A note on the energy of relative equilibria of point vortices

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Analytical formulas are derived for the energy of simple relative equilibria of identical point vortices such as the regular polygons, both open and centered, and the various known configurations consisting of nested regular polygons with or without a vortex at the center. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2795276]

I. INTRODUCTION

The problem of determining relative equilibria of identical point vortices is longstanding.[1] The simplest such equilibria, vortices arranged at the vertices of a regular polygon with or without one at the center, go back to the work by Kelvin and Thomson[2] late in the 19th century, and are wound up with the now defunct theory of vortex atoms. Many years later, Havelock[3] found relative equilibria consisting of two nested, regular polygons. One may place a vortex at the center of such configurations (modulo adjustment of the radii) and produce further relative equilibria. More recently Aref and van Buren[4] found relative equilibria consisting of three nested regular polygons.

An important quantity to be determined for all such configurations is the kinetic energy of the induced flow, which is given by the point vortex Hamiltonian.[1,5] For identical point vortices, this is, in essence, a purely geometric quantity, viz. (the logarithm of) the product of all intravortex distances in the configuration. For N vortices, then, we wish to calculate the product of the $N(N-1)/2$ vortex distances. The main result of this paper is a formula, given as Eq. (14) below, that allows the analytical determination of this product for the analytically known relative equilibria.

Let us establish a bit of notation. Let the vortices be given as N points in the complex plane $\{z_\alpha | \alpha=1,\ldots,N\}$. In order to form a relative equilibrium configuration for N identical point vortices, the z_α must solve the system of equations,

$$z_\alpha^* = \sum_{\beta=1}^{N}{}' \frac{1}{z_\alpha - z_\beta}; \quad \alpha = 1,\ldots,N. \tag{1}$$

Here the asterisk on the left-hand side denotes complex conjugation. The prime on the summation on the right-hand side reminds us to skip the singular term $\beta=\alpha$. Units of length and time have been chosen so that the dimensional factor that would otherwise appear in Eq. (1) is 1. Stated more explicitly, if in dimensional units the circulations of all the vortices are Γ, and the angular frequency of rotation of the configuration is ω, then units of length and time are chosen such that $2\pi\omega/\Gamma=1$. All lengths, such as the radii of the various nested regular polygons considered below, are thus scaled by $\sqrt{\Gamma/2\pi\omega}$.

It follows from Eq. (1) that

$$\sum_{\alpha=1}^{N} z_\alpha = 0. \tag{2}$$

It also follows that

$$\sum_{\alpha=1}^{N} |z_\alpha|^2 = \frac{N(N-1)}{2}. \tag{3}$$

These imply that the "average" of the coordinates of the vortices vanishes,

$$\langle z_\alpha \rangle = \frac{1}{N}\sum_{\alpha=1}^{N} z_\alpha = 0, \tag{4}$$

whereas the average size of the configuration, estimated as

$$\sqrt{\langle |z_\alpha|^2 \rangle} = \sqrt{\frac{1}{N}\sum_{\alpha=1}^{N} |z_\alpha|^2} = \sqrt{\frac{N-1}{2}}, \tag{5}$$

will grow with N.

A convenient summary of the positions in a relative equilibrium $z_1,\ldots,z_N$ is the generating polynomial[6]

$$P(z) = (z-z_1)(z-z_2)\cdots(z-z_N). \tag{6}$$

This polynomial contains much information about the relative equilibrium. For example, if $P(z)$ has real coefficients, the relative equilibrium has an axis of symmetry along the real axis of coordinates. The main point of this paper is to show how the generating polynomial can be used to calculate the energy of a relative equilibrium in simple cases.

For identical vortices the point vortex Hamiltonian is $-\Gamma^2/4\pi$ times the logarithm of the product of vortex separations,[5]

$$\Theta = \prod_{\alpha,\beta=1}^{N}{}' l_{\alpha\beta}. \tag{7}$$

(The prime on the product means $\alpha \neq \beta$.) Note that large values of this product correspond to low energies (vortices are far apart); high energy states correspond to small values of the product. The order of magnitude of Eq. (7) may be estimated by noting that

$$\frac{1}{2}\sum_{\alpha,\beta=1}^{N}{}' l_{\alpha\beta}^2 = \frac{1}{2}\sum_{\alpha,\beta=1}^{N} l_{\alpha\beta}^2 = N\sum_{\alpha=1}^{N}|z_\alpha|^2, \tag{8}$$

where Eq. (2) has been used. Hence, from Eq. (3) the average squared separation is

$$\langle l_{\alpha\beta}^2\rangle = \frac{1}{N(N-1)}\sum_{\alpha,\beta=1}^{N}{}' l_{\alpha\beta}^2 = \frac{2}{N-1}\sum_{\alpha=1}^{N}|z_\alpha|^2 = N. \tag{9}$$

Thus, we may expect

$$\Theta = \prod_{\alpha,\beta=1}^{N}{}' l_{\alpha\beta} \approx N^{N(N-1)/2}. \tag{10}$$

Because of this estimate, we will often work with

$$\begin{aligned}\theta &= \frac{2}{N(N-1)}\log\Theta \\ &= \frac{2}{N(N-1)}\log\prod_{\alpha,\beta=1}^{N}{}' l_{\alpha\beta} \\ &= \frac{2}{N(N-1)}\sum_{\alpha,\beta=1}^{N}{}' \log l_{\alpha\beta} = \langle\log l_{\alpha\beta}^2\rangle.\end{aligned} \tag{11}$$

II. FORMULA FOR THE VORTEX PATTERN ENERGY

There is a general formula for Θ, Eq. (7), in terms of the generating polynomial Eq. (6). This formula, Eq. (14) below, is our main result. To obtain the formula note that for $z \neq z_\alpha$,

$$\frac{P(z)-P(z_\alpha)}{z-z_\alpha} = \prod_{\beta=1}^{N}{}' (z-z_\beta),$$

where the prime on the product means $\beta \neq \alpha$. Taking the limit $z \to z_\alpha$ yields the derivative of P at z_α

$$P'(z_\alpha) = \prod_{\beta=1}^{N}{}' (z_\alpha - z_\beta). \tag{12}$$

Thus,

$$\prod_{\alpha=1}^{N} P'(z_\alpha) = \prod_{\alpha,\beta=1}^{N}{}' (z_\alpha - z_\beta), \tag{13}$$

so that

$$\prod_{\alpha,\beta=1}^{N}{}' l_{\alpha_\beta} = \left|\prod_{\alpha=1}^{N} P'(z_\alpha)\right|. \tag{14}$$

Given the vortex positions, this formula has at least two merits as a way of computing the value of the Hamiltonian. First, it involves evaluating and multiplying just N numbers, viz. the moduli of the derivative of P' at the vortex locations, instead of $N(N-1)/2$ vortex separations. Second, in cases where the vortex locations are known analytically, and the generating polynomial has a simple form, explicit formulas for the Hamiltonian may be obtained. Several of these formulas appear to be new. The study of special cases of Eq. (14) is the subject of the next section.

To bring out the essence of the formula (14) consider the following problem: Calculate the product $\Pi_{\lambda=1}^{n-1}(1-\epsilon^\lambda)$, where ϵ is the nth root of unity $e^{i2\pi/n}$. Using the idea of the main formula (14) we calculate as follows:

$$z^n - 1 = (z-1)\prod_{\lambda=1}^{n-1}(z-\epsilon^\lambda).$$

Thus,

$$\frac{z^n-1}{z-1} = \prod_{\lambda=1}^{n-1}(z-\epsilon^\lambda),$$

and upon taking the limit $z \to 1$ (which is equivalent to finding the derivative of z^n at $z=1$)

$$\prod_{\lambda=1}^{n-1}(1-\epsilon^\lambda) = n. \tag{15}$$

If N identical vortices are at the vertices of a regular polygon, i.e., situated at $z_\alpha = \mathrm{R}e^{i2\pi\alpha/N}$, $\alpha = 1, \ldots, N$, we have

$$\begin{aligned}\prod_{\alpha,\beta=1}^{N}{}' (z_\alpha - z_\beta) &= \prod_{\alpha=1}^{N}\left[z_\alpha^{N-1}\prod_{\beta=1}^{N}{}' (1-e^{i2\pi(\beta-\alpha)/N})\right] = R^{N(N-1)} \\ &\times\left[\prod_{\lambda=1}^{N-1}(1-e^{i2\pi\lambda/N})\right]^N \exp\left(-i\frac{2\pi}{N}\sum_{\alpha=1}^{N}\alpha\right),\end{aligned}$$

or, using Eq. (15) and evaluating the sum in the exponential,

$$\prod_{\alpha,\beta=1}^{N}{}' (z_\alpha - z_\beta) = (-1)^{N-1}N^N R^{N(N-1)}. \tag{16}$$

This is the basis of Eq. (19) below, and the "theme" of this calculation recurs in all that follows.

We mention that Eq. (13) is well known in the theory of polynomial equations[7] as the formula for the discriminant expressed as a symmetric function of the roots.

III. APPLICATION TO ANALYTICALLY KNOWN RELATIVE EQUILIBRIA

There are a number of relative equilibria that exist for all values of N, for which the vortex positions may be given analytically by a formula with N as the sole parameter.

A. Regular polygons

The regular polygon configurations have already been mentioned. In view of the normalization (3) they are given by

$$z_\alpha = R\exp\left(i\frac{2\pi\alpha}{N}\right), \quad R = \sqrt{\frac{N-1}{2}}, \quad \alpha = 1, \ldots, N. \tag{17}$$

The generating polynomial is

$$P(z) = z^N - \left(\frac{N-1}{2}\right)^{N/2}. \tag{18}$$

We have after a brief calculation, either as in the derivation of Eq. (16) or directly from Eq. (14),

$$\Theta_{\text{reg}} = N^N\left(\frac{N-1}{2}\right)^{N(N-1)/2};$$
$$\theta_{\text{reg}} = \frac{2}{N-1}\log N + \log(N-1) - \log 2. \tag{19}$$

B. Centered regular polygons

The centered regular polygons,

$$z_\alpha = R\exp\left(i\frac{2\pi\alpha}{N-1}\right), \quad R = \sqrt{\frac{N}{2}},$$
$$\alpha = 1, \ldots, N-1; \quad z_N = 0, \tag{20}$$

are a simple generalization of the open, regular polygons. Their generating polynomial is

$$P(z) = z\left[z^{N-1} - \left(\frac{N}{2}\right)^{(N-1)/2}\right]. \tag{21}$$

Thus,

$$P'(z) = Nz^{N-1} - R^{N-1}, \quad P'(\mathrm{R}e^{i2\pi\alpha/N-1}) = (N-1)R^{N-1},$$

$$P'(0) = -R^{N-1},$$

where $R=\sqrt{N/2}$. From Eq. (14) we now have

$$\Theta_{\text{cent}} = (N-1)^{N-1}\left(\frac{N}{2}\right)^{N(N-1)/2};$$
$$\theta_{\text{cent}} = \frac{2}{N}\log(N-1) + \log N - \log 2. \tag{22}$$

From Eqs. (19) and (22) it follows that for $N \leqslant 5$, $\theta_{\text{cent}} < \theta_{\text{reg}}$, but for $N \geqslant 6$ we have $\theta_{\text{cent}} > \theta_{\text{reg}}$. This follows by comparing Eqs. (19) and (22), which yields

$$\theta_{\text{reg}} - \theta_{\text{cent}} = \left(\frac{2}{N-1} - 1\right)\log N + \left(1 - \frac{2}{N}\right)\log(N-1)$$
$$= \frac{(N-2)(N-1)\log(N-1) - (N-3)N\log N}{N(N-1)}.$$

The last quantity is positive for $N=2,3,4,5$ but negative for larger values of N.

Results equivalent to Eqs. (19) and (22), and the inequality between them, were reported in Ref. 8 as part of a related investigation, albeit with a somewhat different motivation. Note that a different normalization than Eq. (3) is used in Ref. 8.

C. Collinear configurations

The configurations with all vortices on a line, conveniently taken to be the x-axis, are also "known analytically," albeit with an implicit statement of the positions as the zeros of the Nth Hermite polynomial,

$$H_N(x_\alpha) = 0, \quad \alpha = 1, \ldots, N. \tag{23}$$

It is known that the roots of the Nth Hermite polynomial satisfy the "sum rule" (3). The generating polynomial in this case is simply the Nth Hermite polynomial. From Eqs. (11) and (14) we now have

$$\Theta = \left|\prod_{\alpha=1}^{N} H_N'(x_\alpha)\right|.$$

Now, the Hermite polynomials satisfy the relations $H_N'(z) = 2NH_{N-1}(z)$ and $H_{N+1}(z) = 2zH_N(z) - 2NH_{N-1}(z)$. Thus, when the x_α are zeros of H_N, we have $H_N'(x_\alpha) = 2NH_{N-1}(x_\alpha) = -H_{N+1}(x_\alpha)$, and the formula

$$\Theta = \left|\prod_{\alpha=1}^{N} H_{N+1}(x_\alpha)\right|. \tag{24}$$

D. Double rings

There are further families of solutions where the vortex positions may be given, if not always by a formula, at least by a simple algorithm that depends on N and shows a systematic change with N. Thus, for N even, $N=2n$, we have relative equilibria with two nested, regular n-gons either arranged symmetrically or with the vortices on one staggered with respect to the other.[3] If the radii of the nested, regular polygons are R_1 and R_2, respectively, their ratio $\xi = R_1/R_2$ must solve

$$(n-1)\xi^{n+2} - (3n-1)\xi^n - (3n-1)\xi^2 + n - 1 = 0, \tag{25}$$

for the symmetric case and

$$(n-1)\xi^{n+2} - (3n-1)\xi^n + (3n-1)\xi^2 - (n-1) = 0, \tag{26}$$

for the staggered case. The roots $\xi=-1$ in Eq. (25) for odd n, and $\xi=1$ in Eq. (26) correspond to the regular N-gon and so are really "one-ring" solutions. Note that if ξ solves Eq. (25) for odd n, then $-\xi$ solves Eq. (26). It may be shown by elementary methods that, apart from the trivial solution $\xi = 1$ of Eq. (26), these polynomial equations have just two real, positive solutions that are reciprocals of one another. In some cases the roots of Eqs. (25) and (26) may be expressed algebraically, but in general they are transcendental.

For some of the lower values of n that lead to algebraic values of ξ we find

$$n = 2: \quad \sqrt{5 \pm 2\sqrt{6}};$$

$$n = 3: \quad \tfrac{1}{4}\left(1 + \sqrt{21} \pm \sqrt{2(3+\sqrt{21})}\right);$$

$$n = 4: \quad \sqrt{\tfrac{1}{3}(7 \pm 2\sqrt{10})}; \quad \sqrt{\tfrac{1}{3}(4 \pm \sqrt{7})};$$

$$n=6: \quad \tfrac{1}{2}\sqrt{\tfrac{1}{5}(17+\sqrt{489}\pm\sqrt{378+34\sqrt{489}})};$$

$$\sqrt{\tfrac{1}{10}(17\pm 3\sqrt{21})};$$

$$n=8: \quad \sqrt{\tfrac{1}{14}(15+\sqrt{113}\pm\sqrt{142+30\sqrt{113}})};$$

$$\sqrt{\tfrac{1}{14}(8+\sqrt{274}\pm\sqrt{142+16\sqrt{274}})}.$$

The values listed first are for symmetric configurations. The values listed second are for staggered configurations. For $n=2$ the symmetric configuration consists of four collinear vortices and is the same configuration encountered in (c) above for $N=4$. For $n=2,3$ the staggered configurations reduce to the square and the regular hexagon which we already considered in (a) above.

The two possibilities, symmetric and staggered, for relative equilibria are now neatly summarized by the generating polynomials

$$P_{\mathrm{symm}}(z)=(z^n-R_1^n)(z^n-R_2^n), \tag{27}$$

respectively,

$$P_{\mathrm{stag}}(z)=(z^n-R_1^n)(z^n+R_2^n), \tag{28}$$

where R_1 and R_2 denote the radii of the two rings. In addition to $\xi=R_1/R_2$ being a solution of Eqs. (25) and (26), respectively, the radii R_1 and R_2 must satisfy Eq. (3) or

$$R_1^2+R_2^2=2n-1. \tag{29}$$

For the symmetric case we have

$$P'_{\mathrm{symm}}(z)=2nz^{2n-1}-n(R_1^n+R_2^n)z^{n-1}$$
$$=nz^{n-1}(2z^n-R_1^n-R_2^n).$$

For vortices on the ring of radius R_1, $z_\alpha=R_1e^{i2\pi\alpha/n}$ and $P'_{\mathrm{symm}}(z_\alpha)=nR_1^{n-1}(R_1^n-R_2^n)e^{-i2\pi\alpha/n}$; for vortices on the ring of radius R_2 simply interchange subscripts. From Eq. (14),

$$\Theta_{\mathrm{symm}}=n^{2n}(R_1R_2)^{n(n-1)}(R_1^n-R_2^n)^{2n}$$
$$=\left(\frac{N}{2}\right)^N(R_1R_2)^{N(N-2)/4}(R_1^{N/2}-R_2^{N/2})^N;$$

$$\theta_{\mathrm{symm}}=\frac{2}{N-1}\log\frac{N}{2}+\frac{N-2}{N-1}\log\sqrt{R_1R_2}$$
$$+\frac{2}{N-1}\log|R_1^{N/2}-R_2^{N/2}|. \tag{30}$$

Here R_1 and R_2 must satisfy Eqs. (29) and (25) or

$$(n-1)R_1^{n+2}-(3n-1)R_1^nR_2^2-(3n-1)R_1^2R_2^n+(n-1)R_2^{n+2}=0. \tag{31}$$

As a check on these calculations note that if we formally set $R_2=-R_1=R$ for odd n, where from Eq. (29) $R^2=(N-1)/2$, Eq. (30) reproduces our previous result, Eq. (19).

For example, for $n=6$ we have from the results given previously

$$R_1=2\sqrt{\frac{55}{37+\sqrt{489}-\sqrt{378+34\sqrt{489}}}},$$

$$R_2=\tfrac{1}{2}\sqrt{22-\sqrt{22(-15+\sqrt{489})}}.$$

When these are substituted into Eq. (30) we obtain an algebraic expression for the energy. In practice, of course, evaluating the solutions R_1 and R_2 to high precision is undoubtedly preferable. However, calculating the energy from Eq. (30) is superior to calculating all pairwise separations and multiplying.

For the staggered case we have

$$P'_{\mathrm{stag}}(z)=2nz^{2n-1}+n(R_2^n-R_2^n)z^{n-1}=nz^{n-1}(2z^n+R_2^n-R_1^n).$$

For vortices on the ring of radius R_1, $z_\alpha=R_1e^{i2\pi\alpha/n}$, and $P'_{\mathrm{stag}}(z_\alpha)=nR_1^{n-1}(R_1^n+R_2^n)e^{-i2\pi\alpha/n}$. For vortices on the ring of radius R_2, $z_\alpha=R_2e^{i\pi(2\alpha+1)/n}$, $P'_{\mathrm{stag}}(z_\alpha)=nR_2^{n-1}(R_1^n+R_2^n)\mathrm{e}^{-i\pi(2\alpha+1)/n}$. From Eq. (14) we get

$$\Theta_{\mathrm{stag}}=n^{2n}(R_1R_2)^{n(n-1)}(R_1^n+R_2^n)^{2n}$$
$$=\left(\frac{N}{2}\right)^N(R_1R_2)^{N(N-2)/4}(R_1^{N/2}+R_2^{N/2})^N;$$

$$\theta_{\mathrm{stag}}=\frac{2}{N-1}\log\frac{N}{2}+\frac{N-2}{N-1}\log\sqrt{R_1R_2}$$
$$+\frac{2}{N-1}\log(R_1^{N/2}+R_2^{N/2}). \tag{32}$$

Here R_1 and R_2 must satisfy Eqs. (29) and (26) or

$$(n-1)R_1^{n+2}-(3n-1)R_1^nR_2^2+(3n-1)R_1^2R_2^n-(n-1)R_2^{n+2}=0. \tag{33}$$

Note that if we set $R_2=R_1=R$ (which is a solution to this equation), where from Eq. (29) $R^2=(N-1)/2$, Eq. (32) reproduces our previous result, Eq. (19).

For N odd, $N=2n+1$, we have relative equilibria with two nested, regular n-gons and a vortex at the center. Again, there is a symmetrical and a staggered arrangement. For two n-gons with a vortex at the center the ratio of the radii must solve

$$(n+1)\xi^{n+2}-(3n+1)\xi^n-(3n+1)\xi^2+n+1=0 \tag{34}$$

in the symmetric case, and

$$(n+1)\xi^{n+2}-(3n+1)\xi^n+(3n+1)\xi^2-(n+1)=0 \tag{35}$$

in the staggered case.

The generating polynomials now take the form

$$P_{\mathrm{symc}}(z)=z(z^n-R_1^n)(z^n-R_2^n) \tag{36}$$

for the symmetric case, and

$$P_{\mathrm{stagc}}(z)=z(z^n-R_1^n)(z^n+R_2^n), \tag{37}$$

for the staggered case, where R_1 and R_2 again denote the radii of the two rings. In addition to $\xi=R_1/R_2$ being a solution of Eq. (34) or Eq. (35), respectively, the radii R_1 and R_2 must satisfy Eq. (3) or

$$R_1^2 + R_2^2 = 2n + 1. \tag{38}$$

Calculations that parallel those performed for the "open" double rings now yield

$$\Theta_{\text{symc}} = n^{2n}(R_1R_2)^{n(n+1)}(R_1^n - R_2^n)^{2n} = \left(\frac{N-1}{2}\right)^{N-1}(R_1R_2)^{(N-1)(N+1)/4}(R_1^{(N-1)/2} - R_2^{(N-1)/2})^{N-1}; \tag{39}$$

$$\theta_{\text{symc}} = \frac{2}{N}\log\frac{N-1}{2} + \frac{N+1}{N}\log\sqrt{R_1R_2} + \frac{2}{N}\log|R_1^{(N-1)/2} - R_2^{(N-1)/2}|.$$

As a verification one can set $R_1 = -R_2$ for n odd and reproduce Eq. (22).

Similarly, for the staggered case we obtain

$$\Theta_{\text{stagc}} = n^{2n}(R_1R_2)^{n(n+1)}(R_1^n + R_2^n)^{2n} = \left(\frac{N-1}{2}\right)^{N-1}(R_1R_2)^{(N-1)(N+1)/4}(R_1^{(N-1)/2} + R_2^{(N-1)/2})^{N-1}. \tag{40}$$

Setting $R_1 = R_2$ clearly reproduces (22) as it should.

E. Triple rings

There are, presumably, further relative equilibria consisting of several nested, regular polygons. So far a complete analysis has only been done for three nested, regular polygons and a tabulation of the possibilities is available.[4] As in the case of two polygons, it is only possible to nest three regular polygons if they all have the same number of vertices. When N is divisible by 3, i.e., $N=3n$, we thus get configurations of three nested, regular n-gons. These configurations are again "analytically known" up to a determination of the three radii as solutions to a set of polynomial equations. There are three possibilities. First, all three polygons may be situated symmetrically. We call this the symmetric case, and for arbitrary n there is just one such solution. Second, two of the polygons may be situated symmetrically and the third rotated by π/n with respect to these two. We call this the staggered case. One would expect at least three such configurations in general depending on whether the radius of the staggered polygon is smaller than, between or larger than the other two. It turns out that the equations allow up to five staggered configurations. Third, there are solutions, referred to in Ref. 4 as the degenerate cases, where two of the regular n-gons have the same radius but the $2n$ vortices that are all at the same distance from the center of the configuration do not form a regular $2n$-gon.

The number of distinct vortex triple ring configurations increases monotonically with N. For any $N=3n$ there is always just one symmetric arrangement of the three nested, regular n-gons. The number of staggered arrangements, however, is 1 for $N=6$, 9, 12, and 15, then 3 for $N=18, 21, 24$, and increases to 5 when N equals 27 (i.e., for nine or more vortices per polygon). It remains at 5 for larger N. There are no "degenerate" configurations for $N=6$ but there are two such configurations for $N=9$ or larger.

When N is of the form $3n+1$, there are corresponding configurations of three nested, regular n-gons with a vortex at the center. These have also been analyzed and tabulated.[4] There is always just one symmetric pattern. There is also just one staggered pattern for $N=7, 10, 13, 16, 19$, three staggered configurations for $N=22$ and 25, and five for $N=28$ or larger. There are no "degenerate" centered configurations of $N=7$ or 10, but there are two for $N=13$ or larger.

We simply quote the main results from Ref. 4: Let $N=3n$ and let R_1, R_2, R_3 be the radii of the three nested n-gons. In all cases these radii must obey

$$R_1^2 + R_2^2 + R_3^2 = \frac{9n-3}{2}. \tag{41}$$

In the symmetric case they must further satisfy

$$(2R_1^2 - 5n + 1)R_1^n + (2R_2^2 - 5n + 1)R_2^n + (2R_3^2 - 5n + 1)R_3^n = 0, \tag{42}$$

and

$$\frac{2R_1^2 - n + 1}{R_1^n} + \frac{2R_2^2 - n + 1}{R_2^n} + \frac{2R_3^2 - n + 1}{R_3^n} = 0. \tag{43}$$

The generating polynomial is then

$$P_{3,\text{symm}}(z) = (z^n - R_1^n)(z^n - R_2^n)(z^n - R_3^n). \tag{44}$$

In the staggered case, assuming the numbering chosen so that ring 3 is the one turned half a turn relative to rings 1 and 2, the radii must satisfy

$$(2R_1^2 - 5n + 1)R_1^n + (2R_2^2 - 5n + 1)R_2^n - (2R_3^2 - 5n + 1)R_3^n = 0, \tag{45}$$

and

$$\frac{2R_1^2 - n + 1}{R_1^n} + \frac{2R_2^2 - n + 1}{R_2^n} - \frac{2R_3^2 - n + 1}{R_3^n} = 0, \tag{46}$$

and the generating polynomial is

$$P_{3,\text{stag}}(z) = (z^n - R_1^n)(z^n - R_2^n)(z^n + R_3^n). \tag{47}$$

The formulas for the energy are straightforward generalizations of Eqs. (30) and (32), respectively,

$$\Theta_{3,\text{symm}} = n^{3n}(R_1R_2R_3)^{n(n-1)}[(R_1^n - R_2^n)(R_2^n - R_3^n)(R_3^n - R_1^n)]^{2n}; \tag{48}$$

$$\Theta_{3,\text{stag}} = n^{3n}(R_1R_2R_3)^{n(n-1)}[(R_1^n - R_2^n)(R_2^n + R_3^n)(R_3^n + R_1^n)]^{2n}. \tag{49}$$

For the "degenerate" cases discussed in Ref. 4 the $3n$ vortices are situated on three rings, two of which have the same radius. The two rings of the same radius are rotated relative to the third ring by angles $\pm\phi$ given by

$$\cos n\phi = -\frac{(2R_1^2 - n + 1)R_1^n}{2(2R^2 - n + 1)R^n}. \tag{50}$$

Here R is the common radius of the two rings and R_1 is the radius of the third ring. These radii are determined by solving the polynomial equations

$$2R_1^2 + 4R^2 = 9n - 3, \tag{51}$$

$$(2R_1^2 - 5n + 1)(2R^2 - n + 1)R_1^{2n} = (2R^2 - 5n + 1)(2R_1^2 - n + 1)R^{2n}. \tag{52}$$

The generating polynomial for the configuration is

$$P_{3,\mathrm{deg}}(z) = (z^n - R_1^n)(z^n - (Re^{i\phi})^n)(z^n - (Re^{-i\phi})^n) = (z^n - R_1^n)(z^{2n} - 2z^n R^n \cos n\phi + R^{2n}). \tag{53}$$

The energy of the configuration follows from

$$\Theta_{3,\mathrm{deg}} = 2^{2n} n^{3n} R_1^{n(n-1)} R^{2n(2n-1)} \times (R_1^{2n} - 2R_1^n R^n \cos n\phi + R^{2n})^{2n} (\sin n\phi)^{2n}, \tag{54}$$

where R_1 and R result from solving Eqs. (51) and (52), and $\cos n\phi$, $\sin n\phi$ follow from Eq. (50).

The reader may easily establish the necessary formulas for the centered triple rings using the results in Ref. 4 and the results obtained in this paper [cf. Eq. (61)].

F. General formula for nested, regular polygon equilibria

The pattern seen above in the single, double, and triple ring formulas may, of course, be extended to an arbitrary number of nested, regular polygons, assuming such equilibria exist. Thus, in the general case of s nested regular n-gons we would posit a generating polynomial of the form

$$P(z) = (z^n - \gamma_1^n) \cdots (z^n - \gamma_s^n), \tag{55}$$

where $\gamma_p = R_p e^{i\phi_p}$, $p = 1, \ldots, s$, embodies both the radius, R_p, of the circle containing the vortices on the pth polygon and the angle, ϕ_p, through which this polygon is turned relative to the real axis. The vortex positions are $\gamma_p \epsilon^\alpha$, where $\epsilon = e^{i2\pi/n}$, $\alpha = 0, \ldots, n-1$ for $p = 1, \ldots, s$.

The derivative of $P(z)$ as given by Eq. (55) is

$$P'(z) = nz^{n-1} P(z) \sum_{q=1}^{s} \frac{1}{z^n - \gamma_q^n}. \tag{56}$$

Thus,

$$P'(\gamma_p \epsilon^\alpha) = n\gamma_p^{n-1} \epsilon^{-\alpha} \prod_{q=1}^{s}{}' (\gamma_p^n - \gamma_q^n), \tag{57}$$

where the prime on the product means $q \neq p$. Then

$$\Theta = \left| \prod_{p=1}^{s} \prod_{\alpha=1}^{n} P'(\gamma_p \epsilon^\alpha) \right| = n^{sn} (R_1 \cdots R_s)^{n(n-1)} \prod_{p,q=1}^{s}{}' |R_p^n - R_q^n e^{i(\phi_q - \phi_p)n}|^n = n^{sn} (R_1 \cdots R_s)^{n(n-1)} \times \prod_{1 \le p < q \le s} \{R_p^{2n} - 2R_p^n R_q^n \cos[n(\phi_p - \phi_q)] + R_q^{2n}\}^n. \tag{58}$$

The preceding results, most transparently formulas such as Eqs. (48) and (49), are specializations of this result to the case when all the phase differences $\phi_p - \phi_q$ are 0 or π/n. For nontrivial rotation angles of one or more polygons relative to the real axis, further transformation of the formulas are possible as we see in Eq. (54).

The centered case follows a similar recipe. The generating polynomial is now of the form $P(z) = zQ(z)$, where $Q(z)$ is of the form (55). The vortex positions are $\gamma_p \epsilon^\alpha$, as before, but now also $z = 0$. The derivative of $P(z)$ is

$$P'(z) = Q(z) + nz^n Q(z) \sum_{q=1}^{s} \frac{1}{z^n - \gamma_q^n}. \tag{59}$$

Thus, $P'(0) = Q(0)$, and the calculation of P' for any of the other vortices proceeds as before (except for the power of z in the prefactor now being n rather than $n-1$). Since

$$Q(0) = (-1)^s (\gamma_1 \cdots \gamma_s)^n, \tag{60}$$

we get

$$\Theta_c = n^{sn} (R_1 \cdots R_s)^{n(n+1)} \times \prod_{1 \le p < q \le s} \{R_p^{2n} - 2R_p^n R_q^n \cos[n(\phi_p - \phi_q)] + R_q^{2n}\}^n. \tag{61}$$

We have not developed a formula for the case of nested regular polygons with different numbers of vertices since we are not aware of the existence of relative equilibria of this type (although there is at present no proof that they do not exist).

IV. CONCLUSION

A formula for the energy of a relative equilibrium of identical point vortices has been derived. The formula allows the energy of the equilibrium configuration to be calculated in a simpler way than by multiplying all mutual vortex separations.

Several extensions and generalizations suggest themselves. An analogous formula would hold if the pattern-forming particles have an energy of interaction that varies with their separation by other than the first power. Extensions of the formula hold for relative equilibria of vortices of different strengths. As a particularly interesting example consider the stationary equilibria with vortices of the same absolute strength but opposite sense. These were shown by Bartman[9] to arise as zeros of successive Adler-Moser polynomials, cf. Ref. 6. Thus, if $P(z)$ is the polynomial with roots at the positions of the positive vortices, $z_1, z_2, \ldots, z_n$ say, and if $Q(z)$ is the polynomial with roots at the positions of the negative vortices, $\zeta_1, \zeta_2, \ldots, \zeta_m$, then P and Q are related by a differential equation due to Tkachenko,[1,6]

$$P''Q - 2P'Q' + PQ'' = 0. \tag{62}$$

The relation to the Adler-Moser polynomials then shows that n and m must be successive triangular numbers, and this may also be shown directly.[1]

The Hamiltonian of this system is, in essence, the logarithm of

$$\begin{aligned}&\prod_{\alpha,\beta=1}^{n}{}'|z_\alpha - z_\beta| \prod_{\lambda,\mu=1}^{m}{}'|\zeta_\lambda - \zeta_\mu| \prod_{\alpha=1}^{n}\prod_{\lambda=1}^{m}|z_\alpha - \zeta_\lambda|^{-1} \\ &= \left|\prod_{\alpha=1}^{n} P'(z_\alpha)\prod_{\lambda=1}^{m} Q'(\zeta_\lambda)\right| \left|\prod_{\lambda=1}^{m} P(\zeta_\lambda)\prod_{\alpha=1}^{n} Q(z_\alpha)\right|^{-1/2}.\end{aligned} \tag{63}$$

The second form follows from an obvious extension of our previous analysis (and we have chosen to "symmetrize" the denominator). In this expression Q and P are successive Adler-Moser polynomials.

There are other formulas for the quantity Θ that may be mentioned. The Vandermonde matrix,

$$\mathbf{V} = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ z_1 & z_2 & \ldots & z_N \\ \ldots & \ldots & \ldots & \ldots \\ z_1^{N-1} & z_2^{N-1} & \ldots & z_N^{N-1} \end{bmatrix}, \tag{64}$$

has determinant

$$\det \mathbf{V} = \prod_{1 \leqslant \alpha < \beta \leqslant N} (z_\beta - z_\alpha). \tag{65}$$

Hence,

$$\Theta = |\det \mathbf{V}|^2 = \left|\begin{bmatrix} 1 & 1 & \ldots & 1 \\ z_1 & z_2 & \ldots & z_N \\ \ldots & \ldots & \ldots & \ldots \\ z_1^{N-1} & z_2^{N-1} & \ldots & z_N^{N-1} \end{bmatrix}\right|^2. \tag{66}$$

While this formula also allows computation of the vortex pattern energy, it appears less convenient, and requires more steps of calculation, than the formula in terms of derivatives of the generating polynomial.

Finally we mention that it may be possible to establish general inequalities between the energies of the various nested polygon equilibria, just as we were able to establish the results about the open and centered regular polygons, although the analysis for the multiple-ring equilibria promises to be considerably more complicated. We are currently exploring these issues.

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