[l] \\
F([1]) \arrow[d] & F([0]) \arrow[l] & \\
F([0]), & &
\end{tikzcd}$$

where each column is the chain complex associated to the simplicial vector space $F(S^1)$. Let us observe that one can write $F([n]) \cong \mathcal{F}_n \otimes_{\mathcal{F}} F$. Therefore, $HC_*(F) \cong H_*(L_{**} \otimes_{\mathcal{F}} F)$, where L_{**} is the following bicomplex of right $\mathcal{F}$-modules:

$$\begin{tikzcd}
\arrow[d] & \arrow[d] & \arrow[d] \\
\mathcal{F}_2 \arrow[d] & \mathcal{F}_1 \arrow[l] \arrow[d] & \mathcal{F}_0 \arrow[l] \\
\mathcal{F}_1 \arrow[d] & \mathcal{F}_0 \arrow[l] & \\
\mathcal{F}_0 \,. & &
\end{tikzcd}$$

Let us calculate the value of L_{**} on $[n]$. Since

$$\mathcal{F}_m = K\big[\mathrm{Hom}_{\mathcal{F}}(-, [m])\big]$$

we see that $H_*(L_{**}([n]))$ is the cyclic homology of the cyclic module associated to the cyclic space $S^1 \times \cdots \times S^1$ ($(n+1)$-times). Now one can use standard facts about cyclic homology theory (see Theorem 7.2.3 and Proposition 4.4.8 of [17]) to conclude that $H_*(L_{**}([n]))$ is isomorphic to the exterior algebra $\Lambda^*(x_1,\dots,x_n)$, where each x_i has degree 1. A similar argument as in Example 1.8 shows that $H_i(L_{**}) \cong \Lambda^i \circ \bar{t}$, where $\bar{t}$ is defined as follows: Let us recall that $\mathcal{F}_0$ is the constant functor with value K. There is a canonical transformation $\mathcal{F}_0 \to \nu^* t$ whose value on $[n]$ is a homomorphism which sends an element $\lambda \in K$ to $\lambda_{[n]} \in \nu^* t([n]) = \mathrm{Hom}_{Sets}([n], K)$. Here $\lambda_{[n]} : S \to K$ is the constant map with value λ. By definition $\bar{t}$ fits in the exact sequence:

$$0 \to \mathcal{F}_0 \to \nu^* t \to \bar{t} \to 0. \tag{3.4.1}$$

3.5. THEOREM. – *Let F be a left $\mathcal{F}$-module.*

(i) *Then there exists a spectral sequence*

$$E^2_{pq} = \mathrm{Tor}^{\mathcal{F}}_p\left(\Lambda^q \circ \bar{t}, F\right) \Rightarrow HC_{p+q}(F).$$

For $q = 1$ and $p \geqslant 2$ one has $E^2_{p1} \cong \mathrm{Tor}^{\Gamma}_p(t, \mu^* F)$.

(ii) *If K is a field of characteristic zero, then the spectral sequence degenerates:*

$$HC_{p+q}(F) \cong \bigoplus_{p+q=n} \mathrm{Tor}^{\mathcal{F}}_p\left(\Lambda^q \circ \bar{t}, F\right).$$

(iii) *Furhermore in this case one has an isomorphism between our decomposition and Loday's decomposition* (3.1.1)

$$HC^{(i)}_n(F) \cong \mathrm{Tor}^{\mathcal{F}}_{n-i}\left(\Lambda^i \circ \bar{t}, F\right), \quad n \geqslant i.$$

Proof. – (i) One can apply the $\mathcal{F}$-version of Proposition 1.6 for $C_* = \mathit{Tot}(L_{**})$. By the above calculation the spectral sequence is of the expected form. Since $\mathcal{F}_0$ is projective, the exact sequence (3.4.1) shows that there exists an isomorphism $E^2_{p1} \cong \mathrm{Tor}^{\mathcal{F}}_p(\nu^* t, F)$ provided $p \geqslant 2$. Therefore the last statement is a consequence of Proposition 3.3(iii).

(ii) By Proposition 1.6 it is enough to show that in characteristic zero one has $\mathrm{Ext}^i_{\mathit{mod}\text{-}\mathcal{F}}(\Lambda^n \circ \bar{t}, \Lambda^m \circ \bar{t}) = 0$ if $n < m$ and $i \geqslant 2$. We will work with tensor powers instead of exterior powers, because these are retracts of tensor powers. The short exact sequence (3.4.1) shows that the complex $(\mathcal{F}_0 \to \nu^* t)^{\otimes n}$ is a (nonprojective) resolution of $\bar{t}^{\otimes n}$. Therefore one has a spectral sequence

$$E^{pq}_1 \Rightarrow \mathrm{Ext}^{p+q}_{\mathit{mod}\text{-}\mathcal{F}}\left(\bar{t}^{\otimes n}, \bar{t}^{\otimes m}\right),$$

where E^{pq}_1 is the sum of $\binom{n}{p}$ copies of

$$\mathrm{Ext}^q_{\mathit{mod}\text{-}\mathcal{F}}\left(\nu^* t^{\otimes p} \otimes \mathcal{F}_0^{\otimes n-p}, \bar{t}^{\otimes m}\right).$$

By Proposition 3.3(iv) we know that the functor ν^* commutes with tensor products. Since $\mathcal{F}_0 = K$ one can delete it in the tensor product. Therefore Proposition 3.3(iii) shows that E^{pq}_1 is a sum of several copies of

$$\mathrm{Ext}^q_{\mathit{mod}\text{-}\Gamma}\left(t^{\otimes p}, \mu^*\left(\bar{t}^{\otimes m}\right)\right).$$

One easily sees that $\mu^*(\bar{t}^{\otimes m}) \cong t^{\otimes m}$. Therefore $E_1^{pq} = 0$ if $q > 0$, because $t^{\otimes m}$ is injective. According to (1.10.2) we have

$$\mathrm{Hom}_{mod\text{-}\Gamma}\big(t^{\otimes p}, t^{\otimes n}\big) \cong \mathrm{Hom}_{\Gamma\text{-}mod}\big(t^{*\otimes n}, t^{*\otimes p}\big) = 0,$$

because $p < n$. Therefore $E_1^{pq} = 0$ even when $q = 0$.

(iii) By the argument given in the proof of Theorem 2.7 it is enough to show that $HC_m(\nu^* t^{*\otimes n}) = 0$ for $n \neq m$ and $HC_n(\nu^* t^{*\otimes n}) = K$ for all n. By Proposition 4.3.10 of [17] one can reduce the problem to the cases $n = 0, 1$. For $n = 0$ one obtains the functor $\mathcal{F}_0$. Clearly $\mathcal{F}_0(S^1) \cong K[S^1]$. Therefore $HC_0(\mathcal{F}_0) = K$ and $HC_m(\mathcal{F}_0) = 0$ for $m > 0$, thanks to Theorem 7.2.3 of [17]. Similarly $HC_*(\mathcal{F}_1) = HC_*(K[S^1 \times S^1]) = \Lambda^*(x_1)$, where x_1 has dimension 1. It follows from 1.4 that $\nu^* t^* \oplus \mathcal{F}_0 \cong \mathcal{F}_1$. Therefore $HC_m(\nu^* t^*) = 0$ for $m \neq 1$ and $HC_1(\nu^* t^*) = K$ and we are done. □

4. Hochschild–Kostant–Rosenberg theorem for functors

The goal of this section is to get a partial generalization of the Hochschild–Kostant–Rosenberg theorem (see for Example 3.4.4 of [17]) which will play a crucial role for our calculations in Section 5. In this section we assume that K is a field of characteristic zero.

4.1. Inner Ext and inner Tor

It is well known that in the category of Γ-modules there exist an inner Hom and inner tensor product. Thus for any left Γ-modules T and U and right Γ-modules V there exists left Γ-modules $\mathbf{Hom}(T, U)$ and $V \odot U$, such that for any left Γ-module F and right Γ-module L, one has the functorial isomorphisms:

$$\mathrm{Hom}_{\Gamma\text{-}mod}(F \otimes T, U) \cong \mathrm{Hom}_{\Gamma\text{-}mod}\big(F, \mathbf{Hom}(T, U)\big),$$
$$(L \otimes V) \otimes_\Gamma U \cong L \otimes_\Gamma (V \odot U).$$

One puts $F = \Gamma^n$ and $L = \Gamma_n$ to get

$$\mathbf{Hom}(T, U)([n]) \cong \mathrm{Hom}_{\Gamma\text{-}mod}\big(\Gamma^n \otimes T, U\big),$$
$$(V \odot U)([n]) \cong (\Gamma_n \otimes V) \otimes_\Gamma U.$$

Usually the inner Hom and inner tensor product are defined by these isomorphisms. This is actually true in any category of functors. Since Γ admits sums and products one can describe the inner Hom and inner tensor product more easily. For this purpose we need the functors $\widetilde{\Delta}_n : \Gamma\text{-}mod \to \Gamma\text{-}mod, n \geqslant 0$, and $\Delta_n : \Gamma\text{-}mod \to \Gamma\text{-}mod$ defined by

$$\big(\widetilde{\Delta}_n U\big)([m]) := U\big([n] \vee [m]\big)$$

and

$$(\Delta_n U)([m]) := U\big([n] \times [m]\big).$$

We claim that there exist natural isomorphisms

$$\mathrm{Hom}_{\Gamma\text{-}mod}(\Gamma^n \otimes T, U) \cong \mathrm{Hom}_{\Gamma\text{-}mod}\big(T, \widetilde{\Delta}_n U\big),$$
$$(\Gamma_n \otimes V) \otimes_\Gamma U \cong U \otimes_\Gamma (\Delta_n U).$$

Indeed, one easily checks that these are isomorphic when $T=\Gamma^k$ and $V=\Gamma_k$ and therefore the Yoneda principle (see Section 1.2) shows that they are isomorphic for any T and V. Hence

$$\mathbf{Hom}(T,U)([n])\cong \mathrm{Hom}_{\Gamma\text{-}mod}\big(T,\widetilde{\Delta}_n U\big) \qquad (V\odot U)([n])\cong V\otimes_\Gamma(\Delta_n U).$$

Clearly for the right derived functors of $\mathbf{Hom}(T,U)$ one has a similar description

$$\mathbf{Ext}^*(T,U)([n])\cong \mathrm{Ext}^*_{\Gamma\text{-}mod}\big(T,\widetilde{\Delta}_n U\big). \tag{4.1.1}$$

Moreover, one has the spectral sequence

$$E_2^{pq}=\mathrm{Ext}^p_{\Gamma\text{-}mod}\big(F,\mathbf{Ext}^q(T,U)\big)\Rightarrow \mathrm{Ext}^{p+q}_{\Gamma\text{-}mod}(F\otimes T,U),$$

which is a consequence of the spectral sequence for the composite of functors. We let $\mathbf{Tor}_*(V,U)$ be the left derived functors of $V\odot U$. Then one has an isomorphism

$$\mathbf{Tor}_*(V,U)([n])\cong \mathrm{Tor}^\Gamma_*(V,\Delta_n U) \tag{4.1.2}$$

and a spectral sequence

$$E^2_{pq}=\mathrm{Tor}^\Gamma_p\big(F,\mathbf{Tor}_q(V,U)\big)\Rightarrow \mathrm{Tor}^\Gamma_{p+q}(F\otimes V,U). \tag{4.1.3}$$

4.2. LEMMA. – *For the functors $F,T:\Gamma\to$ Vect one has an isomorphism*

$$\pi_*^{st}(F\otimes T)\cong \pi_*^{st}(F)\otimes T([0])\oplus F([0])\otimes \pi_*^{st}(T).$$

Proof. – This follows from the isomorphism (2.1.2) and from the Eilenberg–Zilber and Künneth theorems. □

4.3. Smooth functors

For any left Γ-module T we let $\Pi_0 T$ be the left Γ-module $[n]\mapsto \pi_0^{st}(\Delta_n T)\cong t\otimes_\Gamma(\Delta_n T)$. Thus $\Pi_0 T\cong t\odot T$. A left Γ-module F is called 0-*smooth* if $\pi_i^{st}(F)=0$ for any $i>0$ and it is called s-*smooth* for $s>0$ if Γ-modules $\Delta_n F$ and $\Pi_0 F$ are $(s-1)$-smooth for any $n\geqslant 0$. Since $\Delta_0 F=F$, we see that any s-smooth functor is $(s-1)$-smooth as well. A left Γ-module F is called ∞-*smooth* or *smooth* for brevity if it is s-smooth for every $s\geqslant 0$.

4.4. LEMMA. – *Let $0\leqslant s\leqslant\infty$. Then the following holds.*

(i) *The direct sum of s-smooth left Γ-modules is s-smooth.*
(ii) *If F and G are s-smooth left Γ-modules, then $F\otimes G$ is s-smooth.*
(iii) *Any projective left Γ-module is smooth.*
(iv) *If F is a s-smooth left Γ-module, then* $\mathrm{Tor}_i^\Gamma(t^{\otimes r},F)=0$, *for $i>0$, and $0\leqslant r\leqslant s+1$.*
(v) *If F is a smooth left Γ-module and V is a right Γ-module of finite type and of finite degree, then* $\mathrm{Tor}_i^\Gamma(V,F)=0$, *for $i>0$.*

Proof. – (i) follows from the fact that the functors Δ_n, $n\geqslant 0$, and Π_0 commute with direct sums.

(ii) follows from Lemma 4.2 for $s=0$. For $s>0$, one works by induction. One observes that the functors $\Delta_n, n\geqslant 0$, commute with pointwise tensor product and

$$\Pi_0(F\otimes T)\cong \Pi_0(F)\otimes T\oplus F\otimes \Pi_0(T),$$

which is a consequence of Lemma 4.2.

(iii) By Proposition 2.2 any projective left Γ-module is 0-smooth. Now take $s>0$. We need to show that projective left Γ-modules are s-smooth. One can restrict ourself to $\Gamma^k, k\geqslant 0$, because they are projective generators. By part (ii) it is enough to consider the case $k=0,1$. Any constant functor is projective, because Γ^0 is constant. Therefore constant functors are 0-smooth. Since the functor $\Delta^n, n\geqslant 0$, takes constant functors to constant functors and since Π_0 vanishes on constant functors, one obtains by induction that constant functors are smooth. Since $\Delta^n(\Gamma^1)=\Gamma^1\otimes\Gamma^1([n])$ and $\pi_0^{st}\Gamma^1\cong K$ we obtain $\Pi_0(\Gamma^1)\cong\Gamma^1$, which shows that Γ^1 is smooth too and the proof of (iii) is finished.

(iv) is clear when $r=0,1$. For $r\geqslant 2$ it follows from the spectral sequence (4.1.2), because (4.1.1) shows that $\mathbf{Tor}_q(t,F)$ is zero for $q>0$ and is $(s-1)$-smooth for $q=0$.

(v) By Proposition 1.12 any such V has finite injective dimension and one can use induction with respect of injective dimension of V. When V is injective then the statement reduces to (iv). Indeed the functors $t^{\otimes n}$ are injective cogenerators, therefore in our circumstances V is a direct summand of a finite sum of functors $t^{\otimes r}$. The general case follows by embedding V in an injective of the same type (see Section 1.11) and considering the long exact sequence for Tor-groups. □

4.5. LEMMA. – *Let A be a smooth commutative algebra of finite type. Then $\mathcal{L}(A,A)$ is smooth.*

Proof. – We need to prove that the functor $F=\mathcal{L}(A,A)$ is s-smooth for any $s\geqslant 0$. In the characteristic zero case the stable homotopy coincides with Harrison homology and hence with André–Quillen homology up to a shift of degree. Thus they vanish for smooth algebras (see [23]). Therefore F is 0-smooth. One observes that one has the following isomorphism

$$\Delta_n F\cong\mathcal{L}\big(A^{\otimes(n+1)},A^{\otimes(n+1)}\big).$$

Therefore $\pi_0^{st}\Delta_n F\cong HH_1(A^{\otimes(n+1)})$. The Künneth theorem for Hochschild homology yields the following isomorphism

$$\Pi_0(F)\cong\mathcal{L}\big(A,\Omega^1_A\big)\otimes\Gamma^1.$$

It is well known that the tensor product of smooth algebras is still smooth and for smooth algebra A the module of differential forms Ω^1_A is a finitely generated projective A-module. Therefore $\mathcal{L}(A,\Omega^1_A)$ is a direct summand of the finite sum of Γ-modules like $\mathcal{L}(A,A)$. According to Lemma 4.4 the proof can be completed by induction. □

4.6. HKR THEOREM FOR FUNCTORS. – *Let F be a smooth left Γ-module and let L be a connected pointed simplicial set. Then*

$$\pi_k\big(F(L)\big)\cong\mathcal{J}_k(H_*L)\otimes_\Gamma F.$$

In particular $\pi_k F(S^d)=0$ if $k\neq jd$ and $\pi_{jd}F(S^d)\cong H_j^{(j)}(F)\cong(\Lambda^j\circ t)\otimes_\Gamma F$ if d is odd and $\pi_{jd}F(S^d)\cong\theta^j\otimes_\Gamma F$ if d is even. Moreiver $H_n^{(i)}(F)=0$ for $i<n$.

Proof. – According to (1.11.1) the functor $\mathcal{J}_k(H_*(L))$ has finite degree. Therefore we can use Lemma 4.4(v) to conclude that all higher Tor groups in Theorem 2.4 vanishe. □

4.7. Generalization for chain complex of left Γ-modules

A chain complex (F_*,∂) of left Γ-modules is called *smooth* if it is smooth componentwise. For example, if (A_*,δ) is a free graded chain algebra, then the argument given in the proof of Lemma 4.5 shows that $\mathcal{L}(A_*,A_*)$ is smooth as chain complex of left Γ-modules. Thanks to

Theorem 4.6 we see that if (F_*, ∂) is smooth, then the chain maps (2.14.1) and (2.14.2) are quasi-isomorphisms.

5. Higher order Hochschild homology and cohomology of mapping spaces

5.1. Definition of higher order Hochschild homology

Let A be a commutative algebra, M be an A-module and Y be a pointed simplicial finite set. Following an idea of Anderson [1] one defines *Y-homology of A with coefficient in M* by

$$H_*^Y(A,M) := \pi_*\big(\mathcal{L}(A,M)(Y)\big).$$

Here $\mathcal{L}(A,M)$ is the let Γ-module introduced in Section 1.7. Theorem 2.4 shows that this definition depends only on the homotopy type of Y. In the case, $Y = S^d$, we call this object *Hochschild homology of order* d and denote it by

$$H_*^{[d]}(A,M), \quad d \geqslant 1.$$

Clearly, for $d = 1$ one recovers usual Hochschild homology. In the particular case $M = A$ we write $HH_*^Y(A)$ and $HH_*^{[d]}(A)$ instead of $H_*^Y(A,M)$ and $H_*^{[d]}(A,A)$. Moreover, for augmented algebra A we write $H_*^Y(A)$ and $H_*^{[d]}(A)$ instead of $H_*^Y(A,K)$ and $H_*^{[d]}(A,K)$. The standard model of S^n has only one 0-simplex and no nondegenerate simplices in dimensions > 0 and $< n$. Therefore

$$H_0^{[d]}(A,M) \cong M \quad \text{and} \quad H_i^{[d]}(A,M) = 0 \quad \text{for } 0 < i < n.$$

The isomorphism (2.1.2) shows that

$$H_d^{[d]}(A,M) \cong H_1^{[1]}(A,M) \cong \Omega_A^1 \otimes_A M.$$

Thanks to Proposition 1.15 the natural maps (2.14.1) and (2.14.2) give rise to the homomorphism

$$HH_{nd}^{[d]}(A) \to \mathit{Sym}_A^n\big(s^d(\Omega_A^1)\big), \quad n \geqslant 1 \tag{5.1.1}$$

Here $\mathit{Sym}_A^*(E)$ denotes the symmetric A-algebra (in the graded sence) generated by an A-module E, while $s^d(E)$ denotes the d-fold suspension of E. Thus the map (5.1.1) has degree zero.

5.2. PROPOSITION. – *Let K be a field of characteristic zero. If d is odd, then for any commutative algebra A and any A-module M one has a natural decomposition.*

$$H_n^{[d]}(A,M) \cong \bigoplus_{i+dj=n} H_{i+j}^{(j)}(A,M).$$

The summand $H_n^{(n)}(A,M)$ is isomorphic to $M \otimes_A \Omega_A^n$. Moreover, for even d one has a natural decomposition

$$H_n^{[d]}(A,M) \cong \bigoplus_{i+dj=n} \operatorname{Tor}_i^\Gamma\big(\theta^j, \mathcal{L}(A,M)\big).$$

The summand corresponding to $j = 1$ is isomorphic to $H_{i+1}^{(1)}(A,M)$, while the summund corresponding to $i = 0$ is isomorphic to $M \otimes_A \mathit{Sym}_A^n(\Omega_A^1)$.

Proof. – This is a consequence of (2.12) and (1.15). □

5.3. PROPOSITION. – *If A is a smooth algebra over a characteristic zero field, then the map* (5.1.1) *is an isomorphism.*

Proof. – This is a consequence of the Hochschild–Kostant–Rosenberg Theorem 4.6, together with Proposition 1.15. □

5.4. Higher order Hochschild homology of truncated polynomial algebras

Let r be a positive integer and let $A = K[x]/(x^{r+1})$. According to Proposition 5.4.15 of [17] one knows that $HH_n^{(q)}(A) = 0$ except when $n = 2q$ or $n = 2q - 1$ for any $q \geqslant 1$ and in these cases it is isomorphic to $A/(x^r) \cong K^r$. Therefore one can use Proposition 5.2 to get $HH_m^{[d]}(A) = 0$ if $m \neq n(d+1)$ or $m \neq n(d+1) - 1$ and

$$HH_m^{[d]}(A) \cong K^r \quad \text{if } m = n(d+1) \text{ or } m = n(d+1) - 1.$$

Here d is odd.

5.5. Higher order Hochschild homology and cohomology of mapping spaces X^{S^d}

We use some ideas from [1] and [2] to relate the groups $HH_*^{[d]}(A)$ to the cohomology of the mapping spaces X^{S^d}. For this we extend the definition of higher order Hochschild homology to commutative graded differential algebras as follows: Let (A_*, ∂) be a commutative graded differential K-algebra with unit and let M_* be a graded differential A-module. In Section 1.7 we defined the chain complex $\mathcal{L}(A_*, M_*)$ of left Γ-modules. Now, one can define $HH_*^{[d]}(A_*, M_*)$ to be the homology of the total complex associated to the simplicial chain complex $\mathcal{L}(A_*, M_*)(S^d)$. If A_* is a free graded algebra, then $\mathcal{L}(A_*, A_*)$ is a smooth chain complex of left Γ-modules and therefore the homomorphism

$$HH_*^{[d]}(A_*) \to H_*\big(Sym_{A_*}^*\big(S^d(\Omega_{A_*}^1)\big)\big),$$

is isomorphism thanks to Section 4.7 and Proposition 1.15.

5.6. THEOREM. – *Let X be a finite-dimensional d-connected CW-complex. Let $A^*(X)$ be the commutative cochain algebra of X in the sense of Sullivan. Then*

$$HH_{-*}^{[d]}\big(A^*(X)\big) \cong H^*\big(X^{S^d}, K\big).$$

For $d = 1$ this is the well-known isomorphism proved in [15].

5.7. The minimal model of X^{S^d}

The following construction is a special case of the general construction due to Heafliger (see for example [10, pp. 308–309]). Let X be a d-connected space and let $\mathcal{X} = (\Lambda^*(V), \partial)$ be a minimal model of X. By our assumptions the graded vector space V has no components in dimension $\leqslant d$. One defines the commutative graded differential algebra $\mathcal{M} = (\Lambda^*(V \oplus \bar{V}), \bar{\partial})$ as follows. Take $\bar{V}$ to be $\Sigma^{-d} V$. For each element $x \in V$, we let $\bar{x}$ be the corresponding element in $\bar{V}$. Then $x \mapsto \bar{x}$ defines an isomorphism $V \to \bar{V}$ of degree $-d$. Let $i : \mathcal{X} \to \mathcal{M}$ be the unique derivation of degree $-d$ extending $x \mapsto \bar{x}$. Then $\bar{\partial}$ is given by

$$\bar{\partial}(x) = \partial x \quad \text{and} \quad \bar{\partial}(\bar{x}) = (-1)^d i \partial x, \ x \in V.$$

Then $\mathcal{M}$ is a minimal model of X^{S^d}. Indeed, the direct inspection shows that for $A = K + Kx, x^2 = 0$, $|x| = d$ the algebra $(\Lambda X, \bar{D})$ of [10] is isomorphic to $\mathcal{M}$.

5.8. Proof of Theorem 5.6

The higher order Hochschild homology of commutative differential graded algebras sends weak equivalences to isomorphisms. Therefore one can use the minimal model $\mathcal{X}$ of X instead of $A^*(X)$. The minimal model $\mathcal{M}$ of X^{S^d} is isomorphic to the differential graded algebra $\Lambda^*_{\mathcal{X}}(\Omega^1_{\mathcal{X}}[-d])$ or $Sym^*_{\mathcal{X}}(\Omega^1_{\mathcal{X}}[-d])$ depending on the parity of d (see (5.4.3) of [17]). Since the underlying graded algebra is free, the homology of $\Lambda^n_{\mathcal{X}}(\Omega^1_{\mathcal{X}}[-d])$ and $Sym^n_{\mathcal{X}}(\Omega^1_{\mathcal{X}}[-d])$ is isomorphic to the higher order Hochschild homology for odd and even d respectively (see Section 5.5), and the result follows.

5.9. *Remarks*. – (i) Theorem 5.6 and Proposition 5.2 shows that the cohomology $H^*(X^{S^d})$ has a natural Hodge decomposition. Moreover the knowledge of this decomposition for $d = 1$ (respectively $d = 2$) completely determines $H^*(X^{S^d})$ for all d odd (respectively even).

(ii) It is well-known that the cohomology $H^*(X^{S^1})$ of a free loop space is isomorphic to the Hochschild homology of the singular cochain algebra $C^*(X)$ of X, provided X is 1-connected. This result is true in any characteristic. Since higher order Hochschild homology is defined only for commutative algebras, we are not able to generalize this isomorphism for the mapping space X^{S^d}, because $C^*(X)$ is not commutative. However in any characteristic there exist spectral sequences

$$E^2_{**} = HH^Y_{-*}\big(H^*(X)\big) \Rightarrow H^*\big(X^Y\big), \qquad E^2_{**} = H^Y_{-*}\big(H^*(X)\big) \Rightarrow H^*\big(Map_*(Y,X)\big)$$

provided Y is a finite complex and X is dim(X)-connected. Here $Map_*(Y,X)$ denotes the space of pointed preserving maps. A homological version of the second spectral sequence was constructed in [1] (see Theorem 2.3 of [1]), and in [5] it was proved that it actually converges (see 4.2 of [5]). A similar argument gives also the first spectral sequence.

(iii) Proof of Theorem 2.1 of [28] shows that one can replace $A^*(X)$ by $H^*(X)$ in Theorem 5.6 if one of the following conditions holds:

(a) $K = \mathbf{R}$ and X is a compact Riemannian symmetric space,

(b) $K = \mathbf{C}$ and X is a compact Kähler manifold.

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(Manuscript received September 3, 1998;
accepted, after revision, June 11, 1999.)

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