Actes, Congrès intern. math., 1970. Tome 1, p. 367 à 369.

A CLASS OF NON-SOLVABLE FINITE GROUPS

by Zvonimir JANKO

Let X be a finite group. If P is a p-subgroup of X different from identity (p is a prime), then the normalizer $N_X(P)$ of P in X is called a p-local subgroup of X.

A subgroup Y of X is called a local subgroup of X if Y is a p-local subgroup of X for some prime p.

The purpose of this work is to determine the structure of every non-solvable finite group G which has the following property:

(S) *Each 2-local subgroup H of G is solvable and all odd order Sylow subgroups of H are cyclic.*

We remark that John G. Thompson has considered in Section 15 of the N-groups paper all non-solvable finite groups X with the property that every local subgroup of X is solvable and every 2-local subgroup of X has cyclic Sylow p-subgroups for all odd primes p. Therefore this work can be considered as a generalization of the Thompson's (unpublished) work. Also the occurrence of the Tits simple group of order $2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$ which is an N-group makes the above problem very complicated. Here an N-group is a non-solvable finite group all of whose local subgroups are solvable.

The only known non abelian finite simple groups with the property (S) are: $L_2(r)$, $r > 3$, $L_3(3)$, M_{11}, $U_3(3)$, $Sz(q)$, $U_3(q)$ where $q = 2^n \geq 4$ and the Tits simple group. We shall call these groups SK-groups,

Let G be a non-abelian finite simple group of the smallest possible order which has the property (S) but which is not isomorphic to any SK-group. We have proved so far that the group G has the following properties:

(1) *Let T be a fixed Sylow 2-subgroup of G. Then T possesses a normal elementary abelian subgroup of order ≥ 8. Also T does not normalize any non-identity odd order subgroup of G.* Here we have used a joint result with J. G. Thompson and also some unpublished results of D. Gorenstein and J. H. Walter about centralizers of involutions.

(2) *A fixed Sylow 2-subgroup T of G is contained in at least two distinct maximal 2-local subgroups of G.* This result in particular rules out the possibility that T is a maximal subgroup of G. Also the chances to determine the structure of T are therefore increased.

(3) *Let H be any 2-local subgroup of G. Then the maximal normal odd order subgroup $O(H)$ of H is equal* 1. This is an immediate consequence of (1) and a result of D. Gorenstein about simple groups of characteristic 2 type.

(4) *The group G does not have a maximal 2-local subgroup M with the following properties*: (*i*) *Every normal abelian 2-subgroup of M is generated by at most two elements.* (*ii*) *M possesses a non-cyclic normal abelian 2-subgroup A such that* $C_G(a) \subseteq M$ *for every involution a of A.* The proof of this result is a straightforward adaptation of the corresponding result in Section 13 of the N-groups paper of J. G. Thompson.

(5) *The group G does not have a maximal 2-local subgroup M such that the maximal normal 2-subgroup* $O_2(M)$ *of M is of symplectic type.* Here a 2-group X is of symplectic type if X is non-cyclic and every characteristic abelian subgroup of X is cyclic. The proof of this result is also a straightforward adaptation of the proof of the corresponding results in Section 13 of the N-groups paper of J. G. Thompson.

(6) *Let T be a fixed Sylow 2-subgroup of G. Let* M_1 *and* M_2 *be two distinct maximal 2-local subgroups of G which contain T. Then we have* $M_1 \cap M_2 = T$.

This difficult result is proved in the following way. Assume that the result (6) is false. Then it is shown at first that G possesses one and only one maximal 2-local subgroup M containing T such that the order of M is divisible by a prime $p \geq 7$. Also $N_G(T) \subseteq M$. Let N be a maximal 2-local subgroup of G containing T which is different from M. Then we have $M \cap N = T \cdot D$, where $D \neq 1$, D is a cyclic odd order subgroup and T is normal in $T \cdot D$. Let E be a Hall 2′-subgroup of M containing D and let F be a Hall 2′-subgroup of N containing D. Then it is shown that E is a Frobenius group with kernel E' and a complement D and F is cyclic. We have $|F/D| = 3$ or 5 or 15. Also D is a Hall subgroup of F. The following result is crucial. For every subgroup D_0 of prime order p of D, D_0 centralizes a four subgroup of T, $C_G(D_0)$ is non-solvable and a Sylow p-subgroup of G is non-cyclic. By the methods of Section 13 of the N-groups paper of J. G. Thompson it is shown that $\Omega_1(Z(T))$ has order ≤ 4 and that $T_0 = \Omega_1(Z(T))$ is a normal subgroup of M. This last result is very strong and leads quickly to a contradiction. It is shown that D possesses a subgroup P of prime order p such that P centralizes T_0. It follows that the order of D is in fact equal p and that $p = 5$ or 7. Also $T_0 = \langle z \rangle$ has order 2, M has no normal four subgroups and $M = C_G(z)$. We have that $T_1 = C_T(D)$ is a Sylow 2-subgroup of $N_G(D) = C_G(D)$ and T_1 is either a dihedral group of order 8 or T_1 is a direct product of a group of order 2 and a dihedral group of order 8. If $p = 5$, then $C_G(D) = D \times L$ where L is isomorphic to A_6, S_5 or S_6 and if $p = 7$, then $C_G(D) = D \times L$ with $L \simeq L_2(7)$. Finally, we also get that $|F/D| = 3$. In all these results the minimality of $|G|$ is used several times. After that we show that $N_G(T) = M$. There are normal elementary abelian 2-subgroups of M of order ≥ 8. Let F be one of these of the smallest possible order. Then for every subgroup F_0 of index 2 of F we show that $C_G(F_0) \subseteq M$. After that a standard consideration of the weak closure of F in T yields a contradiction.

(7) *A Sylow 2-subgroup T of G is self-normalizing in G.* In the proof of this result the minimality of $|G|$ is used together with a result of C. Sims about primitive permutation groups.

(8) *Let M be any maximal 2-local subgroup of G containing the fixed Sylow 2-subgroup T of G. Then* $T = TU$ *where the subgroup U has order* 3 *or* 5 *or* 15. This result is proved by using the methods of Section 13 of the N-groups paper of J. G. Thompson.

As a direct consequence of this result we also get that every maximal 2-local subgroup of G has order $2^a k$ where $a > 0$ and $k = 3$ or 5 or 15.

As a conclusion we may say that the last result heavily restricts the structure of 2-local subgroups of G and so it is hoped that the final contradiction will be reached showing that there is no counter-example G. This will then show that every non-abelian finite simple group with the property (S) is an *SK-group.*

It is also easily seen that G is an N-group.

In the future work the following characterization of the Tits simple group given by D. Parrott (unpublished) will be very useful.

Let X be a finite simple group which possesses an involution z such that the centralizer H of z in X has the following properties:

(*i*) $O_2(H)$ *has order* 2^9 *and class at least* 3.

(*ii*) $H/O_2(H)$ *is a Frobenius group of order* 10 *or* 20.

(*iii*) *If P is a subgroup of order* 5 *of H, then the centralizer of P in $O_2(H)$ is contained in* $Z(O_2(H))$.

Then $H/O_2(H)$ *has order* 20 *and X is isomorphic to the Tits simple group.*

The Ohio State University
Department of Mathematics,
Columbus, Ohio 43210
(U. S. A.)