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UNIFORM POLYHEDRA

By

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UNIFORM POLYHEDRA

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(*Received* 25 *June* 1953)

[Plates 1 to 6]

CONTENTS

Uniform polyhedra have regular faces meeting in the same manner at every vertex. Besides the five Platonic solids, the thirteen Archimedean solids, the four regular star-polyhedra of Kepler (1619) and Poinsot (1810), and the infinite families of prisms and antiprisms, there are at least fifty-three others, forty-one of which were discovered by Badoureau (1881) and Pitsch (1881). The remaining twelve were discovered by two of the present authors (H.S.M.C. and J.C.P.M.) between 1930 and 1932, but publication was postponed in the hope of obtaining a proof that there are no more. Independently, between 1942 and 1944, the third author (M.S.L.-H.) in collaboration with H. C. Longuet-Higgins, rediscovered eleven of the twelve.

We now believe that further delay is pointless; we have temporarily abandoned our hope of obtaining a proof that our enumeration is complete, but we shall be much surprised if any new uniform polyhedron is found in the future. We have classified the known figures with the aid of a systematic notation and we publish drawings (by J.C.P.M.) and photographs of models (by M.S.L.-H.) which include all those not previously constructed.

One remarkable new polyhedron is contained in the present list, having eight edges at a vertex. This is the only one which cannot be derived immediately from a spherical triangle by Wythoff's construction.

1. Introduction

A *polyhedron* is a finite set of polygons such that every side of each belongs to just one other, with the restriction that no subset has the same property. The polygons and their sides are called faces and edges. The faces are not restricted to be convex, and may surround their centres more than once (as, for example, the pentagram, or five-sided star polygon, which surrounds its centre twice). Similarly, the faces at a vertex of the polyhedron may surround the vertex more than once.

A polyhedron is said to be *uniform* if its faces are regular while its vertices are all alike. By this we mean that one vertex can be transformed into any other by a symmetry operation.

A uniform polyhedron whose faces are all alike is said to be *regular*. Four of the five convex regular polyhedra were known to the ancient Egyptians: the tetrahedron, octahedron and cube occur in their architecture, and they seem to have played with icosahedral dice (according to an exhibit in the British Museum). The Etruscans made a dodecahedron before 500 B.C. (Heath 1921, p. 160). These five figures are generally known as the *Platonic solids*, although they were all studied by the early Pythagoreans, if not by Pythagoras himself.

Plato is said to have known one of the uniform polyhedra with faces of two kinds: the cuboctahedron. This and twelve others are more usually ascribed to Archimedes, though his book on them is lost. Five of these thirteen solids were rediscovered by Piero della Francesca (1416–1492), whose manuscript *Libellus de quinque corporibus regularibus* is in the Vatican. This treatise was translated into Italian by Fra Luca Pacioli (1509, pp. 259–266), who added an icosihexahedron (now known as the rhombicuboctahedron). A glass model of this last solid was exquisitely painted by Jacopo de' Barbari in his portrait of Pacioli, which can be seen in the Museo Nazionale in Naples.

The earliest complete enumeration of convex uniform polyhedra was made by Kepler (1619, pp. 116–128), who observed that the definition includes also the *prisms* with square side-faces and the *antiprisms* with equilateral triangular side-faces. For a simple account of all these uniform solids see Ball (1939, pp. 129, 135–140) or Thompson (1925).

Two new uniform polyhedra were discovered by Hess (1878), and many more were enumerated by Badoureau (1881) and Pitsch (1881) working independently in France and Austria. (Badoureau found thirty-seven and Pitsch eighteen.)

Between 1930 and 1932 two of the present authors (H. S. M. C. and J. C. P. M.), by a fairly systematic enumeration, discovered twelve other uniform polyhedra. Publication was, however, postponed, in the hope of obtaining a proof that there are no more. Independently, between 1942 and 1944, the third author (M. S. L.-H.) became interested in the subject through H. C. Longuet-Higgins, who had rediscovered many of the uniform polyhedra, including two not previously published. By essentially the same methods as the other two authors, the third author enumerated all but one of the remaining twelve*; the twelfth, an exceptional case, is that described in § 11 of the present paper. Publication was likewise postponed, and the authors did not learn of one another's work until 1952. In the meantime, five of the twelve were rediscovered by Lesavre & Mercier (1947), who computed their circum-radii but did not publish any drawings.

The authors' enumeration of uniform polyhedra is based on a systematic application of Wythoff's construction to all possible Schwarz triangles (see §§ 3 and 4). All but one of the polyhedra, namely, the one just mentioned, can be so derived, and it is the authors' belief that the enumeration is complete, although a rigorous proof has still to be given.

The vertices of a uniform polyhedron all lie on a sphere whose centre is their centroid (Coxeter 1948, p. 44). Those vertices which are joined to any one vertex lie also on a sphere

* Models of these polyhedra are to be found in the Winchester College Museum.

around this vertex, and therefore lie in a plane. The faces that come together at this vertex form a solid angle whose section by the plane is a polygon called the *vertex figure*, which is regular whenever the polyhedron is regular.

We shall find it convenient to use the symbol $\{p\}$ for the regular p-gon, and $\{p, q\}$ for the regular polyhedron, whose face and vertex figure are $\{p\}$ and $\{q\}$; e.g. the cube is $\{4, 3\}$. This notation is due to Schläfli (1852, p. 213). Strictly, the numbers p and q should satisfy the inequalities

$$p>2, \quad q>2, \quad (p-2)(q-2)<4$$

(Coxeter 1948, p. 5); but for some purposes it is desirable to admit the *dihedron* $\{p, 2\}$, whose faces are two coincident $\{p\}$'s, and the polar polyhedron $\{2, q\}$, whose faces are coincident digons $\{2\}$ corresponding to spherical lunes of angle $2\pi/q$.

Table 1 contains a list of the Platonic and Archimedean solids, with Schläfli symbols for the former and a convenient extension for the latter (Coxeter 1940, p. 394). The names are the customary anglicized version of those used by Kepler (1619, pp. 123–126). Some authors have preferred to call $t\begin{Bmatrix}3\\4\end{Bmatrix}$ the 'great rhombicuboctahedron' because the actual truncation of $\begin{Bmatrix}3\\4\end{Bmatrix}$ has some rectangular faces which need to be distorted into squares. The symbol $t\{2, q\}$ for the prism is appropriate since $t\{p, q\}$ has, at each vertex, one $\{q\}$ and two $\{2p\}$'s. The symbol $s\begin{Bmatrix}2\\q\end{Bmatrix}$ for the antiprism is a little more questionable, as strict analogy with $s\begin{Bmatrix}p\\q\end{Bmatrix}$ would require the recognition of q digonal faces; but we naturally regard these as collapsing to form single edges.

TABLE 1. CONVEX UNIFORM POLYHEDRA

tetrahedron	$\{3, 3\}$	truncated tetrahedron	$t\{3, 3\}$
octahedron	$\{3, 4\}$	truncated octahedron	$t\{3, 4\}$
cube	$\{4, 3\}$	truncated cube	$t\{4, 3\}$
icosahedron	$\{3, 5\}$	truncated icosahedron	$t\{3, 5\}$
dodecahedron	$\{5, 3\}$	truncated dodecahedron	$t\{5, 3\}$
cuboctahedron	$\begin{Bmatrix}3\\4\end{Bmatrix}$	truncated cuboctahedron	$t\begin{Bmatrix}3\\4\end{Bmatrix}$
icosidodecahedron	$\begin{Bmatrix}3\\5\end{Bmatrix}$	truncated icosidodecahedron	$t\begin{Bmatrix}3\\5\end{Bmatrix}$
rhombicuboctahedron	$r\begin{Bmatrix}3\\4\end{Bmatrix}$	snub cube	$s\begin{Bmatrix}3\\4\end{Bmatrix}$
rhombicosidodecahedron	$r\begin{Bmatrix}3\\5\end{Bmatrix}$	snub dodecahedron	$s\begin{Bmatrix}3\\5\end{Bmatrix}$
q-gonal prism	$t\{2, q\}$	q-gonal antiprism	$s\begin{Bmatrix}2\\q\end{Bmatrix}$

The regular $\{p, q\}$ has N_0 vertices, N_1 edges and N_2 faces, where

$$N_0 = \frac{4p}{4-(p-2)(q-2)}, \qquad N_1 = \frac{2pq}{4-(p-2)(q-2)}, \qquad N_2 = \frac{4q}{4-(p-2)(q-2)}$$

(Coxeter 1948, p. 13). In terms of these, the numerical properties of the Archimedean solids may be summarized as in table 2 (see also the beginning of table 7).

TABLE 2. NUMERICAL PROPERTIES

polyhedron		vertices	edges	faces
$\{p, q\}$		N_0	N_1	$N_2\{p\}$
$t\{p, q\}$		$2N_1$	$3N_1$	$N_0\{q\}+N_2\{2p\}$
$\begin{Bmatrix}p\\q\end{Bmatrix}$		N_1	$2N_1$	$N_0\{q\}+N_2\{p\}$
$t\begin{Bmatrix}p\\q\end{Bmatrix}$		$4N_1$	$6N_1$	$N_0\{2q\}+N_1\{4\}+N_2\{2p\}$
$r\begin{Bmatrix}p\\q\end{Bmatrix}$		$2N_1$	$4N_1$	$N_0\{q\}+N_1\{4\}+N_2\{p\}$
$s\begin{Bmatrix}p\\q\end{Bmatrix}$	$(p, q>2)$	$2N_1$	$5N_1$	$N_0\{q\}+2N_1\{3\}+N_2\{p\}$
$s\begin{Bmatrix}2\\q\end{Bmatrix}$	$(N_1=q)$	$2N_1$	$4N_1$	$2\{q\}+2N_1\{3\}$

It is sometimes desirable to modify the definition of the *vertex figure* so as to give it a definite size, independent of the size of the polyhedron. The simplest way to do this is to regard the vertices of the vertex figure as lying at *unit distance* from one vertex of the polyhedron along all the edges meeting at this vertex. Then every uniform polyhedron is characterized by its vertex figure, which is a cyclic polygon having a side $2\cos\pi/p$ for each $\{p\}$ at a vertex of the polyhedron. Thus the vertex figure of $\{p, q\}$ is a $\{q\}$ of side $2\cos\pi/p$; that of $t\{p, q\}$ is an isosceles triangle having one side $2\cos\pi/q$ and two sides $2\cos\pi/2p$; that of $t\begin{Bmatrix}p\\q\end{Bmatrix}$ is a scalene triangle of sides

$$2\cos\frac{\pi}{2q}, \qquad 2\cos\frac{\pi}{4}=\surd 2, \qquad 2\cos\frac{\pi}{2p};$$

that of $\begin{Bmatrix}p\\q\end{Bmatrix}$ is a rectangle of sides $2\cos\pi/p$ and $2\cos\pi/q$; that of $r\begin{Bmatrix}p\\q\end{Bmatrix}$ is a trapezoid whose parallel sides are $2\cos\pi/p$ and $2\cos\pi/q$, while the others are both $\surd 2$; that of $s\begin{Bmatrix}2\\q\end{Bmatrix}$ is a trapezoid of sides $2\cos\pi/q$, 1, 1, 1; and that of $s\begin{Bmatrix}p\\q\end{Bmatrix}$ is a pentagon of sides

$$1, \quad 2\cos\frac{\pi}{p}, \quad 1, \quad 2\cos\frac{\pi}{q}, \quad 1.$$

It is therefore reasonable to regard $s\begin{Bmatrix}2\\3\end{Bmatrix}$ as an alternative symbol for the octahedron, and $s\begin{Bmatrix}3\\3\end{Bmatrix}$ for the icosahedron. Similarly, $\begin{Bmatrix}3\\3\end{Bmatrix}$ is another symbol for the octahedron, $t\{2, 4\}$ or $t\begin{Bmatrix}2\\2\end{Bmatrix}$ for the cube and $s\begin{Bmatrix}2\\2\end{Bmatrix}$ for the tetrahedron.

In our drawings of the vertex figures we shall find it convenient to mark their sides (the 'vertex figures' of the faces) with the values of p instead of the actual lengths $2\cos\pi/p$. These numbers p can be translated into lengths by means of table 3 (which includes some fractional values for use later on). Here, and elsewhere, we use the abbreviation

$$\tau = \tfrac{1}{2}(\surd 5+1).$$

TABLE 3. VERTEX FIGURES OF POLYGONS $\{p\}$

p	$2\cos\pi/p$	p	$2\cos\pi/p$
2	0	3	1
4	$\sqrt{2}$	6	$\sqrt{3}$
5	τ	$\frac{5}{2}$	τ^{-1}
8	$(2+\sqrt{2})^{\frac{1}{2}}$	$\frac{8}{3}$	$(2-\sqrt{2})^{\frac{1}{2}}$
10	$5^{\frac{1}{4}}\tau^{\frac{1}{2}}$	$\frac{10}{3}$	$5^{\frac{1}{4}}\tau^{-\frac{1}{2}}$
12	$(\sqrt{3}+1)/\sqrt{2}$	$\frac{12}{5}$	$(\sqrt{3}-1)/\sqrt{2}$

2. SPHERICAL TESSELLATIONS

By projecting the edges of a uniform polyhedron from its centre on to the concentric unit sphere, we obtain a network of arcs of great circles decomposing the surface into spherical polygons, one for each face of the polyhedron. We shall use the same symbols for such spherical tessellations as for the polyhedra themselves. Thus $\{p, q\}$ now means an arrangement of spherical p-gons, q coming together at each vertex. The tessellations $\{p, q\}$ and $\begin{Bmatrix} p \\ q \end{Bmatrix}$ were described by Abũ'l Wafã, a tenth-century Arab (see Woepcke 1855, pp. 352–357).

One advantage of shifting our attention from solids to spherical tessellations is that the symbol $\{p, q\}$ is just as significant when p or q takes the value 2 as when both are greater than 2. In fact, $\{2\}$ is a spherical digon or lune, and $\{2, q\}$ is an arrangement of q lunes formed by q great semicircles (meridians). Moreover, the faces of $\{p, 2\}$ are the northern and southern hemispheres, regarded as spherical p-gons whose vertices coincide with p points evenly distributed along the equator. The symbol $t\{2, q\}$ for the q-gonal prism is now fully justified; this figure has two vertices on each edge of $\{2, q\}$, just as the truncated cube has two vertices on each edge of the cube.

Let $(p\ q\ r)$ denote a spherical triangle whose angles are

$$\pi/p, \quad \pi/q, \quad \pi/r.$$

We know that every finite group generated by more than two reflexions is generated by reflexions in the sides of such a triangle where p, q, r are integers (Coxeter 1948, pp. 81–82). Since the area

$$(p^{-1}+q^{-1}+r^{-1}-1)\,\pi$$

must be positive, the only possibilities are

$$(2\ 2\ r), \quad (2\ 3\ 3), \quad (2\ 3\ 4), \quad (2\ 3\ 5).$$

We name these *Möbius triangles* because it was Möbius (1849, pp. 360, 661; see also Coxeter 1948, p. 66) who observed that the planes of symmetry of $\{p, q\}$ decompose the sphere into such a network of triangles $(2\ p\ q)$. Since the network contains four triangles for each edge of $\{p, q\}$, the total number of triangles (i.e. the order of the group) is

$$g = 4N_1 = 8pq/\{4-(p-2)\,(q-2)\}.$$

Another way to find this number is to divide 4π by the area of $(2\ p\ q)$.

The whole network is derived from any one of the triangles (the fundamental region) by the various operations of the group. Möbius illustrated this fact by means of his polyhedral kaleidoscope, consisting of three mirrors forming a trihedron whose dihedral angles are $\pi/2$, π/p, π/q. It is most convenient in practice to use mirrors cut into the shape of circular sectors of appropriate angles ϕ, χ, ψ (Coxeter 1948, pp. 24, 293). A small object, representing a point, placed within the trihedron, yields g 'images' (strictly, the object itself and $g-1$ images). When the object is placed on one of the mirrors, the images coincide in pairs, leaving only $\frac{1}{2}g$ points. The number is further reduced when the object is on an edge where two mirrors meet. In fact, as we shall soon see, the images are then the vertices of

$$\{p, q\} \quad \text{or} \quad \{q, p\} \quad \text{or} \quad \begin{Bmatrix} p \\ q \end{Bmatrix}$$

according as the angle at the edge is π/q or π/p or $\pi/2$.

This kaleidoscopic construction for polyhedra is appropriately ascribed to Wythoff (1918) because it was he who first successfully exploited it (in four dimensions).

3. Wythoff's construction

In terms of the spherical triangle (which the three mirrors of the polyhedral kaleidoscope cut out from a sphere drawn round their common point), we are considering the images of one vertex, say the vertex P of the triangle PQR $= (p\ q\ r)$, where the angles at P, Q, R are π/p, π/q, π/r. The polyhedron whose vertices are the images of P is conveniently denoted by

$$p \mid q\ r,$$

which is, of course, the same as $p \mid r\ q$. The vertex P is joined by an edge to its image by reflexion in the opposite side QR, and the other edges meeting at P are derived from this one by the mirrors PQ and PR.

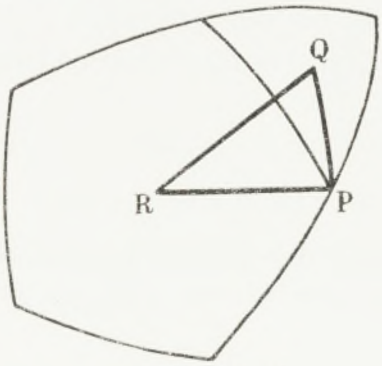


Figure 1

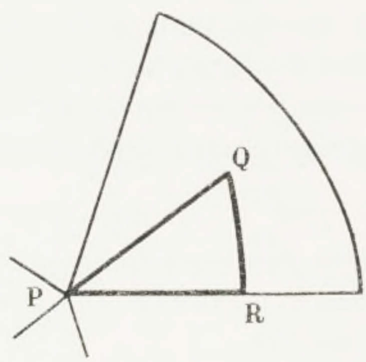


Figure 2

If q and r are greater than 2, as in figure 1, we find a face $\{q\}$ with centre Q, and a face $\{r\}$ with centre R; thus

$$2 \mid q\ r = \begin{Bmatrix} q \\ r \end{Bmatrix}.$$

But if $r = 2$, as in figure 2, the 'face' with centre R is a mere digon, which collapses to form an edge, and the polyhedron is regular:

$$p \mid q\ 2 = \{q, p\}.$$

Since an isosceles triangle $(p\ q\ q)$ is dissected by its symmetrical median into two right-angled triangles $(2p\ q\ 2)$, the polyhedron is again regular when $r = q$:

$$p \mid q\ q = 2p \mid q\ 2 = \{q, 2p\}.$$

Thus $\begin{Bmatrix} p \\ q \end{Bmatrix}$ is $2 \mid p\ q$; and $\{p, q\}$ is $q \mid p\ 2$ (or alternatively if q is even, $\frac{1}{2}q \mid p\ p$).

Instead of the images of a *vertex* of the triangle PQR or $(p\ q\ r)$, we may take the images of a point C on one of the sides, say on the side PQ opposite to the angle π/r. This vertex C of the new polyhedron is joined by edges to its images by reflexion in the sides PR and QR. (Any other edges meeting at C are images of these by reflexion in PQ.)

In order that these two edges may have the same length, the initial point C must be chosen on the bisector of the angle π/r at R. Then the mirrors PR and QR will yield a face $\{2r\}$ with centre R. There is also a face $\{p\}$ with centre P if $p>2$, and a face $\{q\}$ with centre Q if $q>2$. The polyhedron so constructed is denoted by

$$p\ q\,|\,r$$

or $q\ p\,|\,r$; thus

$$p\ q\,|\,2 = \mathrm{r}\begin{Bmatrix}p\\q\end{Bmatrix}, \qquad 2\ q\,|\,r = \mathrm{t}\{r, q\}$$

(see figures 3 and 4, respectively). Changing the notation again, we see that $\mathrm{t}\{p, q\}$ is $2\ q\,|\,p$.

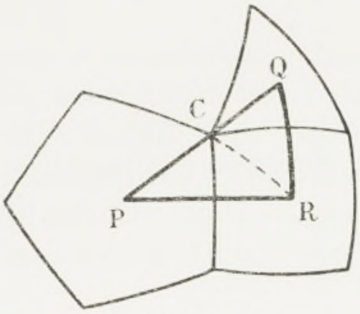


FIGURE 3

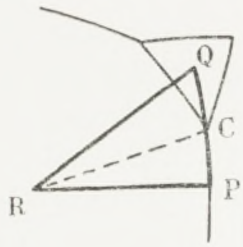


FIGURE 4

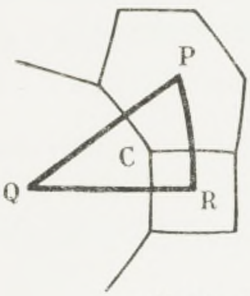


FIGURE 5

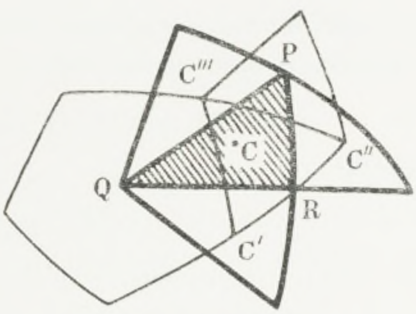


FIGURE 6

Another polyhedron is obtained by taking the images of an interior point of the triangle $(p\ q\ r)$. This point is joined by edges of the polyhedron to its images by reflexion in the three sides of the triangle. In order that these three edges may all have the same length, the initial point must be precisely the *in-centre* of the triangles (see figure 5). A suitable symbol is

$$p\ q\ r\,|,$$

with the understanding that the numbers p, q, r may be freely permuted. There are only three faces at each vertex: a $\{2p\}$ with centre P, a $\{2q\}$ with centre Q, and a $\{2r\}$ with centre R. Thus

$$2\ p\ q\,| = \mathrm{t}\begin{Bmatrix}p\\q\end{Bmatrix}.$$

The remaining Archimedean solids are obtained by taking, instead of the whole group generated by reflexions, the subgroup of index 2 consisting of rotations. In other words, we regard the spherical triangles $(p\ q\ r)$ as being alternately white and black, selecting as vertices of the polyhedron points in all the white triangles, so situated that the points in the three white triangles surrounding any black one form an equilaterial triangle (see figure 6). Thus the faces are $\{p\}$'s, $\{q\}$'s and $\{r\}$'s, each entirely surrounded by triangles; but if one of p, q, r is equal to 2, the consequent digon can be ignored and two of the triangles have a common side.

Making use of the one remaining position for the vertical stroke, we denote this 'snub' polyhedron by

$$|\, p\ q\ r$$

(with free permutation again). Thus

$$|\, 2\ p\ q = \mathrm{s}\begin{Bmatrix} p \\ q \end{Bmatrix}.$$

In particular, the q-gonal antiprism $\mathrm{s}\begin{Bmatrix} 2 \\ q \end{Bmatrix}$ is $|\, 2\ 2\ q$.

Such results are most easily visualized by referring to a drawing or model of the partition of a sphere into Möbius triangles. Suitable drawings are either stereographic projections (Klein 1884, p. 26; Coxeter 1938) or orthogonal projections (Ball 1939, p. 157; Coxeter 1948, p. 66). The ideal model would be a globe with great circles inscribed on it; but an easily made approximation is the polyhedron whose faces are plane triangles having the same vertices as the spherical triangles. If the sides of a spherical triangle are ϕ, χ, ψ, those of the corresponding plane triangle are proportional to

$$\sin \tfrac{1}{2}\phi : \sin \tfrac{1}{2}\chi : \sin \tfrac{1}{2}\psi.$$

Thus a model for the icosahedral family (to which all the most interesting figures belong) is the hexakisicosahedron formed by 120 congruent triangles whose sides are

$$\sin 15^\circ\, 52' : \sin 18^\circ\, 41' : \sin 10^\circ\, 27' = 2733 : 3204 : 1814 = 6 : 7 : 4$$

very nearly.

4. The Schwarz triangles

A very interesting extension of this theory is obtained by considering triangles $(p\ q\ r)$, where p, q, r are rational but not necessarily integral. The area

$$(p^{-1}+q^{-1}+r^{-1}-1)\,\pi$$

must still be positive, but this condition is no longer sufficient to ensure that repeated reflexions in the sides will yield a finite network, i.e. a network covering the sphere a finite number of times, say d times. Those triangles which do yield a finite network we shall call *Schwarz triangles* because it was Schwarz (1873, p. 243) who first listed them. It may be shown that the group generated by reflexions in the sides of a Schwarz triangle is equally well generated by reflexions in the sides of a certain Möbius triangle

$$(2\ 2\ r), \quad (2\ 3\ 3), \quad (2\ 3\ 4), \quad \text{or} \quad (2\ 3\ 5).$$

Accordingly, the Schwarz triangle may be classified as *dihedral*, *tetrahedral*, *octahedral*, or *icosahedral*.

Since the g operations of the group transform the Schwarz triangle into g replicas filling the surface of the sphere d times, and transform the Möbius triangle into the same number filling the surface once, it follows that the Schwarz triangle is covered by just d replicas of the Möbius triangle; e.g. $(2\ \frac{5}{2}\ 5)$ is covered by three $(2\ 3\ 5)$'s, and $(2\ 3\ \frac{5}{2})$ by seven (Coxeter 1948, p. 111). A corner of the Schwarz triangle where the angle is $m\pi/n$ must be filled with m replicas of a Möbius triangle having an angle π/n. Thus a given Schwarz triangle $(p\ q\ r)$ can be recognized as dihedral if two of p, q, r are equal to 2, and otherwise

tetrahedral ($g = 24$), octahedral ($g = 48$) or icosahedral ($g = 120$),

according as the largest numerator occurring is

3, 4 or 5.

To compute d, we merely have to divide $p^{-1}+q^{-1}+r^{-1}-1$ by the corresponding expression for the appropriate Möbius triangle, i.e. to multiply $p^{-1}+q^{-1}+r^{-1}-1$ by

$$\tfrac{1}{4}g = n \quad \text{or} \quad 6 \quad \text{or} \quad 12 \quad \text{or} \quad 30.$$

For instance, n and d for $(2\ 2\ r)$ are the numerator and denominator of the fraction r (or, if r is an integer, they are r and 1).

Schwarz triangles may also be classified into sets of four (or sometimes fewer) *colunar* triangles

$$(p\ q\ r), \quad (p\ q'\ r'), \quad (p'\ q\ r'), \quad (p'\ q'\ r),$$

where

$$p^{-1}+p'^{-1} = 1, \quad q^{-1}+q'^{-1} = 1, \quad r^{-1}+r'^{-1} = 1$$

(Coxeter 1948, p. 112). The sides of such a set of triangles are various arcs of the same three great circles.

There is evidently a Schwarz triangle $(2\ 2\ r)$ for every rational r greater than 1. Other Schwarz triangles are found by systematically dissecting the particular triangles

$$(2\ \tfrac{3}{2}\ \tfrac{3}{2}), \quad (2\ \tfrac{3}{2}\ \tfrac{4}{3}), \quad (2\ \tfrac{3}{2}\ \tfrac{5}{4}),$$

which are colunar to $(2\ 3\ 3)$, $(2\ 3\ 4)$, $(2\ 3\ 5)$ (cf. Coxeter 1948, p. 113, where these were mistakenly called 'the largest triangles of each family'). The results are listed in table 5 on p. 430. (In the second row, d may be any positive integer and n any greater, relatively prime, integer.)

This list agrees with Schwarz's; but he was content to give the smallest of each set of colunars. A simple way to verify its completeness is to consider first all possible triangles $(p\ q\ r)$, where p, q, r take the values

$$2, \quad 3, \quad \tfrac{3}{2}, \quad 4, \quad \tfrac{4}{3}, \quad 5, \quad \tfrac{5}{2}, \quad \tfrac{5}{3}, \quad \tfrac{5}{4},$$

with the restriction that numerators 4 and 5 cannot occur together (for, if they did, they would have to occur together in some Möbius triangle). Taking only the smallest triangle in each set of colunars, and remembering that any spherical triangles must satisfy

$$p^{-1}+q^{-1}+r^{-1} > 1,$$

we obtain all the triangles in Schwarz's own list and also

$$(3\ 3\ \tfrac{5}{3}), \quad (2\ \tfrac{5}{2}\ \tfrac{5}{2}), \quad (3\ \tfrac{5}{2}\ \tfrac{5}{2}), \quad (5\ \tfrac{5}{2}\ \tfrac{5}{2}), \quad (\tfrac{5}{3}\ \tfrac{5}{2}\ \tfrac{5}{2}), \quad (\tfrac{5}{3}\ \tfrac{5}{2}\ 5),$$

which can all be ruled out by the following considerations. The first five are isosceles triangles whose symmetrical medians dissect them into right-angled triangles

$$(2\;3\;\tfrac{10}{3}),\quad (2\;4\;\tfrac{5}{2}),\quad (2\;\tfrac{5}{2}\;6),\quad (2\;\tfrac{5}{2}\;10),\quad (2\;\tfrac{5}{2}\;\tfrac{10}{3}).$$

If repetitions of one of the isosceles triangles could cover the sphere a finite number of times, then repetitions of the corresponding right-angled triangle would do likewise. But these right-angled triangles are inadmissible (having numerators 4 and 5 together, or else one greater than 5); therefore the isosceles triangles are inadmissible too. Finally, $(\frac{5}{3}\;\frac{5}{2}\;5)$ is colunar to $(\frac{5}{4}\;\frac{5}{2}\;\frac{5}{2})$, and this splits into two triangles $(2\;\frac{5}{2}\;\frac{5}{2})$ which we have already seen to be inadmissible.

It has been assumed that p, q and r are all greater than 1, so that spherical triangles with reflex angles are excluded from the foregoing enumeration. This is because, for any reflex-angled triangle, there is a set of colunar triangles whose sides lie in the same great circles and all of whose angles are less than π; thus it can be seen that the admission of reflex-angled triangles will not give rise to any essentially new uniform polyhedra. However, it is sometimes suggestive, as in § 10, to associate a few of the polyhedra with certain reflex-angled triangles rather than with Schwarz triangles colunar to them.

5. Wythoff's construction generalized

The symbols $p\,|\,q\;r$, $p\;q\,|\,r$, $p\;q\;r\,|$ and $|\,p\;q\;r$, defined in § 3, extend in a natural manner to the situation where $(p\;q\;r)$ is a Schwarz triangle instead of a Möbius triangle. The analogy is closest when one of p, q, r is equal to 2. In particular,

$$5\,|\,2\;\tfrac{5}{2} = \{\tfrac{5}{2}, 5\} \quad\text{and}\quad 3\,|\,2\;\tfrac{5}{2} = \{\tfrac{5}{2}, 3\}$$

are the star-faced polyhedra of Kepler (1619, p. 122), which Cayley named the *small stellated dodecahedron* and the *great stellated dodecahedron*; and their reciprocals

$$\tfrac{5}{2}\,|\,2\;5 = \{5, \tfrac{5}{2}\} \quad\text{and}\quad \tfrac{5}{2}\,|\,2\;3 = \{3, \tfrac{5}{2}\}$$

are the star-cornered polyhedra of Poinsot (1810, pp. 39–42), namely, the *great dodecahedron* and the *great icosahedron*. The Schläfli symbol $\{p, q\}$ remains appropriate when $\{p\}$ is the face and $\{q\}$ the vertex figure, $\{\frac{5}{2}\}$ being the star pentagon or *pentagram*.

The analogues of the cuboctahedron $\begin{Bmatrix}3\\4\end{Bmatrix}$ and the icosidodecahedron $\begin{Bmatrix}3\\5\end{Bmatrix}$ are the *dodecadodecahedron* and the *great icosidodecahedron*:

$$2\,|\,\tfrac{5}{2}\;5 = \begin{Bmatrix}\frac{5}{2}\\5\end{Bmatrix} \quad\text{and}\quad 2\,|\,3\;\tfrac{5}{2} = \begin{Bmatrix}3\\\frac{5}{2}\end{Bmatrix}$$

(Hess 1878, p. 267; Pitsch 1881, p. 87 and Plate I).

As before, the faces of $\begin{Bmatrix}p\\q\end{Bmatrix}$ at one vertex are

$$\{p\},\quad \{q\},\quad \{p\},\quad \{q\},$$

so that the vertex figure is a rectangle of sides $2\cos\pi/p$, $2\cos\pi/q$.

There are also *prisms* such as

$$2\;\tfrac{8}{3}\,|\,2 = \mathrm{t}\{2, \tfrac{8}{3}\} \quad\text{and}\quad 2\;\tfrac{5}{2}\,|\,2 = \mathrm{t}\{2, \tfrac{5}{2}\}$$

(Badoureau 1881, Figs. 79 and 80); *antiprisms* such as

$$|\,2\;2\;\tfrac{7}{2}=\mathrm{s}\begin{Bmatrix}2\\ \frac{7}{2}\end{Bmatrix},\quad |\,2\;2\;\tfrac{7}{3}=\mathrm{s}\begin{Bmatrix}2\\ \frac{7}{3}\end{Bmatrix},\quad |\,2\;2\;\tfrac{7}{4}=\mathrm{s}'\begin{Bmatrix}2\\ \frac{7}{3}\end{Bmatrix},\quad |\,2\;2\;\tfrac{8}{3}=\mathrm{s}\begin{Bmatrix}2\\ \frac{8}{3}\end{Bmatrix},\quad |\,2\;2\;\tfrac{8}{5}=\mathrm{s}'\begin{Bmatrix}2\\ \frac{8}{3}\end{Bmatrix}$$

(Badoureau 1881, Figs. 81, 85, 82, 83, 84); *truncations*

$$2\;\tfrac{5}{2}\,|\,5=\mathrm{t}\{5,\tfrac{5}{2}\},\quad\text{and}\quad 2\;\tfrac{5}{2}\,|\,3=\mathrm{t}\{3,\tfrac{5}{2}\}$$

(nos. VI and XIX of Pitsch 1881, p. 86 and Plate II).

In § 3, $\mathrm{t}\{r, q\}$ was shown to be the polyhedron $2\;q\,|\,r$ whose vertex was the intersection of one side PQ of the spherical triangle $(2\;q\;r)$ with the internal bisector of the opposite angle at R. The *quasi-truncation* $\mathrm{t}'\{r, q\}$ may be defined as the polyhedron whose vertex is the intersection of PQ with the *external* bisector at R, i.e. the internal bisector for the triangle $(2\;q\;r')$; thus $\mathrm{t}'\{r, q\}$ is $2\;q\,|\,r'$. If r has an even denominator, a quasi-truncation may exist even when the corresponding truncation does not; if r has an even numerator, both kinds may exist together. Our list includes the following:

$$2\;3\,|\,\tfrac{4}{3}=\mathrm{t}'\{4,3\},\qquad 2\;5\,|\,\tfrac{5}{3}=\mathrm{t}'\{\tfrac{5}{2},5\},\qquad 2\;3\,|\,\tfrac{5}{3}=\mathrm{t}'\{\tfrac{5}{2},3\}$$

(nos. XIV, XVII, XX of Pitsch 1881, p. 86). Similarly $\mathrm{t}'\begin{Bmatrix}q\\ r\end{Bmatrix}$ may be defined as a non-degenerate polyhedron whose vertex is at an excentre of $(2\;q\;r)$, i.e. $\mathrm{t}'\begin{Bmatrix}q\\ r\end{Bmatrix}=2\;q'\;r\,|$ or $2\;q\;r'\,|$. We find in particular

$$2\;3\;\tfrac{4}{3}\,|=\mathrm{t}'\begin{Bmatrix}3\\ 4\end{Bmatrix},\quad 2\;\tfrac{5}{3}\;5\,|=\mathrm{t}'\begin{Bmatrix}\frac{5}{2}\\ 5\end{Bmatrix},\quad 2\;3\;\tfrac{5}{3}\,|=\mathrm{t}'\begin{Bmatrix}3\\ \frac{5}{2}\end{Bmatrix}$$

(nos. X, XII, XIII of Pitsch 1881, p. 86). Lastly, just as the *rhombicuboctahedron-analogue* $\mathrm{r}\begin{Bmatrix}p\\ q\end{Bmatrix}$ can be defined (see § 3) with reference to the internal bisector of the angle at R in the triangle $(p\;q\;2)$, so $\mathrm{r}'\begin{Bmatrix}p\\ q\end{Bmatrix}$ can be defined with reference to the external bisector; thus $\mathrm{r}'\begin{Bmatrix}p\\ q\end{Bmatrix}=p'\;q\,|\,2$ or $p\;q'\,|\,2$. In particular we have

$$\tfrac{3}{2}\;4\,|\,2=\mathrm{r}'\begin{Bmatrix}3\\ 4\end{Bmatrix},\qquad \tfrac{5}{2}\;5\,|\,2=\mathrm{r}\begin{Bmatrix}\frac{5}{2}\\ 5\end{Bmatrix},\qquad 3\;\tfrac{5}{3}\,|\,2=\mathrm{r}'\begin{Bmatrix}3\\ \frac{5}{2}\end{Bmatrix}$$

(Badoureau 1881, Figs. 93, 139, 144). Finally, we have the 'snubs'

$$|\,2\;\tfrac{5}{2}\;5,\quad |\,2\;\tfrac{5}{3}\;5,\quad |\,2\;3\;\tfrac{5}{2},\quad |\,2\;3\;\tfrac{5}{3},\quad |\,2\;\tfrac{3}{2}\;\tfrac{5}{3}$$

(Lesavre & Mercier 1947), concerning which we shall have more to say in § 10.

The reader may wonder why the list of truncations does not include $\mathrm{t}\{\tfrac{5}{2}, q\}$ $(q=5$ or $3)$. In general, the faces of $\mathrm{t}\{p, q\}$ are $\{2p\}$ and $\{q\}$; but when $p=\tfrac{5}{2}$ the truncated face $\{2p\}=\{\tfrac{10}{2}\}$ is a *repeated* pentagon. In fact, $\mathrm{t}\{\tfrac{5}{2}, 5\}$ consists of three coincident dodecahedra, while $\mathrm{t}\{\tfrac{5}{2}, 3\}$ consists of two coincident great dodecahedra along with the icosahedron that has the same vertices and edges (Coxeter 1931, pp. 209–210).

6. Density

These star polyhedra do not all satisfy Euler's formula

$$N_0 - N_1 + N_2 = 2,$$

which connects the numbers of vertices, edges and faces of a convex polyhedron (Coxeter 1948, p. 9). But they do all satisfy a suitably modified version. This involves the *density* of the polyhedron, which is the number of intersections that the faces make with a ray drawn from the centre in a general direction, counting two intersections for each penetration of the core of a pentagram (Coxeter 1948, pp. 102–105), and counting certain retrograde faces negatively.

Cayley (1859, p. 127) observed that every regular polyhedron $\{p, q\}$ satisfies

$$aN_0 - N_1 + cN_2 = 2d,$$

where a is the density of the vertex figure $\{q\}$ (namely, 1 or 2 according as q is an integer or $\frac{5}{2}$), c is the density of the face $\{p\}$, and d is the density of the whole polyhedron.

A further extension was discovered by Hess (1876, p. 15), who allowed for faces of several kinds and for the possibility of an 'overhanging' edge, where the dihedral angle does not enclose the centre of the polyhedron. Of the two faces meeting at an overhanging edge, it is natural to regard the outer one as being 'retrograde', so that its penetration counts for -1 in the computation of d. In fact, the appropriate generalization is

$$\Sigma a - \Sigma b + \Sigma c = 2d, \tag{6·1}$$

where a is the density of the vertex figure (which can be zero if it does not enclose the circumcentre), b (for each edge) is 1 or 0 according as the edge is ordinary or overhanging, c is the density of a face (with a minus sign if the face is retrograde), and d is the total density (counting as many as three intersections for penetration of the innermost core of an octagram $\{\frac{8}{3}\}$ or a decagram $\{\frac{10}{3}\}$) (Brückner 1900, p. 165). Since we are dealing only with polyhedra whose vertices are all alike, the first term Σa can always be replaced by aN_0.

The foregoing conventions ensure that the total density d is the same as the 'area' d of the basic Schwarz triangle, in nearly every case (but see the remarks in § 9 and the end of § 10). For instance, the 'quasi-rhombicuboctahedron' $\mathrm{r}'\begin{Bmatrix}3\\4\end{Bmatrix}$, derived from the Schwarz triangle $(2\ \frac{3}{2}\ 4)$, has 24 vertices, 24 ordinary edges, 24 overhanging edges, 18 squares, 8 retrograde triangles, and satisfies the formula as follows:

$$24 - 24 + 18 - 8 = 2 . 5.$$

7. The regular and quasi-regular polyhedra $p \mid q\ r$

Let us define a *quasi-regular* polyhedron as consisting of regular polygons of two kinds, say $\{q\}$'s and $\{r\}$'s, each entirely surrounded by specimens of the other kind (Coxeter & Whitrow 1950, p. 422). The centres of two adjacent faces form, with either of their common vertices, a triangle whose sides lie in planes of symmetry. By central projection on to a concentric sphere, this yields a Schwarz triangle, say $(p\ q\ r)$. Hence any quasi-regular polyhedron whose faces have distinct centres can be derived from such a spherical triangle by taking as vertices the images of one vertex of the triangle, namely, the one where the angle π/p occurs. Since the construction is not essentially altered when we replace $(p\ q\ r)$ by a colunar triangle, we shall assume $(p\ q\ r)$ to be the smallest triangle in its set of colunars. We let

$$p \mid q\ r$$

denote the polyhedron formed by the images of the vertex P of a triangle PQR whose sides PQ and PR are both $\leqslant \frac{1}{2}\pi$, so that P is a vertex of faces $\{q\}$ and $\{r\}$ whose centres are Q and R (rather than the antipodes of those points). If in addition the angles at P and Q are supplementary, so that two colunar triangles are congruent, we seem to be confronted with a choice between (say) $q' \mid q\ r$ and $q \mid q'\ r$, where

$$q^{-1} + q'^{-1} = 1.$$

In this case the appropriate symbol is $q' \mid q\ r$ with $q' < 2 < q$; for, if the triangle $(q'\ q\ r)$ is PQR with the obtuse angle π/q' at P, we have $\mathrm{PR} < \frac{1}{2}\pi < \mathrm{QR}$.

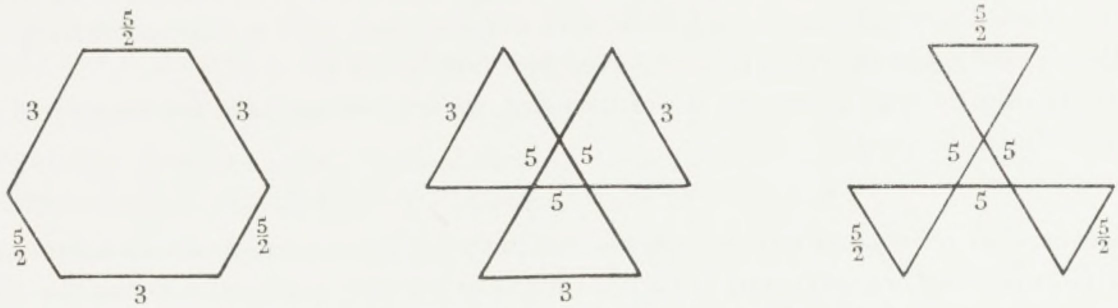


FIGURE 7. The vertex figures of $3 \mid 3\ \frac{5}{2}$, $\frac{3}{2} \mid 3\ 5$, $3 \mid \frac{5}{3}\ 5$.

If $r = 2$ or $r = q$, the polyhedron $p \mid q\ r$ is not only quasi-regular but *regular* (as we saw in § 3):

$$p \mid 2\ q = \{q, p\}, \quad p \mid q\ q = \{q, 2p\}.$$

In other cases, each vertex of $p \mid q\ r$ is surrounded by $\{q\}$'s and $\{r\}$'s, arranged alternately, the number of each being the numerator of the rational number p. If $q < 2$, the symbol $\{q\}$ is to be interpreted as a retrograde $\{q'\}$. This is natural, since a positive rotation through $2\pi/q$ has the same effect as a negative rotation through $2\pi/q'$.

Since every vertex of a Schwarz triangle is a vertex of a Möbius triangle, every quasi-regular polyhedron has the same vertices as a convex regular or quasi-regular solid; e.g. $\begin{Bmatrix}\frac{5}{2}\\5\end{Bmatrix}$ and $\begin{Bmatrix}3\\\frac{5}{2}\end{Bmatrix}$ are both inscribed in the icosidodecahedron $\begin{Bmatrix}3\\5\end{Bmatrix}$ (Badoureau 1881, p. 133).

Inscribed in the dodecahedron $\{5, 3\}$, we find the two ditrigonal icosidodecahedra

$$3 \mid 3\ \tfrac{5}{2} \quad \text{and} \quad \tfrac{3}{2} \mid 3\ 5$$

(Badoureau's Figs. 74 and 75; for the latter, see also Coxeter 1932, p. 517) and the ditrigonal dodecadodecahedron

$$3 \mid \tfrac{5}{3}\ 5$$

(Coxeter 1939, p. 141). These three quasi-regular polyhedra have not only the same twenty vertices but also the same sixty edges; therefore their vertex figures (figure 7) all have the same six vertices. The long, medium and short sides of these irregular hexagons are vertex figures of pentagons, triangles and pentagrams, respectively.

The denominator 2 in the symbol $\frac{3}{2} \mid 3\ 5$ indicates that in this case the vertex figure is a polygon of density $a = 2$. In the symbol $3 \mid \frac{5}{3}\ 5$, the occurrence of $\frac{5}{3}$ rather than $\frac{5}{2}$ indicates that the pentagrams are retrograde, and consequently all the edges are overhanging: $b = 0$. Thus (6·1) (the generalization of Euler's formula) has the following appearance in these three cases:

$$
\begin{array}{llr}
3 \mid 3\ \frac{5}{2}, & 20-60+\ \ 20+2.12 = 2.2; \\
\frac{3}{2} \mid 3\ 5, & 2.20-60+\ \ 20+\ \ 12 = 2.6; \\
3 \mid \frac{5}{3}\ 5, & 20\ \ \ -2.12+\ \ 12 = 2.4.
\end{array}
$$

To illustrate the significance of d, consider the last case. Here the density can be observed directly by coming out from the centre along a pentagonal axis of symmetry. We penetrate one pentagon, then five more, and finally the core of a retrograde pentagram; thus the total number of penetrations is

$$1+5-2 = 4.$$

Our d is not always equal to the E of Badoureau (1881, pp. 101–108), which is 8 for $3 \mid \frac{5}{3}\ 5$. In fact, d and E are different in all but the very simplest cases, such as those considered also by Pitsch (1881, pp. 72–87), whose A agrees with both d and E.

On looking through Schwarz's list of triangles, we see that we have not considered the apparently valid symbols

$$4 \mid \tfrac{3}{2}\ 4, \quad 5 \mid \tfrac{3}{2}\ 5, \quad 5 \mid 3\ \tfrac{5}{3}, \quad \tfrac{5}{3} \mid 3\ 5, \quad \tfrac{5}{3} \mid 3\ \tfrac{5}{2}.$$

However, if we try to carry out the construction in these cases we merely obtain compounds of familar polyhedra superposed in such a way as to have the same vertices and the same edges, like the three coincident dodecahedra formed by '$t\{\frac{5}{2}, 5\}$' (see the end of § 5). These compounds are listed in the first five lines of table 6 (on p. 431), with negative signs for components having retrograde faces. We give also the corresponding analysis of density, and diagrams to show how the vertex figures collapse.

In the identity $4 \mid \frac{3}{2}\ 4 = -\{3, 4\}+\mathit{3}\{4, 2\}$ (at the beginning of table 6) the italic $\mathit{3}$ indicates three *distinct* dihedra (corresponding to the three equatorial squares of the octahedron). Later in the table we see

$$2\ 4 \mid \tfrac{3}{2} = \mathit{3}\{4, 2\}+2\{3, 4\},$$

where the ordinary 2 indicates two *coincident* octahedra.

8. The semi-regular polyhedra $p\ q \mid r$

The bisector of the angle π/r of a Schwarz triangle $(p\ q\ r)$ meets the opposite side in a point whose images are the vertices of a polyhedron which we denote by $p\ q \mid r$, as in § 3. Since the construction is not essentially altered when we replace $(p\ q\ r)$ by its colunar triangle $(p'\ q'\ r)$, we shall assume $(p\ q\ r)$ to be the smaller of these two triangles.

If p and q are greater than 2, the faces surrounding any one vertex are, in general,

$$\{p\}, \quad \{2r\}, \quad \{q\}, \quad \{2r\}.$$

If $p = 2$ while $r \geqslant 2$, we have a truncation (or, for $r = 2$, a prism):

$$2\ \ q \mid r = t\{r, q\}.$$

If $r = 2, 3, 4$ or 5, this is a single polyhedron of density d, the faces at a vertex being

$$\{2r\}, \quad \{q\}, \quad \{2r\};$$

but if $r = \frac{5}{2}$, it splits, as we saw at the end of § 5. In the case of

$$2\ 3\ |\ \tfrac{5}{2} = \{3, 5\} + 2\{5, \tfrac{5}{2}\},$$

analogy with the other cases would make us expect the vertex figure to be an isosceles triangle with sides 1, τ, τ (which we would mark 3, 5, 5 according to the convention at the end of § 1). This would indicate that each pentagonal face is surrounded by pentagons and triangles alternately, which is absurd. Actually the sixty vertices coincide in twelve sets of five, and the five isosceles triangles in different positions combine to form a pentagon with an inscribed pentagram (repeated). This is indicated in table 6 by emphasizing one of the five isosceles triangles.

If $r < 2$ (with $p = 2$), we have a quasi-truncation

$$2\ q\ |\ r = \mathrm{t}'\{r', q\} \qquad (r'^{-1} + r^{-1} = 1)$$

which is a single polyhedron if $r = \frac{4}{3}$ or $\frac{5}{3}$, the faces at a vertex being again

$$\{2r\}, \quad \{q\}, \quad \{2r\}.$$

But if $r = \frac{3}{2}$ or $\frac{5}{4}$ (having an even denominator), we find further cases of splitting:

$$2\ q\ |\ \tfrac{3}{2} \quad (q = 3, 4, 5, \tfrac{5}{2}) \qquad \text{and} \qquad 2\ q\ |\ \tfrac{5}{4} \quad (q = 3 \text{ or } \tfrac{5}{2}).$$

Two of these deserve special mention, because the isosceles triangles in their vertex figures are obtuse-angled, indicating retrograde faces:

$$2\ 5\ |\ \tfrac{3}{2} = -\{5, \tfrac{5}{2}\} + 2\{3, 5\}, \qquad 11 = -3 + 2 \quad + 12;$$
$$2\ 3\ |\ \tfrac{5}{4} = -\{3, \tfrac{5}{2}\} + 2\{\tfrac{5}{2}, 5\}, \qquad 19 = -7 + 2.3 + 20.$$

Here the total density d is obtained by adding the number of retrograde faces to the sum of the component densities. In fact, the retrograde faces, being on the 'far' side of the centre, are regarded as having passed beyond it.

If $p = q$, the bisector of the angle π/r decomposes the isosceles triangle $(p\ p\ r)$ into two right-angled triangles $(2\ p\ 2r)$, so we get nothing fresh:

$$p\ p\ |\ r = 2\ |\ p\ 2r = \begin{Bmatrix} p \\ 2r \end{Bmatrix}.$$

If $r = 2$, we have $\quad p\ q\ |\ 2 = \mathrm{r}\begin{Bmatrix} p \\ q \end{Bmatrix}, \quad p'\ q\ |\ 2 = \mathrm{r}'\begin{Bmatrix} p \\ q \end{Bmatrix} \qquad (p > 2 > p',\ q > 2).$

The faces at a vertex are $\quad \{p\}, \quad \{4\}, \quad \{q\}, \quad \{4\},$

with the squares crossing each other in the case of $p'\ q\ |\ 2$ (because the $\{p\}$'s are then retrograde). Thus the vertex figure is a trapezoid or a crossed trapezoid.

In particular, $\frac{3}{2}\ 3\ |\ 2$ or $\mathrm{r}'\begin{Bmatrix} 3 \\ 3 \end{Bmatrix}$ may be identified with the famous one-sided heptahedron or *tetratrihedron* (Badoureau 1881, Fig. 70) whose faces consist of alternate triangles of the octahedron and three squares lying in planes through the centre, so that the vertex figure is a 'crossed rectangle' consisting of two opposite sides of a square along with the two diagonals. Since, in general, $\mathrm{r}\begin{Bmatrix} p \\ q \end{Bmatrix}$ and $\mathrm{r}'\begin{Bmatrix} p \\ q \end{Bmatrix}$ each have a face for every face and vertex of $\begin{Bmatrix} p \\ q \end{Bmatrix}$, it would be more strictly correct to use the symbol $\mathrm{r}'\begin{Bmatrix} 3 \\ 3 \end{Bmatrix}$ for the two-sided 'covering surface' of the one-sided tetratrihedron. Topologically, this covering surface, formed by eight triangles and six squares, is homeomorphic to the cuboctahedron $\mathrm{r}\begin{Bmatrix}3\\3\end{Bmatrix} = \begin{Bmatrix}3\\4\end{Bmatrix}$. However, we find it more convenient to let $\mathrm{r}'\begin{Bmatrix}3\\3\end{Bmatrix}$ denote the simple tetratrihedron itself. Since some of the faces lie in planes through the centre, it is not much use trying to define a 'density' for such a polyhedron.

FIGURE 8

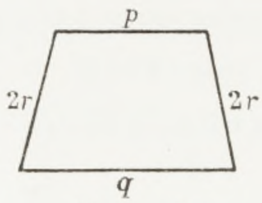


FIGURE 9

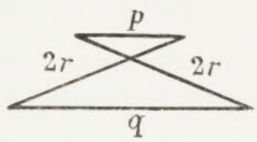


FIGURE 10

Figure 8 (cf. Coxeter 1948, p. 111, Fig. 6·7B) shows the partition of a trirectangular spherical triangle (2 2 2) into fifteen (2 3 5)'s. The centre of the (2 2 2), being on an axis of trigonal symmetry, is a vertex of the dodecahedron $3\,|\,2\;5$ (that is, a point of type **2** in the notation of Coxeter 1948, p. 66, Fig. 4·5A). It follows that the bisectors of the right angles in the triangles $(2\;3\;\frac{5}{2})$, $(2\;\frac{5}{3}\;5)$ and $(2\;\frac{3}{2}\;5)$ meet the opposite sides in points of this type. Hence the sixty vertices that we should expect to find for each of

$$\mathrm{r}\begin{Bmatrix}3\\\frac{5}{2}\end{Bmatrix},\quad \mathrm{r}'\begin{Bmatrix}\frac{5}{2}\\5\end{Bmatrix},\quad \mathrm{r}'\begin{Bmatrix}3\\5\end{Bmatrix}$$

actually coincide by threes at the twenty vertices of a dodecahedron, and the thirty squares are the faces of the compound of five cubes, $\{5, 3\}[5\{4, 3\}]$ (Coxeter 1948, pp. 49, 100). For further details, see table 6.

Turning now to the cases where p, q, r all differ from 2, and $p<q$, we make a further classification according as $p>2$ or $q'<p<2$ or $p=q'$.

When p and q are both greater than 2, $p\;q\,|\,r$ has, at each vertex, the polygons

$$\{p\},\quad \{2r\},\quad \{q\},\quad \{2r\},$$

and the vertex figure is an ordinary symmetrical trapezoid (figure 9). The four actual cases are

$$3\;4\,|\,\tfrac{4}{3},\quad 3\;\tfrac{5}{2}\,|\,3,\quad 3\;5\,|\,\tfrac{5}{3},\quad 3\;\tfrac{5}{2}\,|\,\tfrac{5}{3}.$$

When $q'<p<2$, $p\;q\,|\,r$ has at each vertex

$$\{p'\},\quad \{2r\},\quad \{q\},\quad \{2r\},$$

with the $\{2r\}$'s crossing each other (because the $\{p'\}$'s are retrograde), and the vertex figure is a crossed trapezoid (figure 10). The five actual cases are

$$\tfrac{3}{2}\,4\,|\,4,\quad \tfrac{3}{2}\,5\,|\,5,\quad 3\,\tfrac{5}{3}\,|\,5,\quad \tfrac{5}{3}\,5\,|\,3,\quad \tfrac{3}{2}\,5\,|\,3.$$

Most of these polyhedra were discovered by Badoureau (see table 7). Pitsch (1881, no. XVIII, p. 87 and Plate II) described $3\,\tfrac{5}{2}\,|\,3$; but $3\,\tfrac{5}{3}\,|\,5$ or $\tfrac{5}{3}\,3\,|\,5$ (which has the same vertices, the same edges, and some of the same faces) appears to be new.

Finally, when $p = q' < 2$, the faces at a vertex are

$$\{q\},\quad \{2r\},\quad \{q\},\quad \{2r\},$$

with the $\{2r\}$'s lying in planes through the centre; and the vertex figure is a 'crossed rectangle' consisting of two opposite sides of an ordinary rectangle along with the two diagonals. If the remaining sides of the ordinary rectangle are vertex figures of $\{p\}$'s, so that

$$\cos^2\frac{\pi}{p}+\cos^2\frac{\pi}{q}=\cos^2\frac{\pi}{2r},$$

we may describe $q'\ q\,|\,r$ as consisting of the $\{q\}$'s and equatorial $\{2r\}$'s of $\begin{Bmatrix} p \\ q \end{Bmatrix}$ (Coxeter 1948, pp. 19, 102). In this manner the octahedron $\begin{Bmatrix} 3 \\ 3 \end{Bmatrix}$ yields the tetratrihedron $\tfrac{3}{2}\,3\,|\,2$, and the quasi-regular polyhedra of §§ 1 and 5 yield further quasi-regular polyhedra as follows:

$$\begin{Bmatrix} 3 \\ 4 \end{Bmatrix},\quad \tfrac{3}{2}\,3\,|\,3 \quad\text{and}\quad \tfrac{4}{3}\,4\,|\,3;$$

$$\begin{Bmatrix} 3 \\ 5 \end{Bmatrix},\quad \tfrac{3}{2}\,3\,|\,5 \quad\text{and}\quad \tfrac{5}{4}\,5\,|\,5;$$

$$\begin{Bmatrix} \tfrac{5}{2} \\ 5 \end{Bmatrix},\quad \tfrac{5}{3}\,\tfrac{5}{2}\,|\,3 \quad\text{and}\quad \tfrac{5}{4}\,5\,|\,3;$$

$$\begin{Bmatrix} 3 \\ \tfrac{5}{2} \end{Bmatrix},\quad \tfrac{3}{2}\,3\,|\,\tfrac{5}{3} \quad\text{and}\quad \tfrac{5}{3}\,\tfrac{5}{2}\,|\,\tfrac{5}{3}$$

(Badoureau 1881, Figs. 97, 96, 116, 115, 119, 118, 121, 122).

A polyhedron is said to be *orientable* if a rotatory sense can be assigned to each face in such a way that every two adjacent faces induce opposite senses along their common edge. In the polyhedron $q'\ q\,|\,r$, formed by the $\{q\}$'s and equatorial $\{2r\}$'s of $\begin{Bmatrix} p \\ q \end{Bmatrix}$, the $\{q\}$'s adjacent to a given 'horizontal' $\{2r\}$ are alternately 'above' and 'below' its plane, as we go round the $\{2r\}$. Thus the senses of two consecutive $\{q\}$'s are opposite. It follows that $q'\ q\,|\,r$ is orientable if and only if the $\{q\}$'s of $\begin{Bmatrix} p \\ q \end{Bmatrix}$, or of $\{q, p\}$, can be given positive and negative orientations alternately. By considering all the $\{q\}$'s that surround a vertex of $\{q, p\}$, we see that this can be done if and only if the numerator of p is even (Coxeter 1948, p. 50, with p and q interchanged), which means that $p = 4$. Hence, as P. du Val once remarked (in a letter dated 30 December 1932):

The only orientable polyhedron $q'\ q\,|\,r$ is the octatetrahedron $\tfrac{3}{2}\,3\,|\,3$.

Since $N_0 - N_1 + N_2 = 12 - 24 + 8 + 4 = 0$, this orientable surface is topologically a torus on which is drawn a map of eight triangles and four hexagons (Coxeter 1939, p. 132, Fig. 13).

For surfaces in ordinary space, the distinction between orientable and unorientable is the same as the distinction between two-sided and one-sided. Thus all the other polyhedra $q'\ q\ |\ r$ are one-sided.

When r has an even denominator, the angle π/r of the triangle $(p\ q\ r)$ is bisected by a plane of symmetry of the spherical tessellation. The consequent dissection of the triangle is

$$(p\ x\ 2r)+(x'\ q\ 2r),$$

where
$$\cos\frac{\pi}{x}=-\cos\frac{\pi}{x'}=\tfrac{1}{2}\left(\cos\frac{\pi}{q}-\cos\frac{\pi}{p}\right)\Big/\cos\frac{\pi}{2r} \tag{8·1}$$

(Coxeter 1948, p. 113, with $r_1=r_2$). In this case $p\ q\ |\ r$ splits into

$$x\ |\ p\ 2r+x'\ |\ q\ 2r$$

(see table 6).

9. The even-faced polyhedra $p\ q\ r\ |$ and $p\ q\ {r \atop s}|$

Generalizing figure 5, we see that the in-centres of a network of Schwarz triangles $(p\ q\ r)$ are the vertices of

$$p\ q\ r\ |,$$

which is a single polyhedron whenever p, q, r are either integers or fractions whose denominators are odd, the faces at a vertex being

$$\{2p\},\quad \{2q\},\quad \{2r\}.$$

In particular, $$2\ 3\ 4\ |,\quad 2\ 3\ 5\ |,\quad 2\ 3\ \tfrac{4}{3}\ |,\quad 2\ 3\ \tfrac{5}{3}\ |,\quad 2\ \tfrac{5}{3}\ 5\ |$$

are the truncated cuboctahedron and its analogues:

$$\mathrm{t}\begin{Bmatrix}3\\4\end{Bmatrix},\quad \mathrm{t}\begin{Bmatrix}3\\5\end{Bmatrix},\quad \mathrm{t}'\begin{Bmatrix}3\\4\end{Bmatrix},\quad \mathrm{t}'\begin{Bmatrix}3\\ \frac{5}{2}\end{Bmatrix},\quad \mathrm{t}'\begin{Bmatrix}\frac{5}{2}\\5\end{Bmatrix},$$

as we remarked in § 5.

In two cases, $2\ 3\ \tfrac{4}{3}\ |$ and $2\ \tfrac{5}{3}\ 5\ |$, the densities of the polyhedra differ from those of the corresponding Schwarz triangles: $2\ 3\ \tfrac{4}{3}\ |$ has density 1 and $2\ \tfrac{5}{3}\ 5\ |$ has density 3, compared with densities 7 and 9 respectively for the triangles $(2\ 3\ \tfrac{4}{3})$ and $(2\ \tfrac{5}{3}\ 5)$. This discrepancy can be traced to the fact that the vertex-figures are obtuse-angled, and that in $2\ 3\ \tfrac{4}{3}\ |$, for example, all except the hexagonal faces are retrograde. It is possible to imagine a distorted form of $2\ 3\ \tfrac{4}{3}\ |$ whose vertex figure is acute-angled and whose density is 7. As this is deformed into $2\ 3\ \tfrac{4}{3}\ |$ the hexagonal faces pass through the centre of the polyhedron and the density becomes $|7-8|=1$. Similarly, $2\ \tfrac{5}{3}\ 5\ |$ has all faces except twelve decagons retrograde, giving the density $|9-12|=3$. We note that $2\ 3\ \tfrac{4}{3}\ |$ has the same density as the colunar triangle $(2\ 3\ 4)$, and that $2\ \tfrac{5}{3}\ 5\ |$ has the same density as the colunar triangle $(2\ \tfrac{5}{2}\ 5)$.

If $p=q=r$, so that the triangle $(p\ q\ r)$ is equilateral, we have the regular polyhedron

$$p\ p\ p\ |=\{2p, 3\}\quad (p=2, \tfrac{3}{2}, \tfrac{5}{2}, \tfrac{5}{4}).$$

If $p=q+r$, so that $(p\ q\ r)$ is isosceles, we have

$$p\ p\ r\ |=2\ 2r\ |\ p=\begin{cases}\mathrm{t}\{p, 2r\} & (p\geqslant 2),\\ \mathrm{t}'\{p', 2r\} & (p<2);\end{cases} \tag{9·1}$$

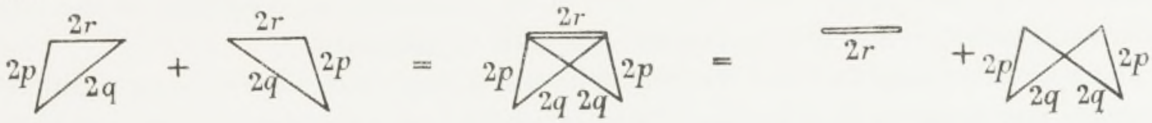
e.g. $2\ 2\ r\,|$ is the prism on a $\{2r\}$, and $3\ 3\ 2\,|$ or $2\ 3\ 3\,|$ is the truncated octahedron. The remaining polyhedra $p\ q\ r\,|$ (with odd denominators), namely,

$$3\ \tfrac{4}{3}\ 4\,| \quad \text{and} \quad 3\ \tfrac{5}{3}\ 5\,|,$$

are among those discovered simultaneously by Badoureau (1881, Figs. 137 and 148) and Pitsch (1881, nos. IX and XI).

If just one of p, q, r, say r, has an even denominator, the face $\{2r\}$ has an odd number of sides. These sides belong to $\{2p\}$'s and $\{2q\}$'s alternately, which seems at first sight to be possible only if $p = q$. The identity (9·1) still holds; but to achieve the proper density we must regard the $t\{p, 2r\}$ or $t'\{p', 2r\}$ as being described twice over. This duplication is the clue to the proper interpretation of $p\ q\ r\,|$ when $p < q$. For, if the odd face $\{2r\}$ is described twice over, the sequence of $\{2p\}$'s and $\{2q\}$'s surrounding it alternately will close up. But then each edge belongs to both a $\{2p\}$ and a $\{2q\}$, as well as to the duplicated $\{2r\}$. We can obtain a single polyhedron, formed by $\{2p\}$'s and $\{2q\}$'s alone, by the simple device of *discarding all the* $\{2r\}$*'s.*

The arrangement of $\{2p\}$'s and $\{2q\}$'s at a vertex is easily seen by superposing the right-handed and left-handed vertex figures and then discarding the common side (marked $2r$). Thus



and the final result is a crossed parallelogram having two sides $2\cos\pi/2p$ and two sides $2\cos\pi/2q$. The four vertices of this crossed parallelogram belong also to a convex trapezoid of sides

$$2\cos\pi/2p, \quad 2\cos\pi/2r, \quad 2\cos\pi/2p \quad \text{and} \quad 2\cos\pi/2s,$$

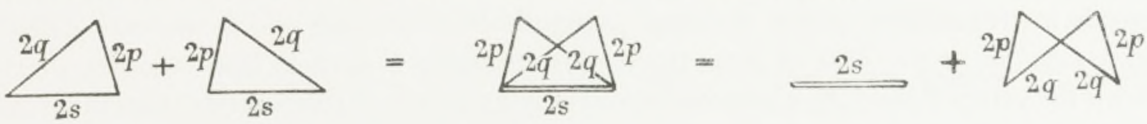
where, by Ptolemy's theorem,

$$\cos\frac{\pi}{2r}\cos\frac{\pi}{2s} = \cos^2\frac{\pi}{2q} - \cos^2\frac{\pi}{2p},$$

that is,

$$2\cos\frac{\pi}{2r}\cos\frac{\pi}{2s} = \cos\frac{\pi}{q} - \cos\frac{\pi}{p}. \tag{9·2}$$

Thus the same crossed parallelogram could have been derived from



and the same polyhedron could have been derived from $p\ q\ s\,|$ by discarding the $\{2s\}$'s. We obtain an appropriate symbol for the polyhedron (whose faces are $\{2p\}$'s and $\{2q\}$'s) by telescoping the two symbols $p\ q\ r\,|$ and $p\ q\ s\,|$ to make

$$p\ q\ {}^{r}_{s}\Big| \quad \text{or} \quad q\ p\ {}^{r}_{s}\Big|.$$

By considering the convex trapezoid and crossed trapezoid that have the same vertices as the crossed parallelogram, we see that this polyhedron has the same vertices and edges as

$$2r\ 2s\,|\,p \quad \text{and} \quad (2r)'\ 2s\,|\,q \quad \text{or} \quad 2r\ (2s)'\,|\,q.$$

This result may alternatively be deduced from the decomposition of the spherical triangle $(p\;q\;r)$ by the bisector of its angle π/r. Comparing (9·2) with (8·1), we see that $x = 2s$.

The advantage of the first method is that it treats r and s symmetrically. The advantage of the second method is that it explains why the s of (9·2) is always rational.

The actual cases are

$$2\;3\;\begin{matrix}\frac{3}{2}\\ \frac{3}{2}\end{matrix}\Bigg|,\quad 2\;4\;\begin{matrix}\frac{3}{2}\\ \frac{4}{2}\end{matrix}\Bigg|,\quad 2\;5\;\begin{matrix}\frac{3}{2}\\ \frac{5}{2}\end{matrix}\Bigg|,\quad 3\;\tfrac{5}{3}\;\begin{matrix}\frac{3}{2}\\ \frac{5}{2}\end{matrix}\Bigg|,$$

$$2\;3\;\begin{matrix}\frac{5}{4}\\ \frac{5}{2}\end{matrix}\Bigg|,\quad 2\;\tfrac{4}{3}\;\begin{matrix}\frac{3}{2}\\ \frac{4}{2}\end{matrix}\Bigg|,\quad 2\;\tfrac{5}{3}\;\begin{matrix}\frac{3}{2}\\ \frac{5}{4}\end{matrix}\Bigg|,\quad 3\;5\;\begin{matrix}\frac{3}{2}\\ \frac{5}{4}\end{matrix}\Bigg|.$$

The first of these is the same as $\frac{4}{3}\;4\,|\,3$, whose four hexagons lie in planes through the centre. The rest are listed in table 7, where we see that all save the last were discovered by Badoureau.

In two cases, $2\;4\;\begin{matrix}\frac{3}{2}\\ \frac{4}{2}\end{matrix}\Big|$ and $2\;\frac{4}{3}\;\begin{matrix}\frac{3}{2}\\ \frac{4}{2}\end{matrix}\Big|$, the symbol $\frac{4}{2}$, although not in its lowest terms, has been used instead of 2, since the corresponding squares have fourfold and not twofold rotational symmetry. The symbol $2\;4\;2\,|$ or $2\;\frac{4}{3}\;2\,|$ would denote a single octagonal or octogrammatic prism. The present polyhedra are derived from an arrangement of three such prisms having some square faces in common. The corresponding Schwarz triangles may similarly be written $(2\;4\;\frac{4}{2})$ and $(2\;\frac{4}{3}\;\frac{4}{2})$, since they are to be considered as occurring in the cubic spherical tessellation and not the fourfold prismatic tessellation.

There are other cases in which the same spherical triangle occurs in two or more different spherical tessellations: for example, $(2\;2\;2)$ occurs in both the cubic and the icosahedral tessellation as well as the diagonal tessellation. But in all other cases the present construction can be shown to lead only to 'compound' polyhedra.

Referring to table 5, we see that the only remaining symbols $p\;q\;r\,|$ are

$$\tfrac{3}{2}\;\tfrac{5}{2}\;5\,|,\quad 3\;\tfrac{5}{4}\;\tfrac{5}{2}\,|,\quad 2\;\tfrac{3}{2}\;\tfrac{5}{2}\,|,\quad 2\;\tfrac{5}{4}\;\tfrac{5}{2}\,|,\quad 2\;\tfrac{3}{2}\;\tfrac{5}{4}\,|,\quad \tfrac{3}{2}\;\tfrac{5}{4}\;\tfrac{5}{3}\,|,$$

where two of p, q, r have even denominators. The splitting in these cases is shown in table 6.

10. The snub polyhedra $|\,p\;q\;r$

We construct $|\,p\;q\;r$ by regarding the spherical triangles $(p\;q\;r)$ as being alternately white and black (see § 3, especially figure 6). The three white triangles that surround a black one contain corresponding points forming an equilaterial triangle which we may called a 'snub face' of $|\,p\;q\;r$. One of these three white triangles is derived from another, sharing with it the vertex P (say), by a rotation through $2\pi/p$ about P. If this rotation takes the chosen point C‴ in the first triangle to C″ in the second, we have an isosceles triangle C‴PC″ whose base C″C‴ (opposite to the angle $2\pi/p$ at P) is one side of the snub face. Solving this isosceles triangle, we find

$$\sin \mathrm{PC}'' \sin\frac{\pi}{p} = \sin \tfrac{1}{2}\mathrm{C}''\mathrm{C}'''.$$

If C is the corresponding point in the black triangle, then PC = PC″. Hence, by considering in turn the other sides of the snub face we deduce

$$\sin \mathrm{PC} \sin\frac{\pi}{p} = \sin \mathrm{QC} \sin\frac{\pi}{q} = \sin \mathrm{RC} \sin\frac{\pi}{r}. \tag{10·1}$$

Thus the sines of the distances of the points C, C′, etc., from the vertices of the corresponding triangles are inversely proportional to the sines of the angles at those vertices (Coxeter 1940, p. 393). (Of course C, being in a black triangle, is *not* a vertex of the $|\,p\;q\;r$.)

The points in the three white triangles are the images of the point C in the black triangle by reflexion in the sides of the black triangle, or in the planes containing these sides. The plane triangle formed by the three images (i.e. the snub face) is clearly similar to the triangle (of half the linear size) formed by the orthogonal projections of C on these three planes. Accordingly, we can describe C as the point whose orthogonal projections form an equilateral triangle. We obtain a natural co-ordinate system by letting x, y, z denote the straight distances of C from the three planes. These, then, are the lengths of three lines CX, CY, CZ, such that

$$\angle \mathrm{YCZ} = \pi - \frac{\pi}{p}, \quad \angle \mathrm{ZCX} = \pi - \frac{\pi}{q}, \quad \angle \mathrm{XCY} = \pi - \frac{\pi}{r}.$$

The condition for the plane triangle XYZ to be equilateral is clearly

$$y^2 + z^2 + 2yz\cos\frac{\pi}{p} = z^2 + x^2 + 2zx\cos\frac{\pi}{q} = x^2 + y^2 + 2xy\cos\frac{\pi}{r},$$

that is,
$$x^2 - ayz = y^2 - bzx = z^2 - cxy,$$

where
$$a = 2\cos\frac{\pi}{p}, \quad b = 2\cos\frac{\pi}{q}, \quad c = 2\cos\frac{\pi}{r}$$

(not to be confused with the a, b, c of § 6). Eliminating z, we obtain for x/y the quartic equation
$$(1-b^2)\,x^4 + (a-bc)\,bx^3y + 2(abc-1)\,x^2y^2 + (b-ac)\,axy^3 + (1-a^2)\,y^4 = 0. \qquad (10\cdot 2)$$

These two methods for locating a vertex of $|\,p\;q\;r$ have been mentioned for their intrinsic interest. But the actual enumeration of snub polyhedra is more easily accomplished by means of the vertex figure. In general, the faces of $|\,p\;q\;r$ consist of

$$\{p\}\text{'s or } \{p'\}\text{'s}, \qquad \{q\}\text{'s or } \{q'\}\text{'s}, \qquad \{r\}\text{'s or } \{r'\}\text{'s},$$

each entirely surrounded by triangles; but if one of p, q, r is equal to 2, the consequent digons can be ignored, and two of the snub faces have a common side. Thus the vertex figure is a cyclic hexagon (or pentagon, or quadrangle, or triangle) of sides

$$a, \quad 1, \quad b, \quad 1, \quad c, \quad 1.$$

When any of p, q, r are less than 2, the corresponding 'negative' sides proceed round the circle in the reverse sense, indicating retrograde faces.

Let the sides a, b, c, 1 of the vertex figure subtend angles 2α, 2β, 2γ, 2δ at its centre, and let ρ denote the radius of the circle in which the vertex figure is inscribed. Then

$$2\rho\sin\alpha = a, \quad 2\rho\sin\beta = b, \quad 2\rho\sin\gamma = c, \quad 2\rho\sin\delta = 1, \quad \alpha+\beta+\gamma+3\delta = \pi.$$

Eliminating α, β, γ, δ, we obtain for ρ the equation

$$[(9-s)^2 - 24t - 4u]\,\rho^8 + [3s(4+t) + t(35+t) + 9(u-12)]\,\rho^6 + [6(18-u) - (27+s)\,(2+t)]\,\rho^4 + (9t+u-12)\,\rho^2 + (1-t) = 0,$$

where
$$s = a^2+b^2+c^2, \quad t = abc, \quad u = b^2c^2 + c^2a^2 + a^2b^2.$$

Writing $\rho^2 = 1/(2-X)$, so that

$$X = 2-\rho^{-2} = 2-4\sin^2\delta = 2\cos 2\delta,$$

we obtain the more elegant equation

$$(1-t)\,X^4+(4-t-u)\,X^3+(3-s)\,(2+t)\,X^2+[(1-t)\,(4-s+t)-3(s-u)]\,X \\ +[(3-s)\,(1-s-2t)-2(1-s+u-t^2)] = 0. \quad (10\cdot3)$$

The antiprisms $s\begin{Bmatrix}2\\p\end{Bmatrix}$ and $s'\begin{Bmatrix}2\\p\end{Bmatrix}$ are given by setting $b=c=0$, so that $s=a^2$, $t=0$, $u=0$, and (10·3) reduces to

$$[(X+1)^2-a^2]^2 = 0,$$

whence $X=-1\pm a$, and

$$\rho^2 = 1/(3\mp a).$$

This is easily seen to be the squared circum-radius of a trapezoid, or a crossed trapezoid, of sides 1, 1, 1, a. The crossed trapezoid is possible only if $a<1$ (so that $\rho>\frac{1}{2}$). Hence, although the 'first' antiprism

$$|\,2\,2\,p = s\begin{Bmatrix}2\\p\end{Bmatrix}$$

occurs for every $p\geqslant 2$, the 'second' antiprism

$$|\,2\,2\,p' = s'\begin{Bmatrix}2\\p\end{Bmatrix}$$

occurs only for $2<p<3$ (that is, $2>p'>\frac{3}{2}$).

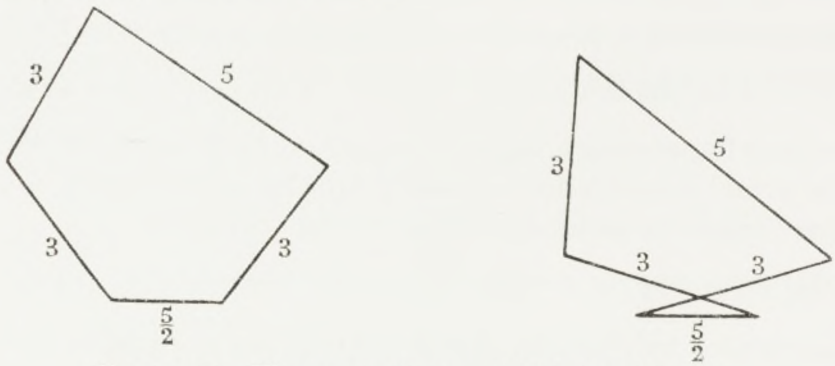


FIGURE 11. The vertex figures of $|\,2\,\frac{5}{2}\,5$ and $|\,2\,\frac{5}{3}\,5$.

Since s, t, u are unchanged when two of a, b, c are reversed in sign, we have one equation (10·3) for each set of four colunar triangles $(p\;q\;r)$; e.g. the triangles

$$(2\,\tfrac{5}{2}\,5), \quad (2\,\tfrac{5}{3}\,5), \quad (2\,\tfrac{5}{2}\,\tfrac{5}{4}), \quad (2\,\tfrac{5}{3}\,\tfrac{5}{4}),$$

for which $s=3$, $t=0$, $u=1$, yield

$$X^4+3X^3-5X+2 = 0.$$

This equation has just two real roots:

$$0\cdot81807\,55760 \quad \text{and} \quad 0\cdot47398\,76869.$$

The corresponding values of $\rho = (2-X)^{-\frac{1}{2}}$ are

$$0\cdot91982\,48671 \quad \text{and} \quad 0\cdot80950\,76943.$$

Drawing circles of these radii, we find the two possible vertex figures shown in figure 11.

Since the latter has retrograde pentagrams, the appropriate symbols for the two polyhedra are

$$|\,2\,\tfrac{5}{2}\,5 \quad \text{and} \quad |\,2\,\tfrac{5}{3}\,5$$

(Lesavre & Mercier 1947, nos. 4 and 5; see also Coxeter 1947).

When $p = q = r$, equation (10·3) becomes

$$[(1+a+a^2)\,X+(1-2a-2a^2)]\,[X+(1+a)]^3 = 0.$$

The simple root $X = (-1+2a+2a^2)/(1+a+a^2)$, implying $\rho^2 = (1+a+a^2)/3$, yields $|\,p\,p\,p$; for instance,

$$|\,2\,2\,2 = \{3, 3\}.$$

The triple root $X = -(1+a)$ yields $|\,p'\,p'\,p$. The splitting of these (with $p = \frac{5}{2}$ or $\frac{5}{4}$, $a = \tau^{-1}$ or $-\tau$) is indicated in table 6.

In all the remaining cases, at least one of a, b, c is ± 1, so that $u = s+t^2-1$, and the equation for X reduces to

$$(1-t)\,X^4+(5-s-t-t^2)\,X^3+(3-s)\,(2+t)\,X^2+(1-t)\,(1-s-2t)\,X+(3-s)\,(1-s-2t) = 0,$$

which factorizes thus:

$$[(1-t)\,X+(3-s)]\,[X^3+(2+t)\,X^2+(1-s-2t)] = 0. \tag{10·4}$$

The linear factor is most easily explained by taking $a = -1$ (that is, $p = \frac{3}{2}$), so that three sides of the vertex figure coincide, leaving a trapezoid of sides b, 1, c, 1, or (if $b = 0$) a triangle of sides 1, c, 1. Then

$$(1-t)\,X+(3-s) = (1+bc)\,X+(2-b^2-c^2),$$

in agreement with the value

$$\rho^2 = (1+bc)/(2+b-c)\,(2-b+c)$$

for the squared circum-radius of such a trapezoid. The actual cases ($|\,2\,2\,\frac{3}{2}$, etc.) are indicated in table 6.

When $a = 1$, the cubic factor of (10·4) yields the equation

$$X^3+(2+bc)\,X^2-(b+c)^2 = 0, \tag{10·5}$$

which has only one real root if $\left(\dfrac{2+bc}{3}\right)^3 < \left(\dfrac{b+c}{2}\right)^2$. The actual cases (with $p = 3$) are exhibited in table 4.

$|\,2\,3\,4$ and $|\,2\,3\,5$ are the snub cube and the snub dodecahedron, as we saw in § 3.

$|\,2\,3\,\frac{5}{2}$, $|\,2\,3\,\frac{5}{3}$, $|\,2\,\frac{3}{2}\,\frac{5}{3}$ are the analogous figures described by Lesavre & Mercier (1947, nos. 2, 3, 1).

$|\,2\,2\,\frac{3}{2}$, $|\,2\,\frac{3}{2}\,3$, $|\,\frac{3}{2}\,3\,\frac{5}{3}$, $|\,\frac{3}{2}\,\frac{5}{3}\,\frac{5}{3}$, $|\,\frac{3}{2}\,\frac{5}{4}\,\frac{5}{4}$, $|\,\frac{3}{2}\,3\,5$ all split as indicated in table 6.

In the case $q = \frac{5}{4}$, $r = 5$, the root $X = 0$ must be discarded, since a circle of diameter $2\rho = \surd 2$ cannot contain a chord of length $c = \tau$.

The remaining polyhedra

$$|\,3\,3\,\tfrac{5}{2}, \quad |\,\tfrac{3}{2}\,\tfrac{3}{2}\,\tfrac{5}{2}, \quad |\,3\,\tfrac{5}{3}\,5, \quad |\,3\,\tfrac{5}{3}\,\tfrac{5}{2}$$

are new. The first two of these, being of the form $|\,p\,p\,r$, are the only non-trivial snub polyhedra possessing a plane of symmetry. Besides the usual twelve pentagrams and sixty

Table 4. Snub polyhedra derived from equation (10·5)

q	r	$bc+2$	$b+c$	equation for X	values of X	values of ρ	polyhedra
2	2	2	0	$X^2(X+2)=0$	0	$2^{-\frac{1}{2}}$	$\|2\ 2\ 3=\{3,4\}$
					-2	$\frac{1}{2}$	$\|2\ 2\ \frac{3}{2}$
2	3	2	1	$(X-\tau^{-1})(X+\tau)(X+1)=0$	τ^{-1}	$5^{-\frac{1}{4}}\tau^{\frac{1}{2}}$	$\|2\ 3\ 3=\{3,5\}$
					$-\tau$	$5^{-\frac{1}{4}}\tau^{-\frac{1}{2}}$	$\|2\ \frac{3}{2}\ \frac{3}{2}=\{3,\frac{5}{2}\}$
					-1	$3^{-\frac{1}{2}}$	$\|2\ \frac{3}{2}\ 3$
2	4	2	$2^{\frac{1}{2}}$	$X^3+2X^2-2=0$	0·83928 67552	0·92819 13780	$\|2\ 3\ 4=\mathrm{s}\begin{Bmatrix}3\\4\end{Bmatrix}$
2	5	2	τ	$X^3+2X^2-\tau^2=0$	0·94315 12592	0·97273 28506	$\|2\ 3\ 5=\mathrm{s}\begin{Bmatrix}3\\5\end{Bmatrix}$
$\frac{5}{2}$	3	τ^2	τ	$(X+\tau)(X^2+X-\tau)=0$	$-\tau$	$5^{-\frac{1}{4}}\tau^{-\frac{1}{2}}$	$\|\frac{3}{2}\ 3\ \frac{5}{3}$
					0·86676 03992	0·93937 52830	$\|3\ 3\ \frac{5}{2}$
					−1·86676 03992	0·50854 14639	$\|\frac{3}{2}\ \frac{3}{2}\ \frac{5}{2}$
$\frac{4}{3}$	4	0	0	$X^3=0$	0	$2^{-\frac{1}{2}}$	$\|3\ \frac{4}{3}\ 4=\mathrm{r}'\begin{Bmatrix}3\\3\end{Bmatrix}$
$\frac{5}{3}$	5	1	1	$X^3+X^2-1=0$	0·75487 76662	0·89617 74018	$\|3\ \frac{5}{3}\ 5$
2	$\frac{5}{2}$	2	τ^{-1}	$X^3+2X^2-\tau^{-2}=0$	0·39902 06457	0·79032 75737	$\|2\ 3\ \frac{5}{2}$
					−0·50556 05785	0·63175 33395	$\|2\ 3\ \frac{5}{3}$
					−1·89346 00671	0·50679 47854	$\|2\ \frac{3}{2}\ \frac{5}{3}$
$\frac{5}{3}$	$\frac{5}{2}$	τ	0	$X^2(X+\tau)=0$	0	$2^{-\frac{1}{2}}$	$\|3\ \frac{5}{3}\ \frac{5}{2}$
					$-\tau$	$5^{-\frac{1}{4}}\tau^{-\frac{1}{2}}$	$\|\frac{3}{2}\ \frac{5}{3}\ \frac{5}{3}$
$\frac{5}{4}$	5	$-\tau^{-1}$	0	$X^2(X-\tau^{-1})=0$	0	$2^{-\frac{1}{2}}$	(none)
					τ^{-1}	$5^{-\frac{1}{4}}\tau^{\frac{1}{2}}$	$\|\frac{3}{2}\ \frac{5}{4}\ \frac{5}{4}$
3	$\frac{5}{4}$	τ^{-2}	$-\tau^{-1}$	$(X-\tau^{-1})(X^2+X+\tau^{-1})=0$	τ^{-1}	$5^{-\frac{1}{4}}\tau^{\frac{1}{2}}$	$\|\frac{3}{2}\ 3\ 5$

'snub' triangles, they have each forty more triangles, lying by pairs in twenty planes (the face-planes of an icosahedron).

Similarly, $| 3\ \frac{5}{3}\ \frac{5}{2}$ has twenty-four pentagrams, lying by pairs in twelve planes (the face-planes of a dodecahedron). The same sixty vertices may be regarded as belonging to another $| 3\ \frac{5}{3}\ \frac{5}{2}$, enantiomorphous to the given one. The roles of the two types of pentagram are interchanged (see figure 12).

In two cases, $| 2\ \frac{3}{2}\ \frac{3}{2}$ and $| 2\ \frac{3}{2}\ \frac{5}{3}$, the vertex figure is a pentagram (regular or irregular) and the snub faces are retrograde. The density is now given by subtracting the d of table 5 from the number of snub faces (cf. p. 418). Thus the density of the 'quasi-snub tetrahedron'

$$| 2\ \tfrac{3}{2}\ \tfrac{3}{2} = \{3, \tfrac{5}{2}\}$$

is not 5 but $7 = 12-5$, while that of

$$| 2\ \tfrac{3}{2}\ \tfrac{5}{3}$$

(Lesavre & Mercier's no. 1) is not 23 but $37 = 60-23$, agreeing with (6·1) in the form

$$2.60-150+80+2.12 = 2.37.$$

Similarly, the density of the new polyhedron

$$| \tfrac{3}{2}\ \tfrac{3}{2}\ \tfrac{5}{2}$$

is not 22 but $38 = 60-22$, agreeing with (6·1) in the form

$$2.60-120+100-2.12 = 2.38.$$

The density of each of these polyhedra is the same as that of one of the reflex colunar triangles. Thus $| 2\ \frac{3}{2}\ \frac{3}{2}$ has the same density, 7, as $(\frac{2}{3}\ 3\ 3)$. Indeed, we might consider the polyhedron as being derived from this triangle in the first place. In the vertex figure the snub faces would then be considered to have the same sense as the other triangles, instead of being retrograde, but the symbol $\frac{2}{3}$ would represent a complete negative revolution about the centre, so that the vertex would be surrounded altogether just once. The polyhedron might equally well be derived from the triangle $(2\ 3\ \frac{3}{4})$, whose density is also 7. In the vertex figure the symbol $\frac{3}{4}$ would denote a revolution through $-\frac{2}{3}\pi$ about the centre. Similarly, $| 2\ \frac{3}{2}\ \frac{5}{3}$ could be derived from any of the reflex-angled triangles $(\frac{2}{3}\ 3\ \frac{5}{2})$, $(2\ \frac{3}{4}\ \frac{5}{3})$ and $(2\ 3\ \frac{5}{7})$, each of which has density 37; and $| \frac{3}{2}\ \frac{3}{2}\ \frac{5}{2}$ could be derived from either $(3\ 3\ \frac{5}{8})$ or $(3\ \frac{3}{4}\ \frac{5}{3})$, each of which has the density 38.

As may be seen from figure 91 and figure 120, plate 5, $| \frac{3}{2}\ \frac{3}{2}\ \frac{5}{2}$ contains groups of ten edges which appear to intersect in a point. That they really are concurrent may be proved as follows. Each group of ten consists of five left-handed and five right-handed edges. Each left-handed edge is the reflexion of each right-handed edge in a plane of symmetry and so must intersect it. The two groups must therefore belong to the two systems of generators of a quadric surface. But it can be seen that two adjacent left-handed edges, for example, belong to the same vertex figure, and must therefore intersect. Thus the quadric surface degenerates to a cone, through whose vertex all the edges pass.

11. A polyhedron having eight faces at each vertex

As we saw in §10, the pentagrams of a given $| 3\ \frac{5}{3}\ \frac{5}{2}$ belong also to another, derived by reflexion in a certain plane (represented by the vertical line in figure 12). Three sides of the vertex figure are three sides of a square, whose fourth side belongs to the reflected vertex figure. Hence the 160 triangles of the two enantiomorphous $|\,3\,\frac{5}{3}\,\frac{5}{2}$'s are the faces of twenty concentric octahedra, and those faces of one octahedron which belong to one $|\,3\,\frac{5}{3}\,\frac{5}{2}$ consist of one 'special' face and its three neighbours. Each of the twenty octahedra has one pair of opposite faces each belonging to the same $|\,3\,\frac{5}{3}\,\frac{5}{2}$ as all its three neighbours. In other words, the octahedron has two opposite 'special' faces, and its eight faces fall into two connected sets of four, each set belonging to one $|\,3\,\frac{5}{3}\,\frac{5}{2}$. The forty 'special' triangles, along

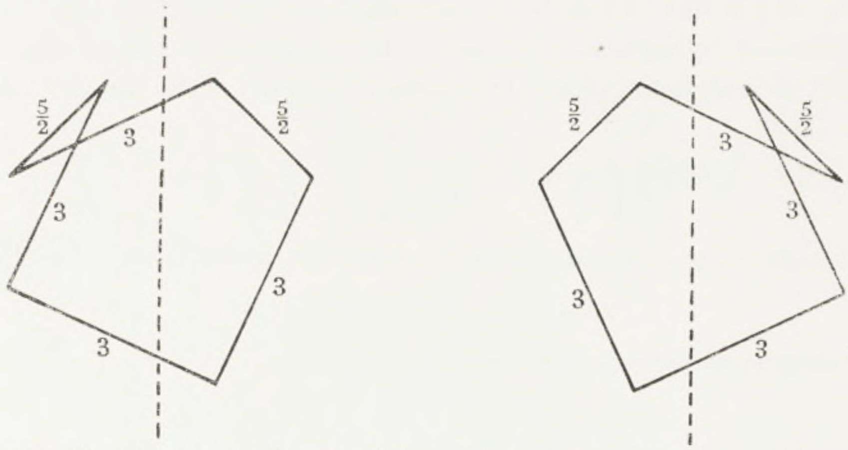


FIGURE 12. The vertex figures of the laevo and dextro varieties of $|\,3\,\frac{5}{3}\,\frac{5}{2}$.

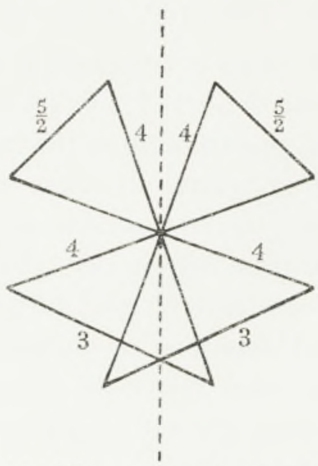


FIGURE 13. The vertex figure of $|\,\frac{3}{2}\,\frac{5}{3}\,3\,\frac{5}{2}$.

with the sixty equatorial squares of the twenty octahedra and the twenty-four common pentagrams of the two $|\,3\,\frac{5}{3}\,\frac{5}{2}$'s, form a single polyhedron whose vertex figure is shown in figure 13.

This is the only known polyhedron that has more than six faces at every vertex. It is also interesting to note that the faces of all three kinds occur in coplanar pairs: twelve pairs of pentagrams, twenty pairs of triangles and thirty pairs of diametral squares. If the faces at a vertex are taken in succession, the four squares occur alternately with the other faces. Moreover, the squares have no rotational symmetry; the only transformation (besides the identical transformation) which transforms any square into itself is the reflexion in the centre of the polyhedron. Analogy suggests that the squares be regarded as 'snub' faces, so that an appropriate symbol for this strange figure is

$$|\,\frac{3}{2}\,\frac{5}{3}\,3\,\frac{5}{2}.$$

12. Conclusion

Using a moderately systematic procedure, we have obtained the five Platonic solids, the thirteen Archimedean solids, the four Kepler-Poinsot star-polyhedra, the prisms and antiprisms, and fifty-three other uniform polyhedra. These, with their faces and vertex figures, are shown in our plates (figures 15 to 128) and described in table 7. (Figures 15 to 92 follow approximately the order of table 7.)

The number of vertices, say N_0, is obvious from Wythoff's construction. The number of edges, say N_1, is given by

$$2N_1 = N_0 N_{01},$$

where N_{01} is the number of edges at each vertex, or the number of vertices of the vertex figure. Similarly, the number of faces having n sides is given by

$$nN_2 = N_0 N_{02},$$

where N_{02} is the number of such faces at each vertex, or the number of sides of the vertex figure having the appropriate length (table 3). The density d is given by (6·1), and agrees with table 5 except in the cases explained at the end of §§ 9 and 10. We do not attempt to assign a density in those cases where some faces pass through the centre, nor in those where the vertex figure is a crossed parallelogram.

In the next column of table 7 (after the density d) we give the circum-radius for edge 2, which is $\operatorname{cosec}\phi$, where 2ϕ is the angle subtended by an edge at the centre. Since the circum-radius of the vertex figure is

$$\rho = \cos\phi$$

(Coxeter 1948, p. 22) we have

$$\operatorname{cosec}\phi = (1-\rho^2)^{-\frac{1}{2}}.$$

It is interesting to observe that, whenever the squared circum-radius $\operatorname{cosec}^2\phi$ is a quadratic surd $m+n\sqrt{2}$ or $m+n\sqrt{5}$ (where m and n are rational), there is a *conjugate* polyhedron in which n is replaced by $-n$. This corresponds to an interchange of octagons and octagrams, or pentagons and pentagrams, or decagons and decagrams. Conjugate polyhedra are isomorphic (Coxeter 1948, p. 106). Some polyhedra, such as $\begin{Bmatrix}\frac{5}{2}\\5\end{Bmatrix}$, are self-conjugate, so that $\operatorname{cosec}^2\phi$ is rational (usually an integer). However, this kind of correspondence is not universal; it breaks down for the snub polyhedra.

It is remarkable that we have obtained all but one of the known uniform polyhedra by applying Wythoff's construction to the various Schwarz triangles. The existence of $|\,\frac{3}{2}\,\frac{5}{3}\,3\,\frac{5}{2}$ indicates that there is no general reason for the restriction to triangles. We can only say that higher spherical polygons would have to satisfy various conditions which are almost always incompatible. In support of our contention that our list (table 7) is probably complete, we may mention that it includes all the uniform polyhedra previously obtained by other authors, using different methods.

The most systematic of these earlier constructions is that of Badoureau (1881, pp. 104–158), who considered each of the Platonic and Archimedean solids in turn with a view to seeing whether any star-polyhedra can have the same vertices. We may summarize his results (and some similar cases where one star-polyhedron is inscribed in another) by remarking that the polyhedra listed in each of the following lines all have the same vertices; those in each of the subgroups (separated by semi-colons) have also the same edges:

$$4\,|\,2\,3,\quad \tfrac32\,3\,|\,2.$$

$$2\,|\,3\,4,\quad \tfrac32\,3\,|\,3,\quad \tfrac43\,4\,|\,3.$$

$$2\,3\,|\,4;\quad \tfrac32\,4\,|\,2,\quad 3\,4\,|\,\tfrac43,\quad 2\,\tfrac43\begin{smallmatrix}\frac32\\ \frac42\end{smallmatrix}\Big|.$$

$$2\,3\,|\,\tfrac43;\quad 3\,4\,|\,2,\quad \tfrac32\,4\,|\,4,\quad 2\,4\begin{smallmatrix}\frac32\\ \frac42\end{smallmatrix}\Big|.$$

$$5\,|\,2\,3,\quad \tfrac52\,|\,2\,5;\quad \tfrac52\,|\,2\,3,\quad 5\,|\,2\,\tfrac52.$$

$$3\,|\,2\,5;\quad 3\,|\,2\,\tfrac52;\quad 3\,|\,3\,\tfrac52,\quad \tfrac32\,|\,3\,5,\quad 3\,|\,\tfrac53\,5.$$

$$2\,|\,3\,5,\quad \tfrac32\,3\,|\,5,\quad \tfrac54\,5\,|\,5;\quad 2\,|\,\tfrac52\,5,\quad \tfrac53\,\tfrac52\,|\,3,\quad \tfrac54\,5\,|\,3;\quad 2\,|\,3\,\tfrac52,\quad \tfrac32\,3\,|\,\tfrac53,\quad \tfrac53\,\tfrac52\,|\,\tfrac53.$$

$$2\,3\,|\,5;\quad 3\,5\,|\,\tfrac53,\quad \tfrac32\,5\,|\,3,\quad 3\,\tfrac53\begin{smallmatrix}\frac32\\ \frac52\end{smallmatrix}\Big|.$$

$$2\,3\,|\,\tfrac53;\quad 3\,\tfrac53\,|\,5,\quad 3\,\tfrac52\,|\,3,\quad 3\,5\begin{smallmatrix}\frac32\\ \frac54\end{smallmatrix}\Big|.$$

$$2\,\tfrac52\,|\,5;\quad 3\,\tfrac53\,|\,2,\quad 3\,\tfrac52\,|\,\tfrac53,\quad 2\,\tfrac53\begin{smallmatrix}\frac32\\ \frac54\end{smallmatrix}\Big|.$$

$$2\,5\,|\,\tfrac53;\quad 3\,5\,|\,2,\quad \tfrac32\,5\,|\,5,\quad 2\,5\begin{smallmatrix}\frac32\\ \frac52\end{smallmatrix}\Big|.$$

$$\tfrac52\,5\,|\,2,\quad \tfrac53\,5\,|\,3,\quad 2\,3\begin{smallmatrix}\frac52\\ \frac54\end{smallmatrix}\Big|.$$

The polyhedra described by Pitsch (1881, pp. 86, 87) are, in his order of Roman numerals,

$$2\,q\,|\,2,\quad |\,2\,2\,q,\quad 3\,4\,|\,\tfrac43,\quad 3\,5\,|\,\tfrac53,\quad —,\quad 2\,\tfrac52\,|\,5,\quad 3\,\tfrac52\,|\,\tfrac53,$$
$$\tfrac52\,5\,|\,2,\quad 3\,\tfrac43\,4\,|,\quad 2\,3\,\tfrac43\,|,\quad 3\,\tfrac53\,5\,|,\quad 2\,\tfrac53\,5\,|,\quad 2\,3\,\tfrac53\,|,\quad 2\,3\,|\,\tfrac43,$$
$$2\,|\,\tfrac52\,5,\quad 2\,|\,3\,\tfrac52,\quad 2\,5\,|\,\tfrac53,\quad 3\,\tfrac52\,|\,3,\quad 2\,\tfrac52\,|\,3,\quad 2\,3\,|\,\tfrac53,\quad 3\,|\,3\,\tfrac52.$$

(One is tempted to identify his V with $\mathrm{r}'\begin{Bmatrix}3\\ \frac52\end{Bmatrix} = 3\,\tfrac53\,|\,2$; but actually he described instead $\mathrm{r}\begin{Bmatrix}3\\ \frac52\end{Bmatrix} = 3\,\tfrac52\,|\,2$, which splits!) Many of these polyhedra were known to Brückner (1900, pp. 122—202). His illustrations, and those of Badoureau, are listed in table 7. The polyhedra of Lesavre & Mercier (1947) are, in their order,

$$|\,2\,\tfrac32\,\tfrac53,\quad |\,2\,3\,\tfrac52,\quad |\,2\,3\,\tfrac53,\quad |\,2\,\tfrac52\,5,\quad |\,2\,\tfrac53\,5.$$

Thus there remain seven polyhedra which are announced here for the first time:

$$|\,3\,3\,\tfrac52,\quad 3\,\tfrac53\,|\,5,\quad |\,3\,\tfrac53\,5,\quad 3\,5\begin{smallmatrix}\frac32\\ \frac54\end{smallmatrix}\Big|,\quad |\,3\,\tfrac53\,\tfrac52,\quad |\,\tfrac32\,\tfrac32\,\tfrac52,\quad |\,\tfrac32\,\tfrac53\,3\,\tfrac52.$$

Some of these polyhedra are 'edge-stellations', that is to say, their edges may be obtained by producing the edges of other polyhedra. Thus, as is well known, the regular polyhedra $5\,|\,2\,\tfrac52$ and $3\,|\,2\,\tfrac52$ are edge-stellations of $3\,|\,2\,5$ and $\tfrac52\,|\,2\,5$ respectively. It is less obvious that $\tfrac53\,\tfrac52\,|\,\tfrac53$ is an edge-stellation of $\tfrac54\,5\,|\,5$ and that $3\,\tfrac52\,|\,\tfrac53$ is an edge-stellation of $\tfrac32\,5\,|\,5$. These last two relations follow from the property that, when the edges of the decagons are produced to become edges of decagrams, the edges of the pentagons become edges of pentagrams (the edges of the stellated polygons being in the same ratio to the edges of the original polygons). It is clear, also, that any truncation containing two octagons or two decagons at a vertex can be stellated to give another truncation with two octagrams or two decagrams at a vertex. Thus $2\ 3\,|\,\frac{4}{3}$, $2\ 5\,|\,\frac{5}{3}$ and $2\ 3\,|\,\frac{5}{3}$ are edge-stellations, respectively, of $2\ 3\,|\,4$, $2\ 3\,|\,5$ and $2\ \frac{5}{2}\,|\,5$. (These relations can readily be appreciated from the wire models shown in plate 2.)

We may say that one polyhedron is *vertex-inscribed* in another if it has the same vertices but longer edges, and that it is *edge-inscribed* in another polyhedron if the latter is one of its edge-stellations. Then combining our results with those of Badoureau mentioned above we have the following 'chains':

$5\,\|\,2\ \frac{5}{2}$	is vertex-inscribed in	$\frac{5}{2}\,\|\,2\ 5$
$\frac{5}{2}\,\|\,2\ 5$	is edge-inscribed in	$3\,\|\,2\ \frac{5}{2}$
$3\,\|\,2\ \frac{5}{2}$	is vertex-inscribed in	$3\,\|\,2\ 5$
$3\,\|\,2\ 5$	is edge-inscribed in	$5\,\|\,2\ \frac{5}{2}$
etc.		

and

$\frac{5}{3}\ \frac{5}{2}\,\|\,\frac{5}{3}$	is vertex-inscribed in	$\frac{5}{4}\ 5\,\|\,5$
$\frac{5}{4}\ 5\,\|\,5$	is edge-inscribed in	$\frac{5}{3}\ \frac{5}{2}\,\|\,\frac{5}{3}$
etc.		

Both these chains are cyclic and so infinite. We have also a short chain in the octahedral group:

$3\ 4\,\|\,\frac{4}{3}$	is vertex-inscribed in	$2\ 3\,\|\,4$
$2\ 3\,\|\,4$	is edge-inscribed in	$2\ 3\,\|\,\frac{4}{3}$
$2\ 3\,\|\,\frac{4}{3}$	is vertex-inscribed in	$\frac{3}{2}\ 4\,\|\,4$

and the following remarkable chain in the icosahedral group:

$3\ 5\,\|\,\frac{5}{3}$	is vertex-inscribed in	$2\ 3\,\|\,5$
$2\ 3\,\|\,5$	is edge-inscribed in	$2\ 5\,\|\,\frac{5}{3}$
$2\ 5\,\|\,\frac{5}{3}$	is vertex-inscribed in	$\frac{3}{2}\ 5\,\|\,5$
$\frac{3}{2}\ 5\,\|\,5$	is edge-inscribed in	$3\ \frac{5}{2}\,\|\,\frac{5}{3}$
$3\ \frac{5}{2}\,\|\,\frac{5}{3}$	is vertex-inscribed in	$2\ \frac{5}{2}\,\|\,5$
$2\ \frac{5}{2}\,\|\,5$	is edge-inscribed in	$2\ 3\,\|\,\frac{5}{3}$
$2\ 3\,\|\,\frac{5}{3}$	is vertex-inscribed in	$3\ \frac{5}{3}\,\|\,5$

This last chain includes all the truncations of the icosahedral group which contain decagons or decagrams among their faces. One may imagine the eight polyhedra inscribed each inside the next, all the decagons and decagrams lying in the same set of twelve planes. The arrangement of the polygons in one of these planes is shown in figure 14.

Our notation extends readily to tessellations filling the Euclidean plane. Eleven simple tessellations having finite, non-overlapping faces (analogous to the Platonic and Archimedean solids) were described by Kepler (1619, pp. 116–120). Many other 'assemblages' were discovered by Badoureau (1881, pp. 163–170). To these we have added four ('conjugate' to Badoureau's Figs. 61, 65, 66, 67, which are Kepler's S, V, N, M) and one other: the snub tessellation $|\,\frac{4}{3}\ 4\ \infty$. Our list, which we believe to be complete, is given in table 8.

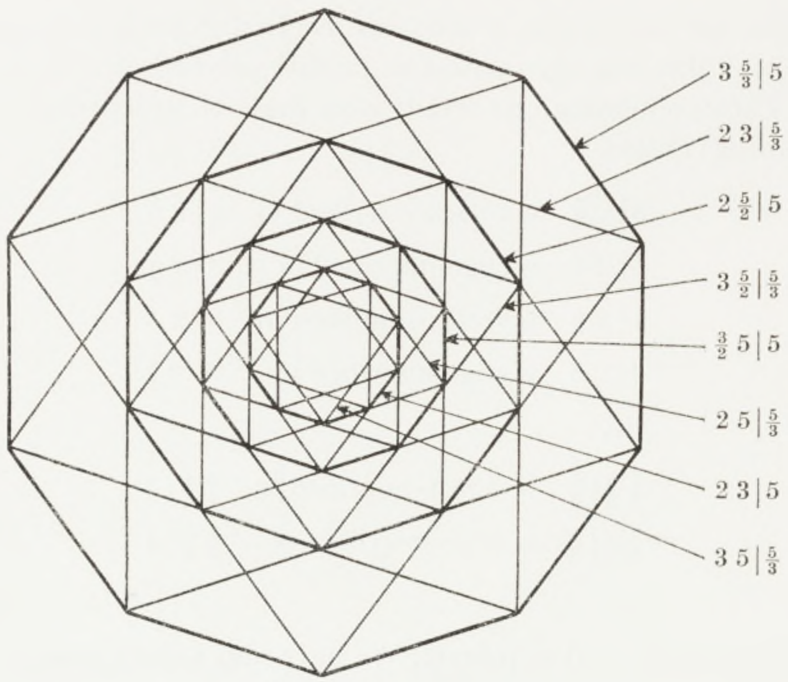


FIGURE 14

In most cases there are infinitely many faces of each kind; but $2\,\infty\,|\,2$ and $|\,2\ 2\,\infty$ have each just two apeirogons $\{\infty\}$, bounding a strip of squares in the former and of triangles in the latter. Alternate strips of these two kinds are used to form the two nameless tessellations at the end of the table.

TABLE 5. THE SCHWARZ TRIANGLES

density		density	
1	$(2\ 3\ 3)$, $(2\ 3\ 4)$, $(2\ 3\ 5)$	16	$(3\ \frac{5}{4}\ \frac{5}{2})$
d	$(2\ 2\ n/d)$	17	$(2\ \frac{3}{2}\ \frac{5}{2})$
2	$(\frac{3}{2}\ 3\ 3)$, $(\frac{3}{2}\ 4\ 4)$, $(\frac{3}{2}\ 5\ 5)$, $(\frac{5}{2}\ 3\ 3)$	18	$(\frac{3}{2}\ 3\ \frac{5}{3})$, $(\frac{5}{3}\ \frac{5}{3}\ \frac{5}{2})$
3	$(2\ \frac{3}{2}\ 3)$, $(2\ \frac{5}{2}\ 5)$	19	$(2\ 3\ \frac{5}{4})$
4	$(3\ \frac{4}{3}\ 4)$, $(3\ \frac{5}{3}\ 5)$	21	$(2\ \frac{5}{4}\ \frac{5}{2})$
5	$(2\ \frac{3}{2}\ \frac{3}{2})$, $(2\ \frac{3}{2}\ 4)$	22	$(\frac{3}{2}\ \frac{3}{2}\ \frac{5}{2})$
6	$(\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2})$, $(\frac{5}{2}\ \frac{5}{2}\ \frac{5}{2})$, $(\frac{3}{2}\ 3\ 5)$, $(\frac{5}{4}\ 5\ 5)$	23	$(2\ \frac{3}{2}\ \frac{5}{3})$
7	$(2\ 3\ \frac{4}{3})$, $(2\ 3\ \frac{5}{2})$	26	$(\frac{3}{2}\ \frac{5}{3}\ \frac{5}{3})$
8	$(\frac{3}{2}\ \frac{5}{2}\ 5)$	27	$(2\ \frac{5}{4}\ \frac{5}{3})$
9	$(2\ \frac{5}{3}\ 5)$	29	$(2\ \frac{3}{2}\ \frac{5}{4})$
10	$(3\ \frac{5}{3}\ \frac{5}{2})$, $(3\ \frac{5}{4}\ 5)$	32	$(\frac{3}{2}\ \frac{5}{4}\ \frac{5}{3})$
11	$(2\ \frac{3}{2}\ \frac{4}{3})$, $(2\ \frac{3}{2}\ 5)$	34	$(\frac{3}{2}\ \frac{3}{2}\ \frac{5}{4})$
13	$(2\ 3\ \frac{5}{3})$	38	$(\frac{3}{2}\ \frac{5}{4}\ \frac{5}{4})$
14	$(\frac{3}{2}\ \frac{4}{3}\ \frac{4}{3})$, $(\frac{3}{2}\ \frac{5}{2}\ \frac{5}{2})$, $(3\ 3\ \frac{5}{4})$	42	$(\frac{5}{4}\ \frac{5}{4}\ \frac{5}{4})$

TABLE 6. DEGENERATE CASES OF WYTHOFF'S CONSTRUCTION

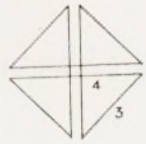
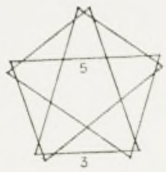
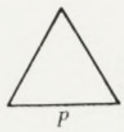
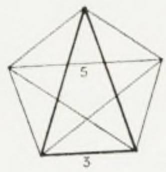
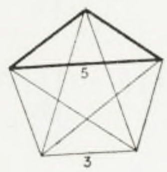
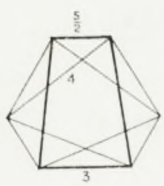
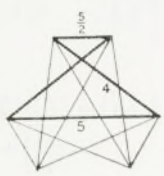
compound	density	vertex figure
$4\,\vert\,\frac{3}{2}\,4 = -\{3, 4\} + 3\{4, 2\}$	$2 = -1+3$	
$5\,\vert\,\frac{3}{2}\,5 = -\{3, 5\} + \{5, \frac{5}{2}\}$	$2 = -1+3$	
$5\,\vert\,3\,\frac{5}{3} = \{3, \frac{5}{2}\} - \{\frac{5}{2}, 5\}$	$4 = 7-3$	
$\frac{5}{2}\,\vert\,3\,5 = \{3, 5\} + \{5, \frac{5}{2}\}$	$4 = 1+3$	
$\frac{5}{3}\,\vert\,3\,\frac{5}{2} = \{3, \frac{5}{2}\} + \{\frac{5}{2}, 5\}$	$10 = 7+3$	
$2\,p\,\Big\vert\,\frac{p}{2} = 3\{p, 3\} \quad (p = 3, 5, \frac{5}{2})$	3, 3, 21	
$2\,3\,\vert\,\frac{5}{2} = \{3, 5\} + 2\{5, \frac{5}{2}\}$	$7 = 1+2.3$	
$2\,4\,\vert\,\frac{3}{2} = 3\{4, 2\} + 2\{3, 4\}$	$5 = 3+2$	
$2\,5\,\vert\,\frac{3}{2} = -\{5, \frac{5}{2}\} + 2\{3, 5\}$	$11 = -3+2+12$	
$2\,\frac{5}{2}\,\vert\,\frac{3}{2} = \{\frac{5}{2}, 5\} + 2\{3, \frac{5}{2}\}$	$17 = 3+2.7$	
$2\,3\,\vert\,\frac{5}{4} = -\{3, \frac{5}{2}\} + 2\{\frac{5}{2}, 5\}$	$19 = -7+2.3+20$	
$3\,\frac{5}{2}\,\vert\,2 = (3\,\vert\,3\,\frac{5}{2}) + 5\{4, 3\}$	$7 = 2+5$	
$\frac{5}{3}\,5\,\vert\,2 = (3\,\vert\,\frac{5}{3}\,5) + 5\{4, 3\}$	$9 = 4+5$	
$\frac{3}{2}\,5\,\vert\,2 = -(\frac{3}{2}\,\vert\,3\,5) + 5\{4, 3\}$	$11 = -6+5+12$	

TABLE 6 (*cont.*)

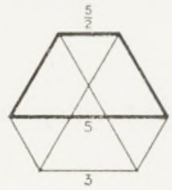
$3\ 5\,|\,\tfrac{3}{2} = 3\{3,\ 5\}+\{5,\ \tfrac{5}{2}\}$ $\qquad 6=3+3$

$\tfrac{3}{2}\ 5\,|\,\tfrac{5}{2} = -\{3,\ 5\}+3\{5,\ \tfrac{5}{2}\}$ $\qquad 8=-1+3.3$

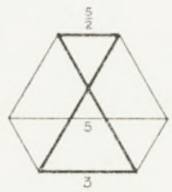
$3\ \tfrac{5}{2}\,|\,\tfrac{5}{4} = \{3,\ \tfrac{5}{2}\}+3\{\tfrac{5}{2},\ 5\}$ $\qquad 16=7+3.3$

$3\ \tfrac{5}{3}\,|\,\tfrac{3}{2} = 3\{3,\ \tfrac{5}{2}\}-\{\tfrac{5}{2},\ 5\}$ $\qquad 18=3.7-3$

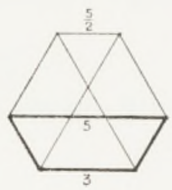
$\tfrac{5}{2}\ 5\,|\,\tfrac{3}{2} = (3\,|\,3\ \tfrac{5}{2})+(\tfrac{3}{2}\,|\,3\ 5)$ $\qquad 8=2+6$



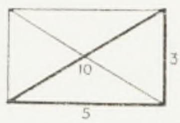
$3\ \tfrac{5}{3}\,|\,\tfrac{5}{2} = (\tfrac{3}{2}\,|\,3\ 5)+(3\,|\,\tfrac{5}{3}\ 5)$ $\qquad 10=6+4$



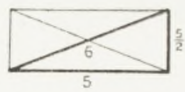
$3\ 5\,|\,\tfrac{5}{4} = (3\,|\,3\ \tfrac{5}{2})-(3\,|\,\tfrac{5}{3}\ 5)$ $\qquad 10=2-4+12$



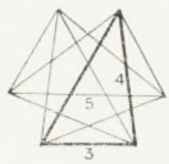
$$\tfrac{3}{2}\ \tfrac{5}{2}\ 5\,| = (\tfrac{3}{2}\ 3\,|\,5)+(\tfrac{5}{4}\ 5\,|\,5) = 2\begin{Bmatrix}3\\5\end{Bmatrix}+\mathit{6}\{10,\ 2\}$$ $\qquad 8=2+6$



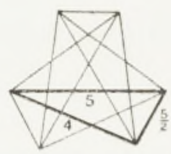
$$3\ \tfrac{5}{4}\ \tfrac{5}{2}\,| = (\tfrac{5}{3}\ \tfrac{5}{2}\,|\,3)+(\tfrac{5}{4}\ 5\,|\,3) = 2\begin{Bmatrix}\tfrac{5}{2}\\5\end{Bmatrix}+\mathit{10}\{6,\ 2\}$$ $\qquad 16=2.3+10$



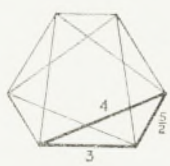
$2\ \tfrac{3}{2}\ \tfrac{5}{2}\,| = \mathit{5}\{4,\ 3\}+2(\tfrac{3}{2}\,|\,3\ 5)$ $\qquad 17=5+2.6$



$2\ \tfrac{5}{4}\ \tfrac{5}{2}\,| = \mathit{5}\{4,\ 3\}-2(3\,|\,\tfrac{5}{3}\ 5)$ $\qquad 21=5-2.4+2.12$



$2\ \tfrac{3}{2}\ \tfrac{5}{4}\,| = -\mathit{5}\{4,\ 3\}+2(3\,|\,3\ \tfrac{5}{2})$ $\qquad 29=-5+2.2+30$



$$\tfrac{3}{2}\ \tfrac{5}{4}\ \tfrac{5}{3}\,| = (\tfrac{3}{2}\ 3\,|\,\tfrac{5}{3})+(\tfrac{5}{3}\ \tfrac{5}{2}\,|\,\tfrac{5}{3}) = 2\begin{Bmatrix}\tfrac{5}{2}\\3\end{Bmatrix}+\mathit{6}\{\tfrac{10}{3},\ 2\}$$ $\qquad 32=2.7+6.3$

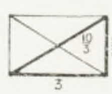


TABLE 6 (*cont.*)

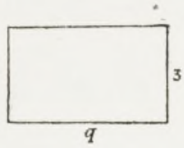
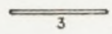
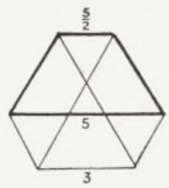
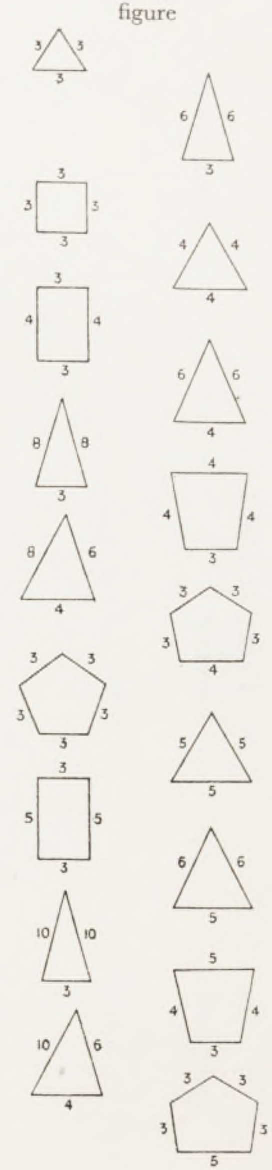
compound	density	vertex figure
$\mid \frac52\,\frac52\,\frac52 = 3(3 \mid 3\,\tfrac52)$	$6 = 3.2$	
$\mid \frac54\,\frac54\,\frac54 = -3(\tfrac32 \mid 3\,5)$	$42 = -3.6+60$	
$\mid \frac32\,q\,q = 2\begin{Bmatrix}3\\ q\end{Bmatrix}\quad (q=3,4,5,\tfrac52)$	2, 2, 2, 14	
$\mid 2\,2\,\frac32 = 2\{3, 2\}$	2	
$\mid 2\,\frac32\,3 = 3\{3, 3\}$	3	
$\mid 2\,\frac32\,4 = 2\{3, 4\} + 3\{4, 2\}$	$5 = 2+3$	
$\mid \frac32\,3\,5 = 6s\begin{Bmatrix}2\\ 5\end{Bmatrix} = 3\{3, 5\} + \{5, \tfrac52\}$	$6 = 3+3$	
$\mid \frac32\,\frac52\,5 = (3 \mid 3\,\tfrac52) + (\tfrac32 \mid 3\,5)$	$8 = 2+6$	
$\mid 2\,\frac32\,5 = 2\{3, 5\} - \{5, \tfrac52\}$	$11 = 2-3+12$	
$\mid 2\,\frac32\,\frac52 = 2\{3, \tfrac52\} + \{\tfrac52, 5\}$	$17 = 2.7+3$	
$\mid \frac32\,3\,\frac53 = 6s'\begin{Bmatrix}2\\ \frac52\end{Bmatrix} = 3\{3, \tfrac52\} - \{\tfrac52, 5\}$	$18 = 6.3 = 3.7-3$	
$\mid \frac32\,\frac53\,\frac53 = -4\{3, \tfrac52\} - 2\{\tfrac52, 5\}$	$26 = -4.7-2.3+60$	
$\mid \frac32\,\frac54\,\frac54 = -4\{3, 5\} + 2\{5, \tfrac52\}$	$38 = -4+2.3+60-2.12$	

Table 7. Uniform polyhedra

polyhedron	N_0	N_1	N_2	d	circum-radius for edge 2	Badoureau's* figures	Brückner's* figures	figure no.	plate no.
$3\,\vert\,2\,3 = \{3, 3\}$	4	6	$4\{3\}$	1	$\sqrt{\frac{3}{2}}$	—	84	15	—
$2\,3\,\vert\,3 = \mathrm{t}\{3, 3\}$	12	18	$4\{3\} + 4\{6\}$	1	$\sqrt{\frac{11}{2}}$	28	VI, 22	16	—
$4\,\vert\,2\,3 = \{3, 4\}$	6	12	$8\{3\}$	1	$\sqrt{2}$	24	85	17	1, 93
$3\,\vert\,2\,4 = \{4, 3\}$	8	12	$6\{4\}$	1	$\sqrt{3}$	21	86	18	—
$2\,\vert\,3\,4 = \begin{Bmatrix}3\\4\end{Bmatrix}$	12	24	$8\{3\} + 6\{4\}$	1	2	30, 39	VI, 24	19	1, 94
$2\,4\,\vert\,3 = \mathrm{t}\{3, 4\}$	24	36	$8\{6\} + 6\{4\}$	1	$\sqrt{10}$	40	VI, 23	20	—
$2\,3\,\vert\,4 = \mathrm{t}\{4, 3\}$	24	36	$8\{3\} + 6\{8\}$	1	$\sqrt{(7+4\sqrt{2})}$	38	VI, 27	21	—
$3\,4\,\vert\,2 = \mathrm{r}\begin{Bmatrix}3\\4\end{Bmatrix}$	24	48	$8\{3\} + (12+6)\{4\}$	1	$\sqrt{(5+2\sqrt{2})}$	42	VI, 29	22	3, 105
$2\,3\,4\,\vert = \mathrm{t}\begin{Bmatrix}3\\4\end{Bmatrix}$	48	72	$8\{6\} + 12\{4\} + 6\{8\}$	1	$\sqrt{(13+6\sqrt{2})}$	41	VI, 30	23	—
$\vert\,2\,3\,4 = \mathrm{s}\begin{Bmatrix}3\\4\end{Bmatrix}$	24	60	$(8+24)\{3\} + 6\{4\}$	1	2·68742 67475	36, 37, 43	VI, 33	24	—
$5\,\vert\,2\,3 = \{3, 5\}$	12	30	$20\{3\}$	1	$5^{\frac{1}{4}}\tau^{\frac{1}{2}}$	77, 78	87	25	—
$3\,\vert\,2\,5 = \{5, 3\}$	20	30	$12\{5\}$	1	$3^{\frac{1}{2}}\tau$	71, 72, 73	88	26	—
$2\,\vert\,3\,5 = \begin{Bmatrix}3\\5\end{Bmatrix}$	30	60	$20\{3\} + 12\{5\}$	1	2τ	45	VI, 26	28	1, 95
$2\,5\,\vert\,3 = \mathrm{t}\{3, 5\}$	60	90	$20\{6\} + 12\{5\}$	1	$\sqrt{\frac{29+9\sqrt{5}}{2}}$	46	VI, 25	27	—
$2\,3\,\vert\,5 = \mathrm{t}\{5, 3\}$	60	90	$20\{3\} + 12\{10\}$	1	$\sqrt{\frac{37+15\sqrt{5}}{2}}$	44	VI, 28	29	—
$3\,5\,\vert\,2 = \mathrm{r}\begin{Bmatrix}3\\5\end{Bmatrix}$	60	120	$20\{3\} + 30\{4\} + 12\{5\}$	1	$\sqrt{(11+4\sqrt{5})}$	48	VI, 31	30	3, 107
$2\,3\,5\,\vert = \mathrm{t}\begin{Bmatrix}3\\5\end{Bmatrix}$	120	180	$20\{6\} + 30\{4\} + 12\{10\}$	1	$\sqrt{(31+12\sqrt{5})}$	47	VI, 32	31	—
$\vert\,2\,3\,5 = \mathrm{s}\begin{Bmatrix}3\\5\end{Bmatrix}$	60	150	$(20+60)\{3\} + 12\{5\}$	1	4·31167 47491	49	VI, 34	32	—

vertex figure



* Those of Badoureau's drawings marked † are partial drawings only; those of Brückner's illustrations marked ‡ are of slightly different varieties. Those marked § are incorrect.

$2\,\frac{n}{d}\,\vert\,2 = \mathrm{t}\left\{2, \frac{n}{d}\right\}$	$2n$	$3n$	$2\left\{\frac{n}{d}\right\} + n\{4\}$	d	$\sqrt{\left(1+\operatorname{cosec}^2 \frac{\pi d}{n}\right)}$	79, 80	VIII, 17	33	2, 99
$\vert\,2\,2\,\frac{n}{d} = \mathrm{s}\left\{\begin{matrix}2\\ n/d\end{matrix}\right\}$ $(d<\frac{1}{2}n)$	$2n$	$4n$	$2\left\{\frac{n}{d}\right\} + 2n\{3\}$	d	$\sqrt{\left(1+\frac{1}{4}\operatorname{cosec}^2 \frac{\pi d}{2n}\right)}$	81, 83, 85	VIII, 13, 16	34	5, 117
$\vert\,2\,2\,\frac{n}{d} = \mathrm{s}'\left\{\begin{matrix}2\\ n/(n-d)\end{matrix}\right\}$ $(\frac{1}{2}n<d<\frac{2}{3}n)$	$2n$	$4n$	$2\left\{\frac{n}{n-d}\right\} + 2n\{3\}$	d	$\sqrt{\left(1+\frac{1}{4}\operatorname{cosec}^2 \frac{\pi d}{2n}\right)}$	82†, 84†	VIII, 20§	35	5, 118
$\frac{3}{2}\,3\,\vert\,3$	12	24	$8\{3\} + 4\{6\}$	—	2	97	VIII, 1	37	1, 94
$\frac{3}{2}\,4\,\vert\,4$	24	48	$8\{3\} + 6\{4\} + 6\{8\}$	2	$\sqrt{(5+2\sqrt{2})}$	104	VIII, 36‡	38	3, 105
$3\,\vert\,3\,\frac{5}{2}$	20	60	$20\{3\} + 12\{\frac{5}{2}\}$	2	$\sqrt{3}$	74	IX, 15	39	1, 98
$3\,\frac{5}{2}\,\vert\,3$	60	120	$20\{3\} + 20\{6\} + 12\{\frac{5}{2}\}$	2	$\sqrt{\frac{17+3\sqrt{5}}{2}}$	—	XII, 13	40	3, 109
$\vert\,3\,3\,\frac{5}{2}$	60	180	$(40+60)\{3\} + 12\{\frac{5}{2}\}$	2	2·91638 06615	—	—	41	5, 119
$\frac{3}{2}\,5\,\vert\,5$	60	120	$20\{3\} + 12\{5\} + 12\{10\}$	2	$\sqrt{(11+4\sqrt{5})}$	130	—	42	3, 107
$\frac{3}{2}\,3\,\vert\,2 = \mathrm{r}'\left\{\begin{matrix}3\\ 3\end{matrix}\right\}$	6	12	$4\{3\} + 3\{4\}$	—	$\sqrt{2}$	70	—	36	1, 93
$5\,\vert\,2\,\frac{5}{2} = \{\frac{5}{2}, 5\}$	12	30	$12\{\frac{5}{2}\}$	3	$5^{\frac{1}{4}}\tau^{-\frac{1}{2}}$	—	X, 5	43	—
$\frac{5}{2}\,\vert\,2\,5 = \{5, \frac{5}{2}\}$	12	30	$12\{5\}$	3	$5^{\frac{1}{4}}\tau^{\frac{1}{2}}$	—	IX, 7	44	—
$2\,\vert\,\frac{5}{2}\,5 = \left\{\begin{matrix}\frac{5}{2}\\ 5\end{matrix}\right\}$	30	60	$12\{\frac{5}{2}\} + 12\{5\}$	3	2	117	IX, 13	45	1, 97
$2\,\frac{5}{2}\,\vert\,5 = \mathrm{t}\{5, \frac{5}{2}\}$	60	90	$12\{\frac{5}{2}\} + 12\{10\}$	3	$\sqrt{\frac{17+5\sqrt{5}}{2}}$	—	IX, 2	47	2, 101
$\frac{5}{2}\,5\,\vert\,2 = \mathrm{r}\left\{\begin{matrix}\frac{5}{2}\\ 5\end{matrix}\right\}$	60	120	$12\{\frac{5}{2}\} + 30\{4\} + 12\{5\}$	3	$\sqrt{7}$	139†	XII, 14	48	4, 116
$2\,5\,\frac{3}{2}\,\frac{5}{2}\,\vert$	60	120	$30\{4\} + 12\{10\}$	—	$\sqrt{(11+4\sqrt{5})}$	131	XII, 5	46	3, 107
$\vert\,2\,\frac{5}{2}\,5$	60	150	$12\{\frac{5}{2}\} + 60\{3\} + 12\{5\}$	3	2·54887 97641	—	—	49	5, 121
$3\,4\,\vert\,\frac{4}{3}$	24	48	$8\{3\} + 6\{4\} + 6\{\frac{8}{3}\}$	4	$\sqrt{(5-2\sqrt{2})}$	91	IX, 8	50	3, 106
$\frac{4}{3}\,4\,\vert\,3$	12	24	$4\{6\} + 6\{4\}$	—	2	96	—	51	1, 94
$3\,\frac{4}{3}\,4\,\vert$	48	72	$8\{6\} + 6\{8\} + 6\{\frac{8}{3}\}$	4	$\sqrt{7}$	137	XI, 12	52	4, 111

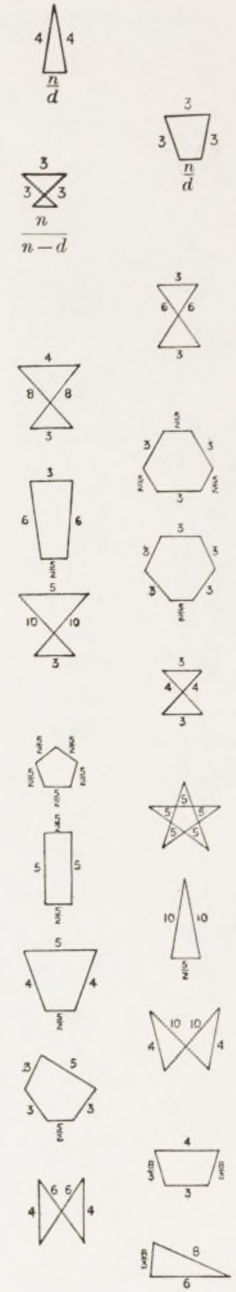
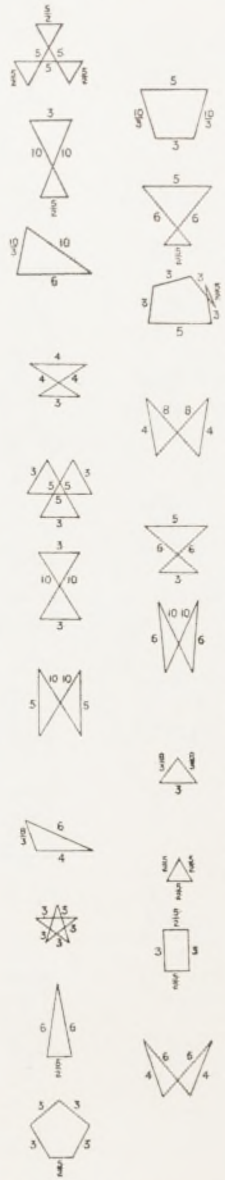


TABLE 7. (*cont.*)

polyhedron	N_0	N_1	N_2	d	circum-radius for edge 2	Badoureau's* figures	Brückner's* figures	figure no.	plate no.
$3 \mid \frac{5}{3}\, 5$	20	60	$12\{\frac{5}{2}\} + 12\{5\}$	4	$\sqrt{3}$	76†	—	53	1, 98
$3\, 5 \mid \frac{5}{3}$	60	120	$20\{3\} + 12\{5\} + 12\{\frac{10}{3}\}$	4	$\sqrt{\frac{17-3\sqrt{5}}{2}}$	109	XII, 15	54	3, 110
$3\, \frac{5}{3} \mid 5$	60	120	$20\{3\} + 12\{\frac{5}{2}\} + 12\{10\}$	4	$\sqrt{\frac{17+3\sqrt{5}}{2}}$	—	—	55	3, 109
$\frac{5}{3}\, 5 \mid 3$	60	120	$20\{6\} + 12\{\frac{5}{2}\} + 12\{5\}$	4	$\sqrt{7}$	140†	—	56	4, 116
$3\, \frac{5}{3}\, 5 \mid$	120	180	$20\{6\} + 12\{10\} + 12\{\frac{10}{3}\}$	4	4	148†	IX, 19‡	57	4, 113
$\mid 3\, \frac{5}{3}\, 5$	60	180	$(20+60)\{3\} + 12\{\frac{5}{2}\} + 12\{5\}$	4	2·25379 58256	—	—	58	6, 123
$\frac{3}{2}\, 4 \mid 2 = r'\begin{Bmatrix}3\\4\end{Bmatrix}$	24	48	$8\{3\} + (12+6)\{4\}$	5	$\sqrt{(5-2\sqrt{2})}$	93†	—	59	3, 106
$2\, 4\, \frac{3}{2}\, \frac{4}{2} \mid$	24	48	$12\{4\} + 6\{8\}$	—	$\sqrt{(5+2\sqrt{2})}$	105†	—	60	3, 105
$\frac{3}{2} \mid 3\, 5$	20	60	$20\{3\} + 12\{5\}$	6	$\sqrt{3}$	75†	—	61	1, 98
$\frac{3}{2}\, 5 \mid 3$	60	120	$20\{3\} + 20\{6\} + 12\{5\}$	6	$\sqrt{\frac{17-3\sqrt{5}}{2}}$	111	—	62	3, 110
$\frac{3}{2}\, 3 \mid 5$	30	60	$20\{3\} + 6\{10\}$	—	2τ	116	VII, 35	63	1, 95
$3\, 5\, \frac{3}{2}\, \frac{5}{4} \mid$	60	120	$20\{6\} + 12\{10\}$	—	$\sqrt{\frac{17+3\sqrt{5}}{2}}$	—	—	64	3, 109
$\frac{5}{4}\, 5 \mid 5$	30	60	$12\{5\} + 6\{10\}$	—	2τ	115	—	65	1, 95
$2\, 3 \mid \frac{4}{3} = t'\{4, 3\}$	24	36	$8\{3\} + 6\{\frac{8}{3}\}$	7	$\sqrt{(7-4\sqrt{2})}$	106§	VIII, 5‡	66	2, 100
$2\, 3\, \frac{4}{3} \mid = t'\begin{Bmatrix}3\\4\end{Bmatrix}$	48	72	$8\{6\} + 12\{4\} + 6\{\frac{8}{3}\}$	1	$\sqrt{(13-6\sqrt{2})}$	136†	VIII, 23‡	67	4, 112
$3 \mid 2\, \frac{5}{2} = \{\frac{5}{2}, 3\}$	20	30	$12\{\frac{5}{2}\}$	7	$3^{\frac{1}{2}}\tau^{-1}$	—	IX, 16	68	—
$\frac{5}{2} \mid 2\, 3 = \{3, \frac{5}{2}\}$	12	30	$20\{3\}$	7	$5^{\frac{1}{4}}\tau^{-\frac{1}{2}}$	—	XI, 24	69	—
$2 \mid 3\, \frac{5}{2} = \begin{Bmatrix}3\\ \frac{5}{2}\end{Bmatrix}$	30	60	$20\{3\} + 12\{\frac{5}{2}\}$	7	$2\tau^{-1}$	120†	XI, 9	70	1, 96
$2\, \frac{5}{2} \mid 3 = t\{3, \frac{5}{2}\}$	60	90	$20\{6\} + 12\{\frac{5}{2}\}$	7	$\sqrt{\frac{29-9\sqrt{5}}{2}}$	—	IX, 9	71	2, 103
$2\, 3\, \frac{5}{4}\, \frac{5}{2} \mid$	60	120	$20\{6\} + 30\{4\}$	—	$\sqrt{7}$	141†	—	72	4, 116
$\mid 2\, 3\, \frac{5}{2}$	60	150	$(20+60)\{3\} + 12\{\frac{5}{2}\}$	7	1·63216 13496	—	—	73	6, 124

vertex figure



* Those of Badoureau's drawings marked † are partial drawings only; those of Brückner's illustrations marked ‡ are of slightly different varieties. Those marked § are incorrect.

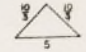
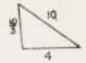
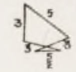
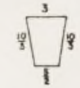
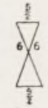
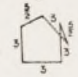
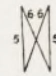
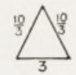
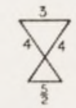
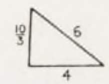
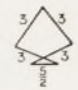
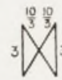
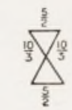
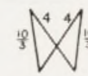
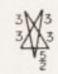
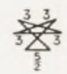
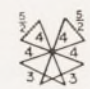
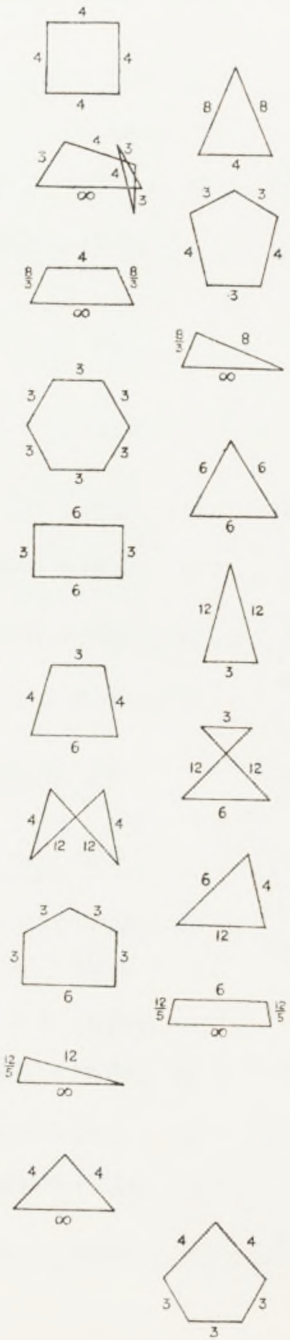
$2\ 5 \mid \frac{5}{3} = t'\{\frac{5}{2}, 5\}$	60	90	$12\{5\} + 12\{\frac{10}{3}\}$	9	$\sqrt{\frac{17-5\sqrt{5}}{2}}$	132§	XII, 23	74	2, 102	
$2\ \frac{5}{3}\ 5 \mid = t'\left\{\begin{matrix}\frac{5}{2}\\5\end{matrix}\right\}$	120	180	$12\{10\}+30\{4\}+12\{\frac{10}{3}\}$	3	$\sqrt{11}$	147†	VII, 33‡	75	4, 114	
$\mid 2\ \frac{5}{3}\ 5$	60	150	$12\{\frac{5}{2}\}+60\{3\}+12\{5\}$	9	1·70326 04562	—	—	76	5, 122	
$3\ \frac{5}{2} \mid \frac{5}{3}$	60	120	$20\{3\}+12\{\frac{5}{2}\}+12\{\frac{10}{3}\}$	10	$\sqrt{(11-4\sqrt{5})}$	143†	XII, 18	77	3, 108	
$\frac{5}{3}\ \frac{5}{2} \mid 3$	30	60	$10\{6\} + 12\{\frac{5}{2}\}$	—	2	119	XII, 1	78	1, 97	
$3\ \frac{5}{3}\ \frac{3/2}{5/2} \mid$	60	120	$20\{6\} + 12\{\frac{10}{3}\}$	—	$\sqrt{\frac{17-3\sqrt{5}}{2}}$	110†	—	79	3, 110	
$\mid 3\ \frac{5}{3}\ \frac{5}{2}$	60	180	$(20+60)\{3\}+(12+12)\{\frac{5}{2}\}$	10	$\sqrt{2}$	—	—	80	6, 127	
$\frac{5}{4}\ 5 \mid 3$	30	60	$10\{6\} + 12\{5\}$	—	2	118†	—	81	1, 97	
$2\ \frac{4}{3}\ \frac{3/2}{4/2} \mid$	24	48	$12\{4\} + 6\{\frac{8}{3}\}$	—	$\sqrt{(5-2\sqrt{2})}$	92†	—	82	3, 106	
$2\ 3 \mid \frac{5}{3} = t'\{\frac{5}{2}, 3\}$	60	90	$20\{3\} + 12\{\frac{10}{3}\}$	13	$\sqrt{\frac{37-15\sqrt{5}}{2}}$	—	IX, 21	83	2, 104	
$3\ \frac{5}{3} \mid 2 = r'\left\{\begin{matrix}3\\\frac{5}{2}\end{matrix}\right\}$	60	120	$20\{3\}+30\{4\}+12\{\frac{5}{2}\}$	13	$\sqrt{(11-4\sqrt{5})}$	144†	—	84	3, 108	
$2\ 3\ \frac{5}{3} \mid = t'\left\{\begin{matrix}3\\\frac{5}{2}\end{matrix}\right\}$	120	180	$20\{6\}+30\{4\}+12\{\frac{10}{3}\}$	13	$\sqrt{(31-12\sqrt{5})}$	149†	—	87	4, 115	
$\mid 2\ 3\ \frac{5}{3}$	60	150	$(20+60)\{3\}+12\{\frac{5}{2}\}$	13	1·29004 04746	—	—	88	6, 125	
$\frac{3}{2}\ 3 \mid \frac{5}{3}$	30	60	$20\{3\} + 6\{\frac{10}{3}\}$	—	$2\tau^{-1}$	121†	—	85	1, 96	
$\frac{5}{3}\ \frac{5}{2} \mid \frac{5}{3}$	30	60	$12\{\frac{5}{2}\} + 6\{\frac{10}{3}\}$	—	$2\tau^{-1}$	122†	—	86	1, 96	
$2\ \frac{5}{3}\ \frac{3/2}{5/4} \mid$	60	120	$30\{4\}+12\{\frac{10}{3}\}$	—	$\sqrt{(11-4\sqrt{5})}$	145†	—	89	3, 108	
$\mid 2\ \frac{3}{2}\ \frac{5}{3}$	60	150	$(20+60)\{3\}+12\{\frac{5}{2}\}$	37	1·16000 30093	—	—	90	6, 126	
$\mid \frac{3}{2}\ \frac{3}{2}\ \frac{5}{2}$	60	180	$(40+60)\{3\}+12\{\frac{5}{2}\}$	38	1·16138 96003	—	—	91	5, 120	
$\mid \frac{3}{2}\ \frac{5}{3}\ 3\ \frac{5}{2}$	60	240	$40\{3\}+60\{4\}+24\{\frac{5}{2}\}$	—	$\sqrt{2}$	—	—	92	6, 128	

TABLE 8. UNIFORM TESSELLATIONS

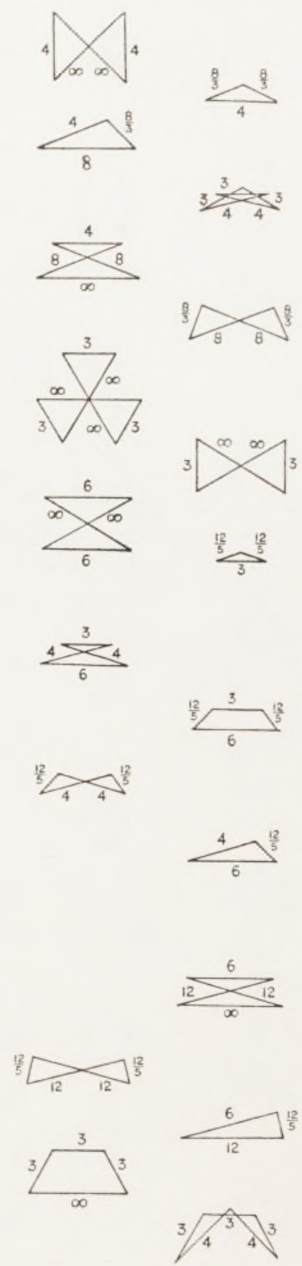
tessellation	Kepler's figure	Badoureau's figure
$4 \mid 2\ 4 = \{4, 4\}$	E	—
$2\ 4 \mid 4 = t\{4, 4\}$	V	65
$\mid \frac{4}{3}\ 4\ \infty$	—	—
$\mid 2\ 4\ 4 = s\begin{Bmatrix}4\\4\end{Bmatrix}$	N	66
$4\ \infty \mid \frac{4}{3}$	—	151
$\frac{4}{3}\ 4\ \infty \mid$	—	153
$6 \mid 2\ 3 = \{3, 6\}$	D	—
$3 \mid 2\ 6 = \{6, 3\}$	F	—
$2 \mid 3\ 6 = \begin{Bmatrix}3\\6\end{Bmatrix}$	P	62
$2\ 3 \mid 6 = t\{6, 3\}$	S	61
$3\ 6 \mid 2 = r\begin{Bmatrix}3\\6\end{Bmatrix}$	Ii	64
$\frac{3}{2}\ 6 \mid 6$	Ii	64
$2\ 6\ {\frac{3}{2} \atop 3} \mid$	Ii	64
$2\ 3\ 6 \mid = t\begin{Bmatrix}3\\6\end{Bmatrix}$	Mm	63
$\mid 2\ 3\ 6 = s\begin{Bmatrix}3\\6\end{Bmatrix}$	L	—
$6\ \infty \mid \frac{6}{5}$	—	154
$\frac{6}{5}\ 6\ \infty \mid$	—	157
$2\ \infty \mid 2 = t\{2, \infty\}$	—	59
—	(M)	67

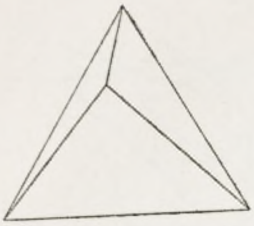
vertex figure



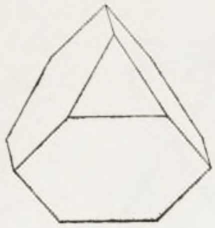
tessellation	Kepler's figure	Badoureau's figure
$\frac{4}{3}\ 4 \mid \infty$	E	—
$2\ 4 \mid \frac{4}{3} = t'\{4, 4\}$	—	—
$2\ \frac{4}{3}\ 4 \mid = t'\begin{Bmatrix}4\\4\end{Bmatrix}$	—	152
$\mid 2\ \frac{4}{3}\ \frac{4}{3} = s'\begin{Bmatrix}4\\4\end{Bmatrix}$	—	—
$\frac{4}{3}\ \infty \mid 4$	—	151
$\frac{4}{3}\ 4\ {2 \atop \infty} \mid$	—	151
$\frac{3}{2} \mid 3\ \infty$	D	—
$\frac{3}{2}\ 3 \mid \infty$	P	62
$\frac{6}{5}\ 6 \mid \infty$	P	62
$2\ 3 \mid \frac{6}{5} = t'\{6, 3\}$	—	—
$\frac{3}{2}\ 6 \mid 2 = r'\begin{Bmatrix}3\\6\end{Bmatrix}$	—	156
$3\ 6 \mid \frac{6}{5}$	—	156
$2\ \frac{6}{5}\ {\frac{3}{2} \atop 3} \mid$	—	156
$2\ 3\ \frac{6}{5} \mid = t'\begin{Bmatrix}3\\6\end{Bmatrix}$	—	155
$\frac{6}{5}\ \infty \mid 6$	—	154
$\frac{6}{5}\ 6\ {3 \atop \infty} \mid$	—	154
$3\ \frac{6}{5}\ 6 \mid$	—	158
$\mid 2\ 2\ \infty = s\begin{Bmatrix}2\\\infty\end{Bmatrix}$	—	60
—	—	—

vertex figure

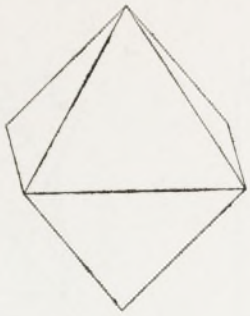




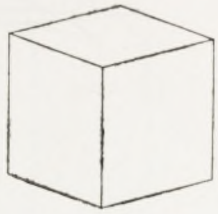
15



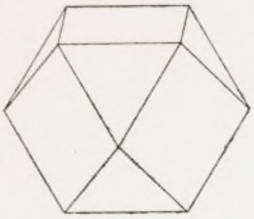
16



17



18



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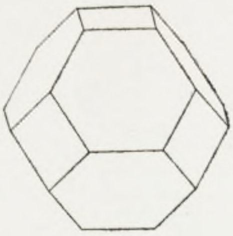
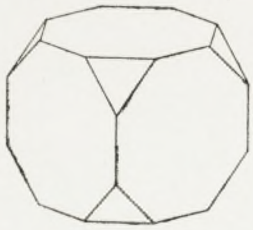
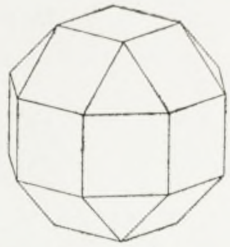


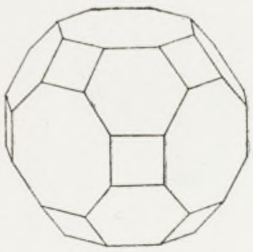
20



21

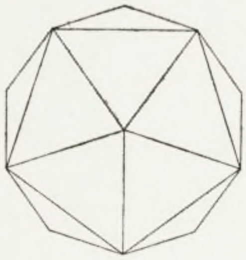


22



23

24



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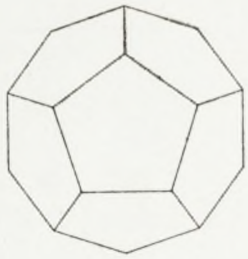
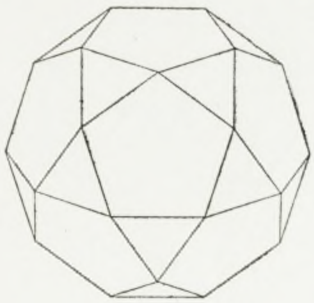


26

27



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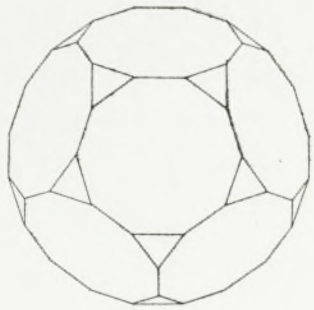
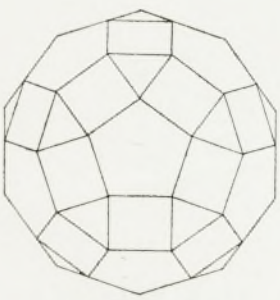
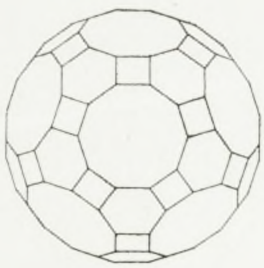


29



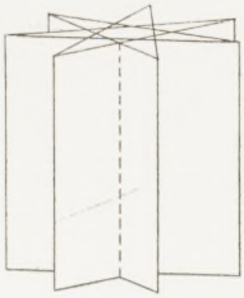
30

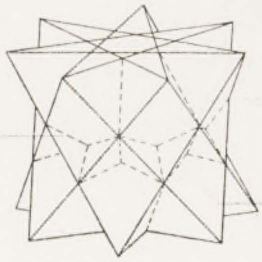


31

32

FIGURES 15 to 32. The Platonic and Archimedean solids.


33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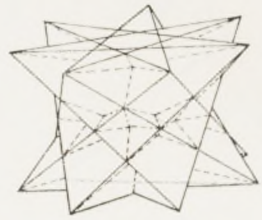
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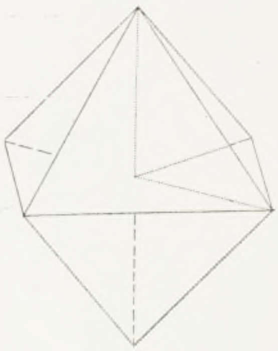
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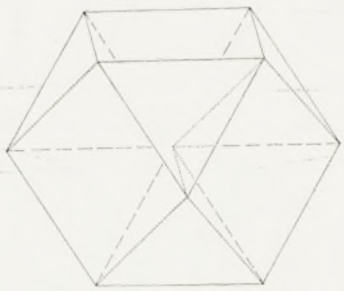
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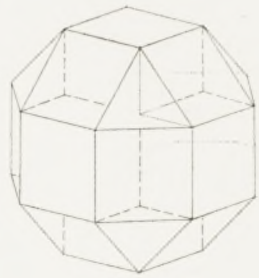
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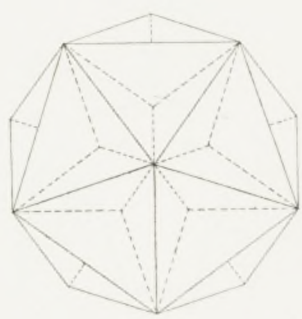
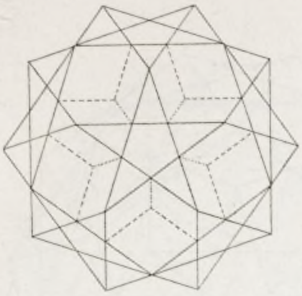
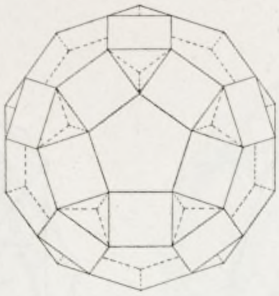

44

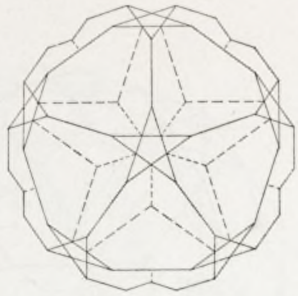
FIGURES 33 to 44. $2\,\frac{7}{3}\,|\,2$, $|\,2\,2\,\frac{7}{3}$, $|\,2\,2\,\frac{7}{4}$; $\frac{3}{2}\,3\,|\,2$, $\frac{3}{2}\,3\,|\,3$, $\frac{3}{2}\,4\,|\,4$: $3\,|\,3\,\frac{5}{2}$, $3\,\frac{5}{2}\,|\,3$, $|\,3\,3\,\frac{5}{2}$; $\frac{3}{2}\,5\,|\,5$, $5\,|\,2\,\frac{5}{2}$, $\frac{5}{2}\,|\,2\,5$.



45



46



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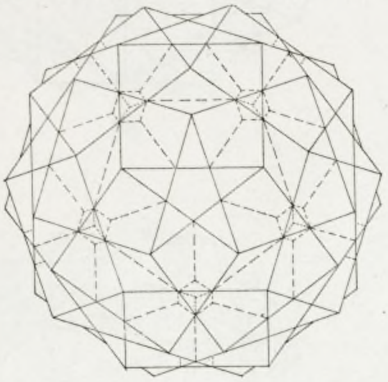

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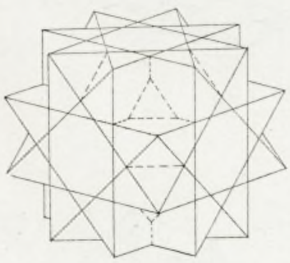

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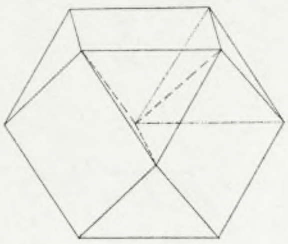
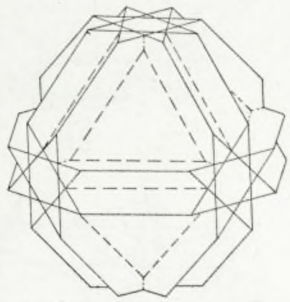


51



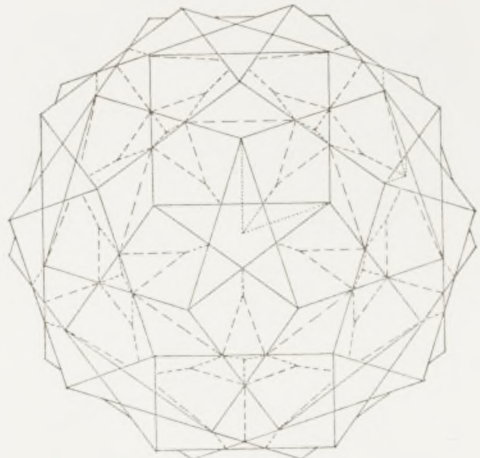
52

53

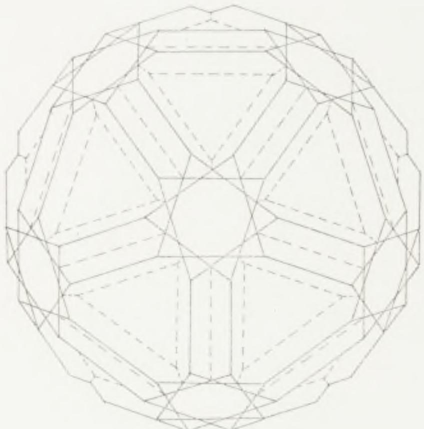
54

FIGURES 45 to 54. $2\,|\,\frac{5}{2}\,5$, $2\,5\,{\frac{3}{2} \atop \frac{5}{2}}\,\Big|$, $2\,\frac{5}{2}\,|\,5$; $\frac{5}{2}\,5\,|\,2$, $|\,2\,\frac{5}{2}\,5$; $3\,4\,|\,\frac{4}{3}$, $\frac{4}{3}\,4\,|\,3$, $3\,\frac{4}{3}\,4\,|$; $3\,|\,\frac{5}{3}\,5$, $3\,5\,|\,\frac{5}{3}$.

55

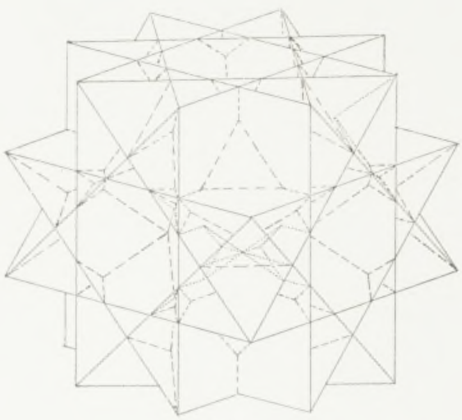


56

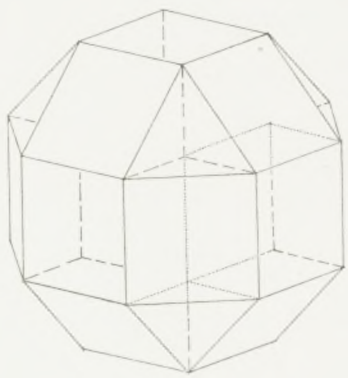


57

58

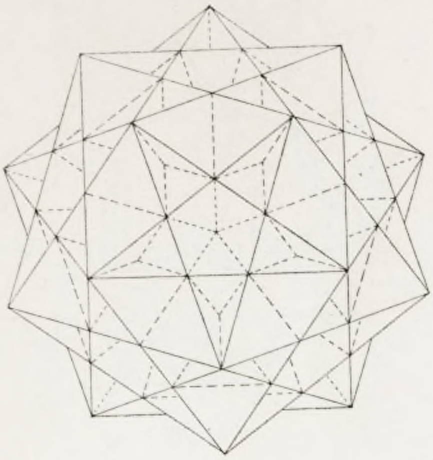


59

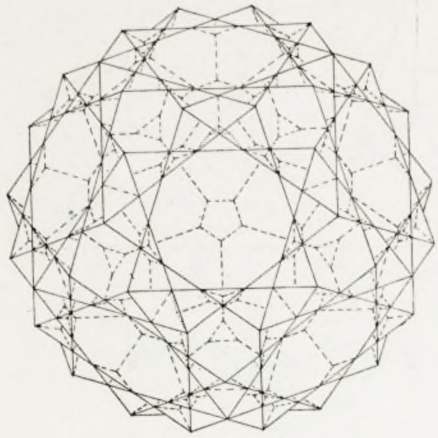


60

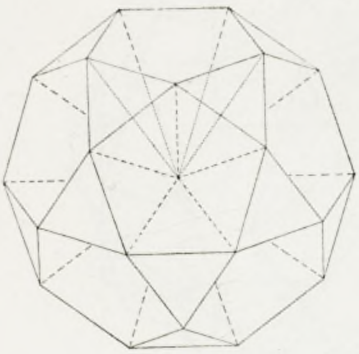
FIGURES 55 to 60. $3\,\frac{5}{3}\,|\,5$, $\frac{5}{3}\,5\,|\,3$; $3\,\frac{5}{3}\,5\,|$, $|\,3\,\frac{5}{3}\,5$; $\frac{3}{2}\,4\,|\,2$, $2\,4\,{\frac{3}{2} \atop \frac{4}{2}}\Big|$.



61



62



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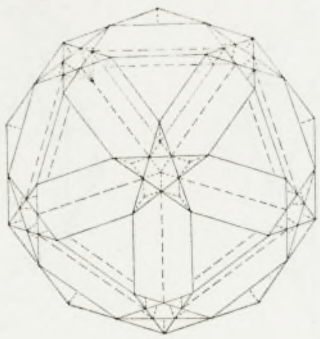

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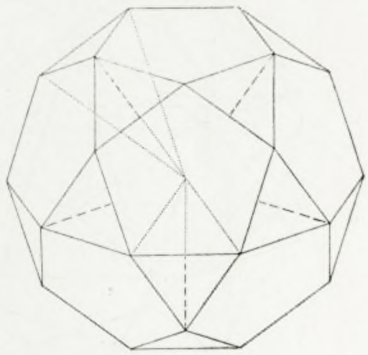

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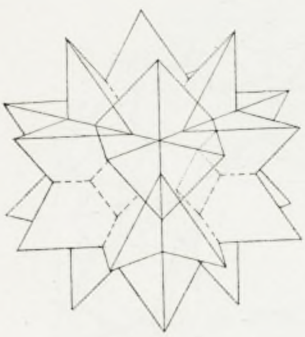

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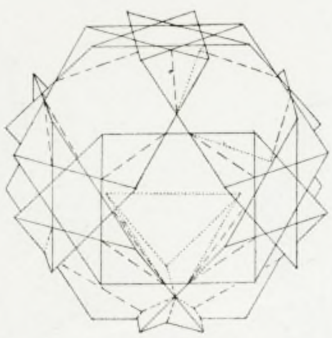

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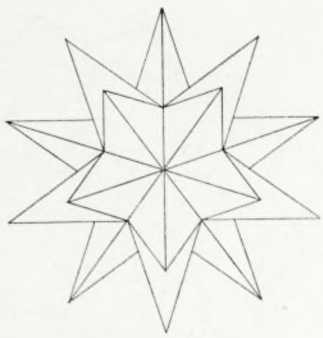
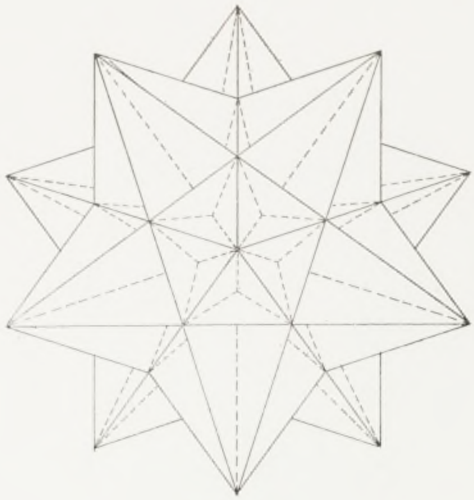


68

FIGURES 61 to 68. $\frac{3}{2}\,|\,3\;5$, $\frac{3}{2}\;5\,|\,3$; $\frac{3}{2}\;3\,|\,5$, $3\;5\;{\frac{3}{2} \atop \frac{5}{4}}\,\Big|$, $\frac{5}{4}\;5\,|\,5$; $2\;3\,|\,\frac{4}{3}$, $2\;3\;\frac{4}{3}\,|$, $3\,|\,2\;\frac{5}{2}$.



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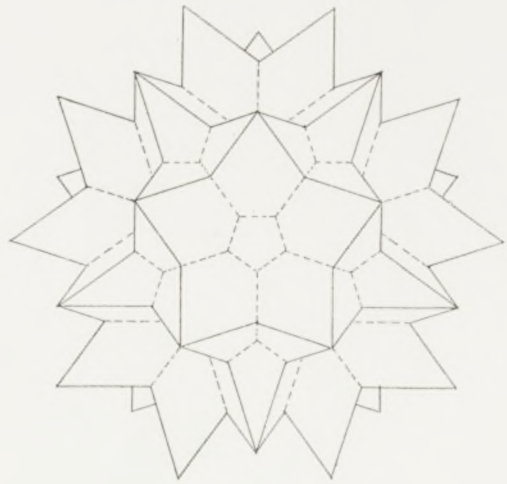

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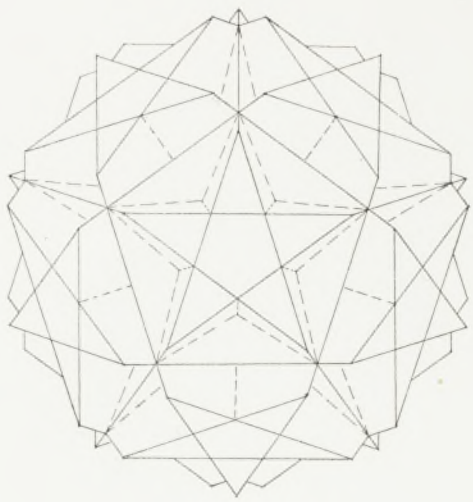

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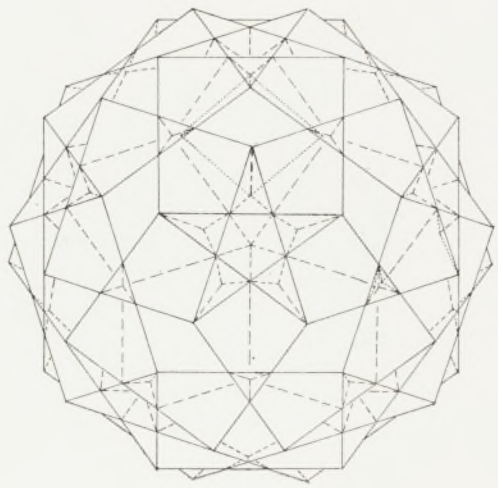

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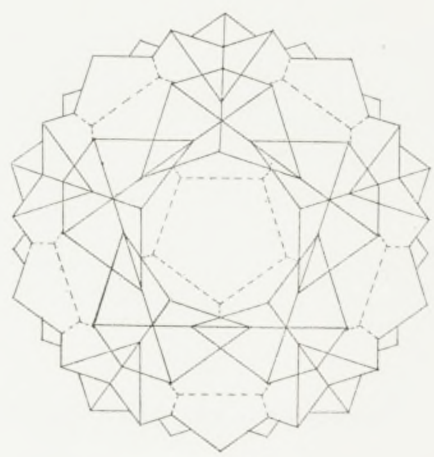
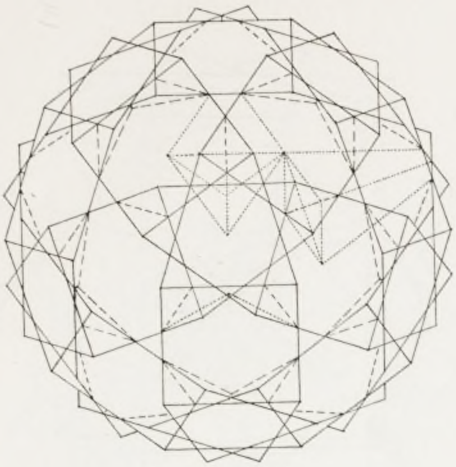


74

FIGURES 69 to 74. $\frac{5}{2}\,|\,2\;3$, $2\,|\,3\;\frac{5}{2}$; $2\;\frac{5}{2}\,|\,3$, $2\;3\;\begin{smallmatrix}\frac{5}{4}\\ \frac{5}{2}\end{smallmatrix}\,\Big|$; $|\,2\;3\;\frac{5}{2}$, $2\;5\,|\,\frac{5}{3}$.



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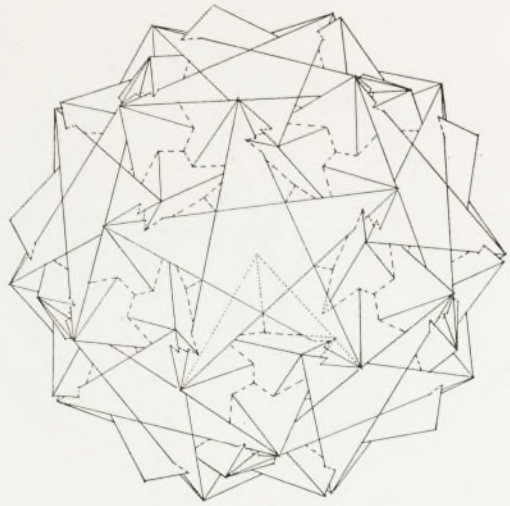

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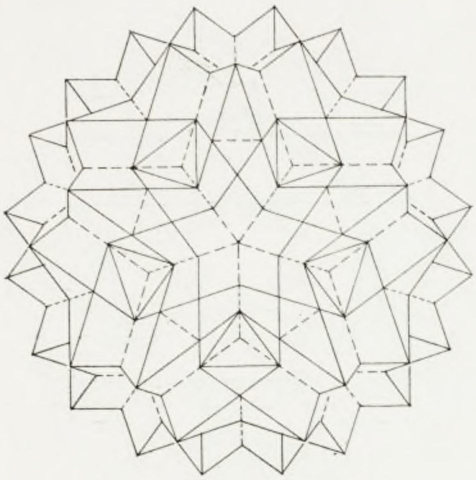

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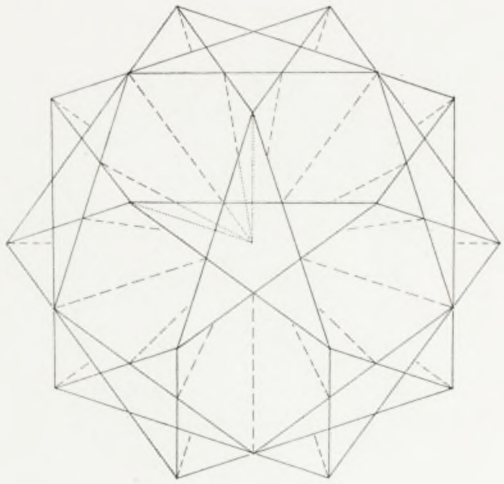

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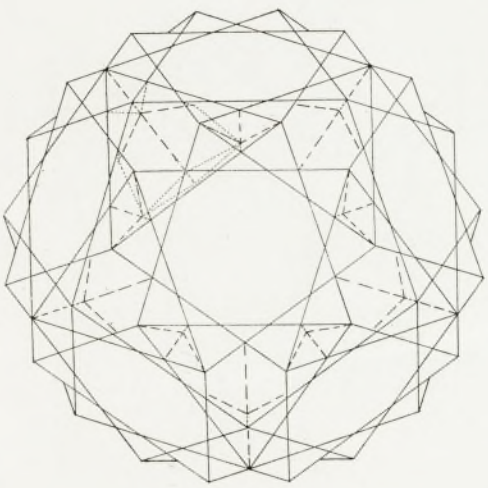

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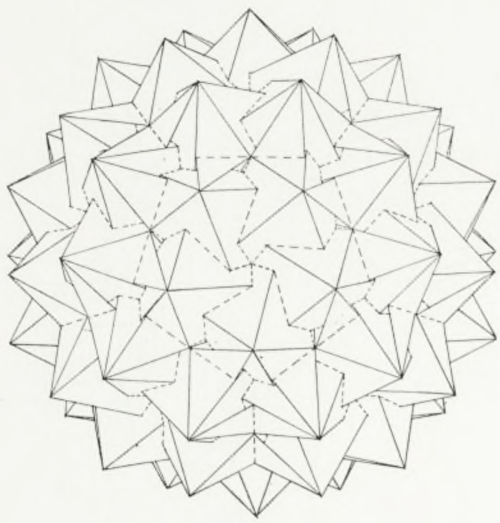
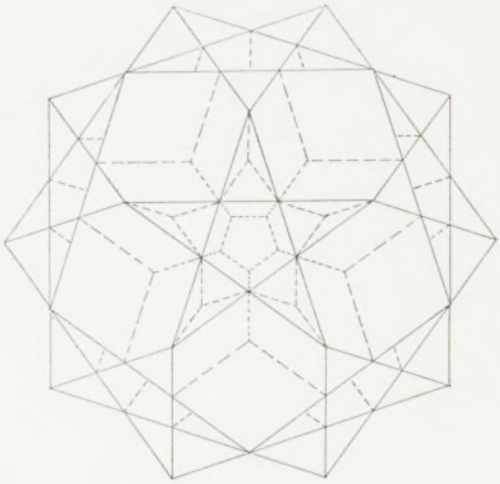


80

FIGURES 75 to 80. $2\,\frac{5}{3}\,5\,|$, $|\,2\,\frac{5}{3}\,5$; $3\,\frac{5}{2}\,|\,\frac{5}{3}$, $\frac{5}{3}\,\frac{5}{2}\,|\,3$; $3\,\frac{5}{3}\,{}^{\frac{3}{2}}_{\frac{5}{2}}\Big|$, $|\,3\,\frac{5}{3}\,\frac{5}{2}$.



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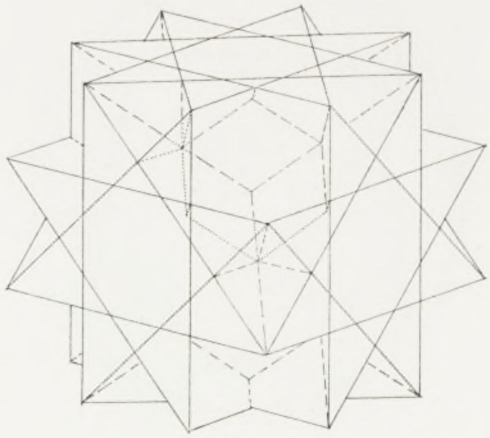

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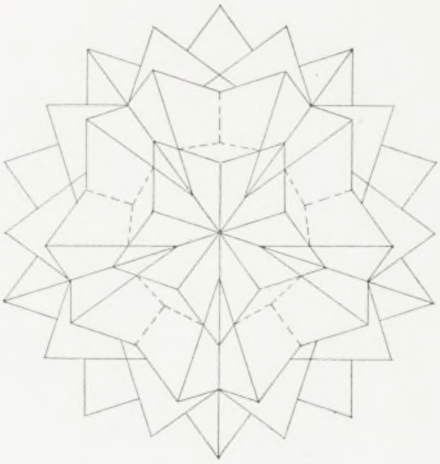

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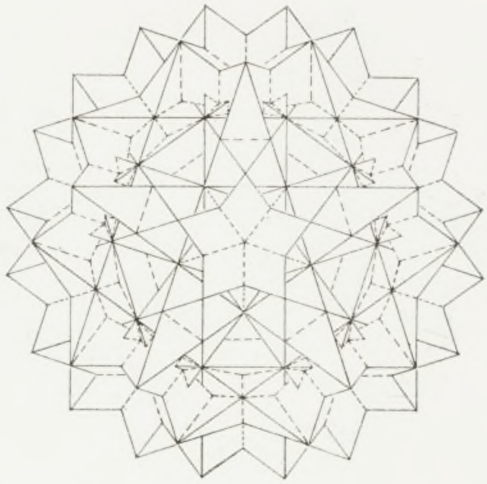

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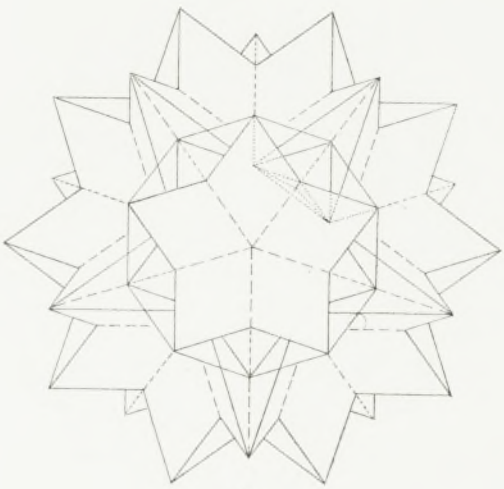

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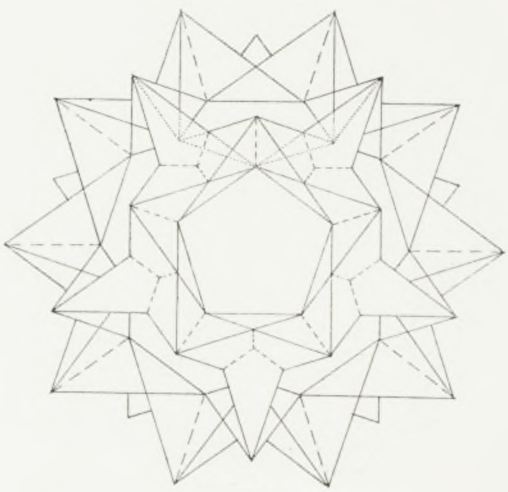
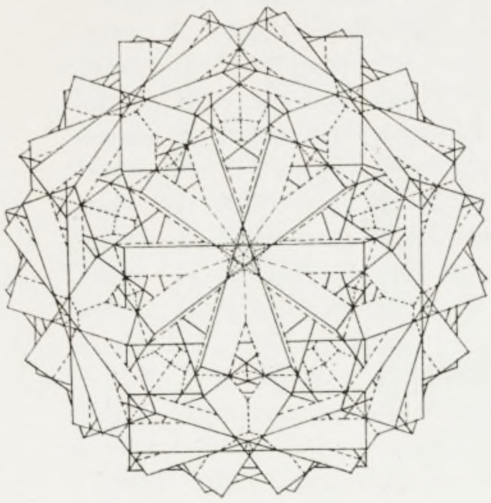


86

FIGURES 81 to 86. $\frac{5}{4}\,5\,|\,3$, $2\,\frac{4}{3}\begin{smallmatrix}\frac{3}{2}\\ \frac{4}{2}\end{smallmatrix}\Big|$; $2\,3\,|\,\frac{5}{3}$, $3\,\frac{5}{3}\,|\,2$; $\frac{3}{2}\,3\,|\,\frac{5}{3}$, $\frac{5}{3}\,\frac{5}{2}\,|\,\frac{5}{3}$.



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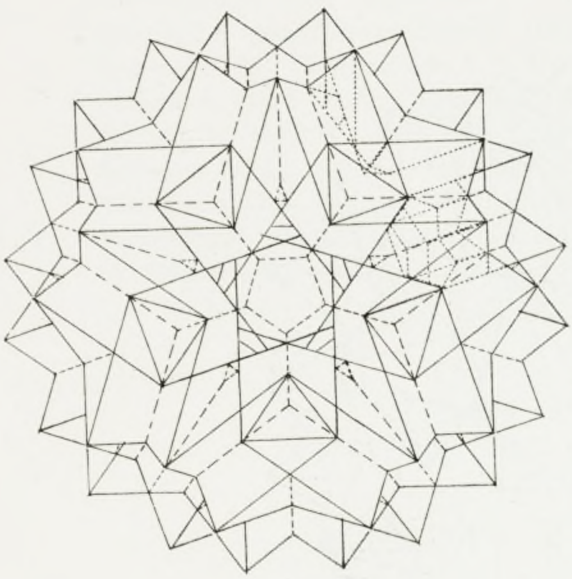
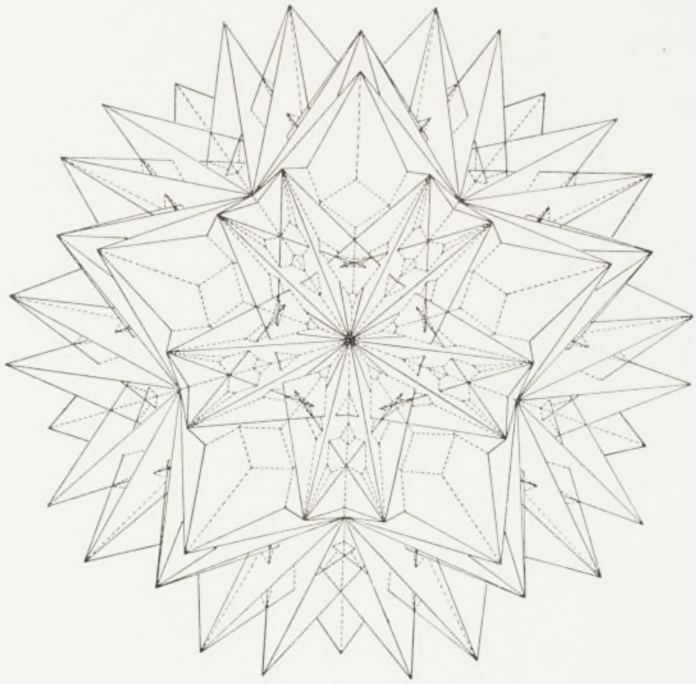


89

90

FIGURES 87 to 90. $2\ 3\ \frac{5}{3}\,|$, $|\,2\ 3\ \frac{5}{3}$; $2\ \frac{5}{3}\ \begin{matrix}\frac{3}{2}\\ \frac{5}{4}\end{matrix}\,\Big|$, $|\,2\ \frac{3}{2}\ \frac{5}{3}$.



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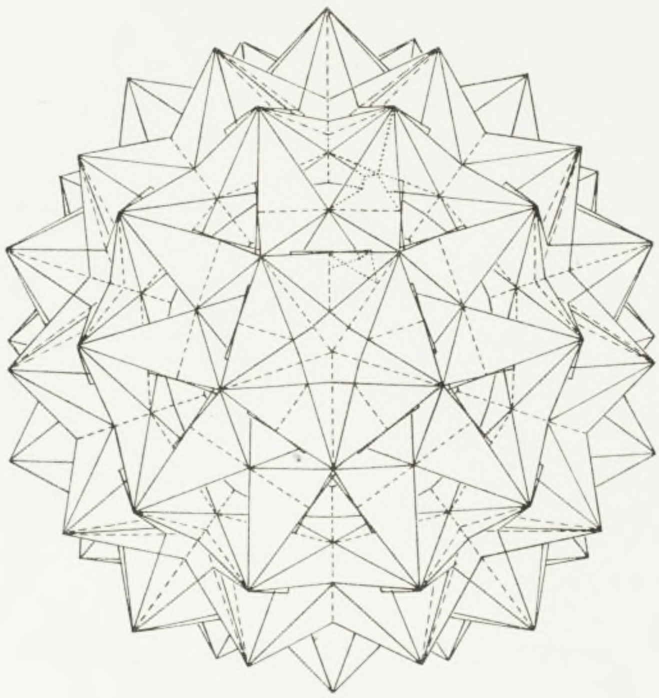


92

FIGURES 91, 92. $|\frac{3}{2}\,\frac{3}{2}\,\frac{5}{2}$; $|\frac{3}{2}\,\frac{5}{3}\,3\,\frac{5}{2}$.

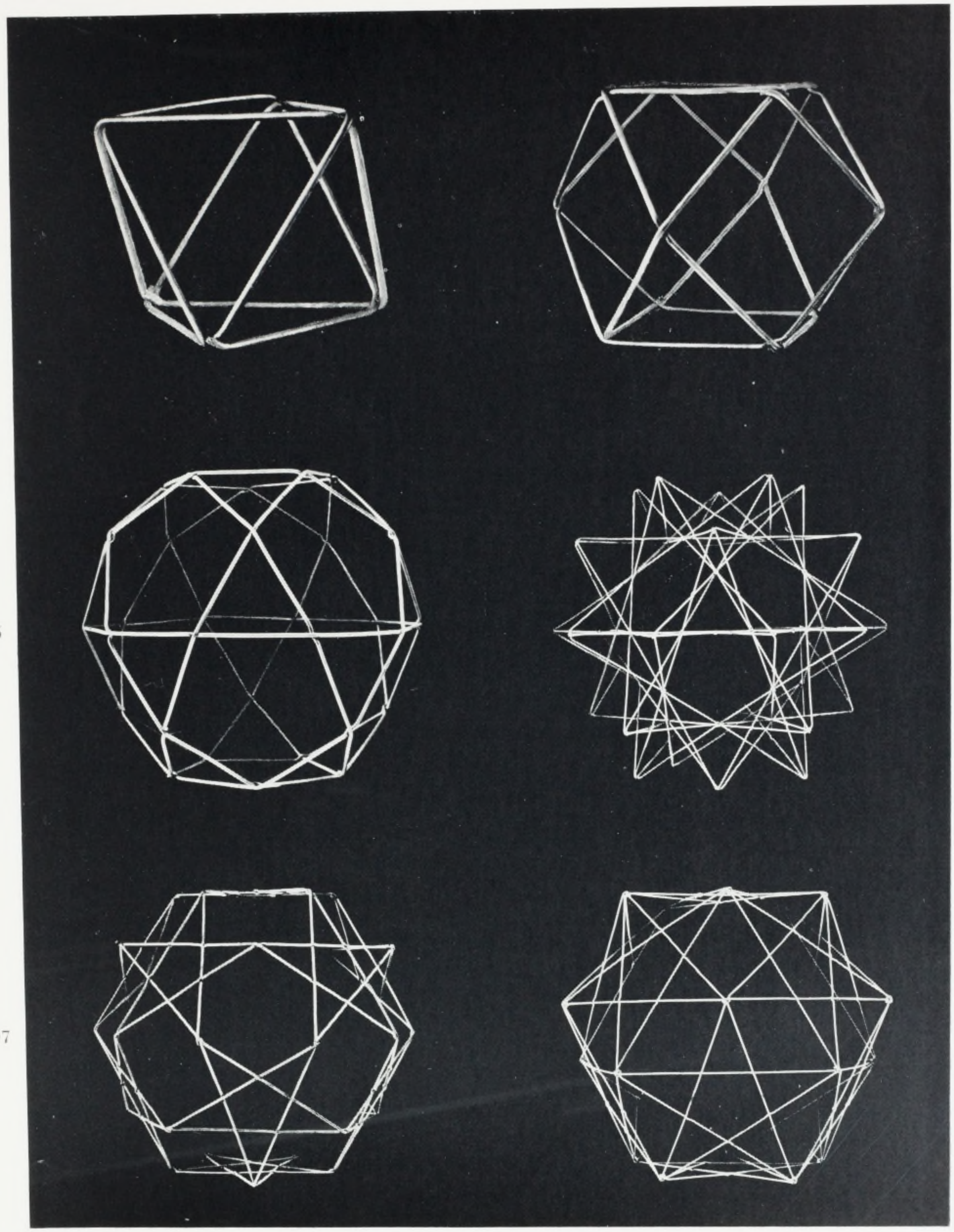
References

Badoureau, A. 1881 *J. Éc. Polyt., Paris*, **49**, 47–172.
Ball, W. W. R. 1939 *Mathematical recreations and essays*. London: Macmillan.
Brückner, M. 1900 *Vielecke und Vielflache*. Leipzig: Teubner.
Catalan, E. C. 1865 *J. Éc. Polyt., Paris*, **41**, 25–32.
Cauchy, A. L. 1813 *J. Éc. Polyt., Paris*, **16**, 68–86.
Cayley, A. 1859 *Phil. Mag.* (4), **17**, 123–128.
Coxeter, H. S. M. 1931 *Proc. Camb. Phil. Soc.* **27**, 201–211.
Coxeter, H. S. M. 1932 *Proc. Camb. Phil. Soc.* **28**, 509–521.
Coxeter, H. S. M. 1938 *Amer. Math. Mon.* **45**, 522–525.
Coxeter, H. S. M. 1939 *Trans. Amer. Math. Soc.* **45**, 73–150.
Coxeter, H. S. M. 1940 *Math. Z.* **46**, 380–407.
Coxeter, H. S. M. 1947 *Math. Rev.* **8**, 481–482.
Coxeter, H. S. M. 1948 *Regular polytopes*. London: Methuen.
Coxeter, H. S. M. & Whitrow, G. J. 1950 *Proc. Roy. Soc.* A, **201**, 417–437.
Heath, T. L. 1921 *A history of Greek mathematics*. Oxford: Clarendon Press.
Hess, E. 1876 *Schr. Ges. ges. Naturw. Marburg*, **11**, 1–15.
Hess, E. 1878 *Schr. Ges. ges. Naturw. Marburg*, **11**, 261–271.
Kepler, J. 1619 *Opera omnia*, **5**. Frankfurt (1864): Heyder.
Klein, F. 1884 *Lectures on the icosahedron*. London (1913): Kegan Paul, Trench, Trübner and Co.
Lesavre, J. & Mercier, R. 1947 *C.R. Acad. Sci., Paris*, **224**, 785–786.
Möbius, A. F. 1849 *Ges. Werke*, **2**. Leipzig (1886): Hirzel.
Pacioli, L. 1509 *Divina proportione*. Buenos Aires (1946): Losada.
Pitsch, J. 1881 *Z. Realschulw.* **6**, 9–24, 64, 72–89, 216.
Poinsot, L. 1810 *J. Éc. Polyt., Paris*, **10**, 16–48.
Schläfli, L. 1852 *Ges. math. Abh.* **1**. Basel (1950): Birkhäuser.
Schwarz, H. A. 1873 *Ges. math. Abh.* **2**. Berlin (1890).
Thompson, D'A. W. 1925 *Proc. Roy. Soc.* A, **107**, 181–188.
Woepcke, F. 1855 *J. Asiat.* (5), **5**, 309–359.
Wythoff, W. A. 1918 *Proc. Acad. Sci. Amst.* **20**, 966–970.

Notes on the plates

Plates 1 to 6 illustrate models of all the non-convex uniform polyhedra, apart from the Kepler-Poinsot polyhedra. Only the sides of the faces are shown, the faces themselves and their subsidiary intersections being omitted. Some of the models, therefore, represent more than one polyhedron; for example, figure 107, plate 3, represents either $3\;5\,|\,2$, $\frac{3}{2}\;5\,|\,5$ or $2\;5\;\begin{smallmatrix}\frac{3}{2}\\ \frac{5}{2}\end{smallmatrix}\Big|$.

The models are constructed of galvanized iron wire ('garden wire'), with the exception of two, figures 120 and 128, which are of cotton thread strung in a wire frame (the frame is just visible in the photographs). No solder has been used in the wire models; for the most part the wires are simply sprung together, although in some cases they have been secured by twisting at the vertices. Where two edges theoretically intersect the corresponding wires are kinked. The two polyhedra $|\,\frac{3}{2}\;\frac{3}{2}\;\frac{5}{2}$ and $|\,\frac{3}{2}\;\frac{5}{3}\;3\;\frac{5}{2}$, of which the models are of cotton, are the only cases in which more than six edges may intersect in a point.



93 94

95 96

97 98

FIGURES 93 to 98. Quasi-regular and semi-regular polyhedra.

93. $4\,|\,2\;3,\;\frac{3}{2}\;3\,|\,2.$

94. $2\,|\,3\;4,\;\frac{3}{2}\;3\,|\,3,\;\frac{4}{3}\;4\,|\,3.$

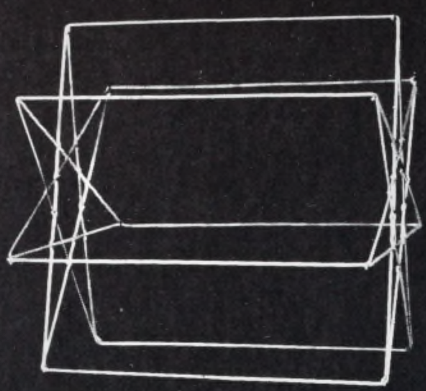
95. $2\,|\,3\;5,\;\frac{3}{2}\;3\,|\,5,\;\frac{5}{4}\;5\,|\,5.$

96. $2\,|\,3\;\frac{5}{2},\;\frac{5}{3}\;\frac{5}{2}\,|\,\frac{5}{3},\;\frac{3}{2}\;3\,|\,\frac{5}{3}.$

97. $2\,|\,\frac{5}{2}\;5,\;\frac{5}{4}\;5\,|\,3,\;\frac{5}{3}\;\frac{5}{2}\,|\,3.$

98. $3\,|\,3\;\frac{5}{2},\;3\,|\,\frac{5}{3}\;5,\;\frac{3}{2}\,|\,3\;5.$

(*Facing p.* 450)

99


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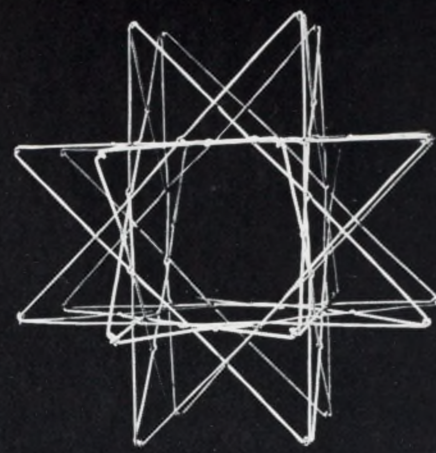

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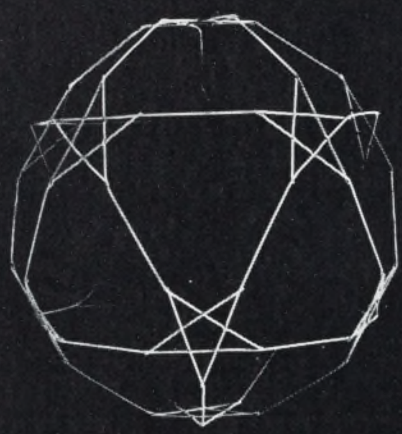

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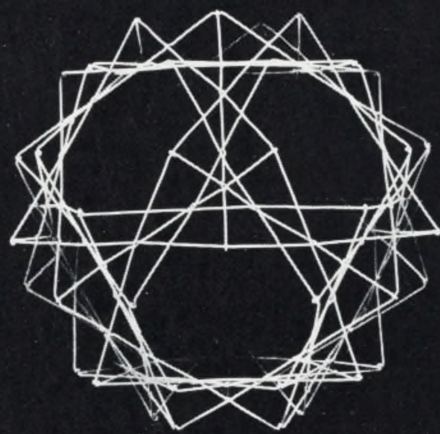

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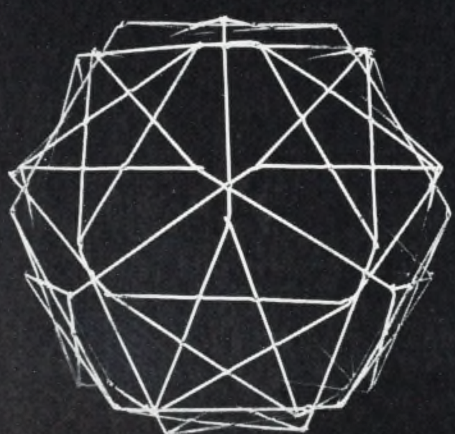

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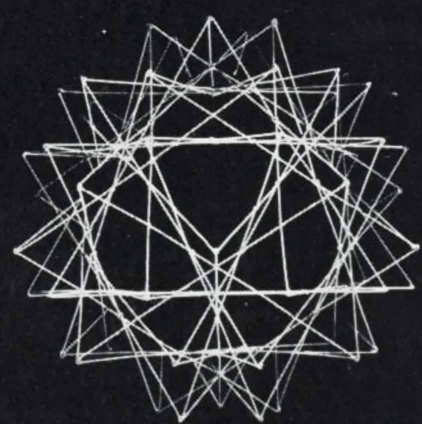
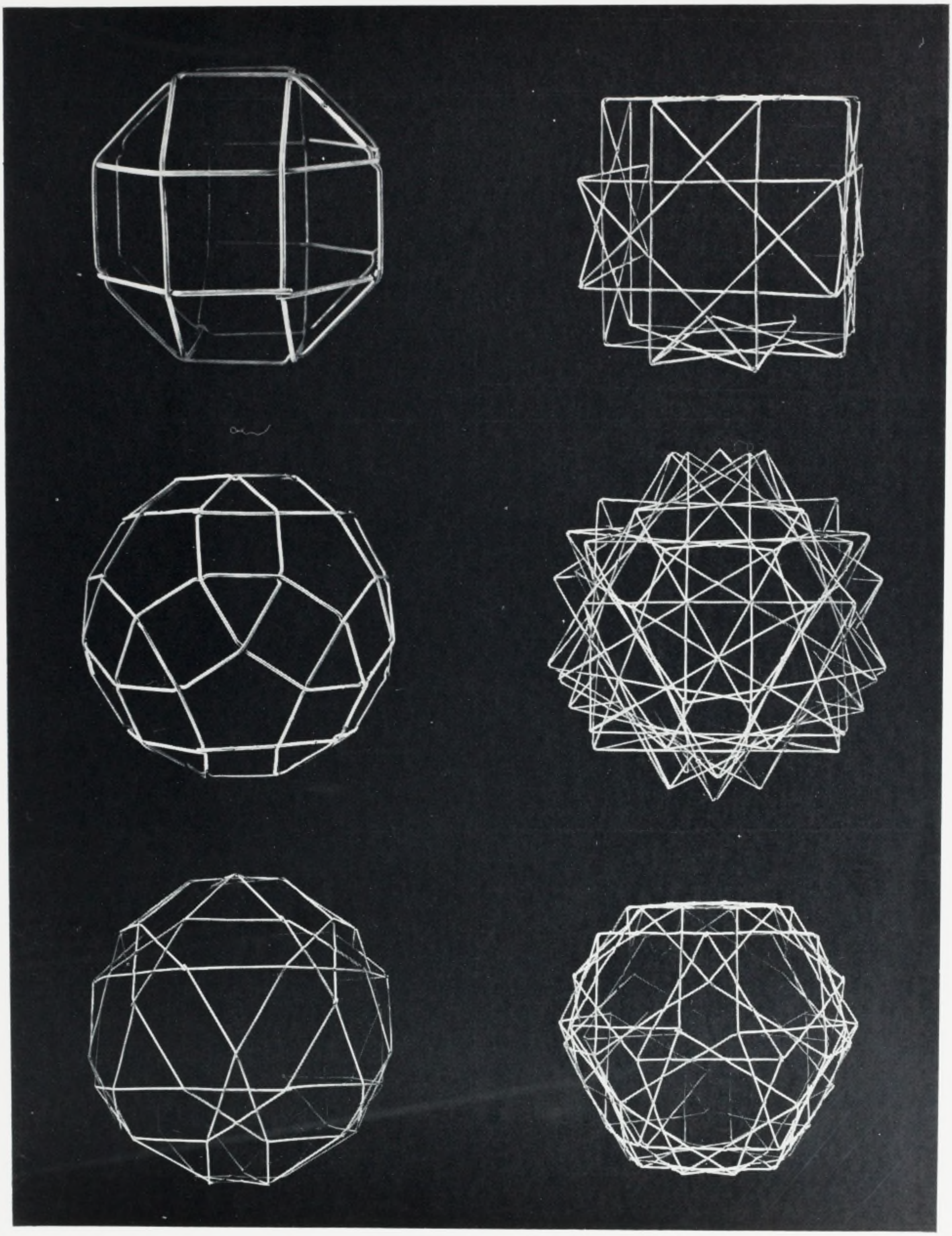


FIGURES 99 to 104. Semi-regular polyhedra.

99. $2\ \frac{7}{3}\,|\,2$. 100. $2\ 3\,|\,\frac{4}{3}$. 101. $2\ \frac{5}{2}\,|\,5$.

102. $2\,|\,5\frac{5}{3}$. 103. $2\ \frac{5}{2}\,|\,3$. 104. $2\ 3\,|\,\frac{5}{3}$.



105 106

107 108

109 110

FIGURES 105 to 110. Semi-regular and even-faced polyhedra.

105. $3\ 4\,|\,2,\ \frac{3}{2}\ 4\,|\,4,\ 2\ 4\ {}^{\frac{3}{2}}_{\frac{4}{2}}\big|$.

106. $3\ 4\,|\,\frac{4}{3},\ \frac{3}{2}\ 4\,|\,2,\ 2\ \frac{4}{3}\ {}^{\frac{3}{2}}_{\frac{4}{2}}\big|$.

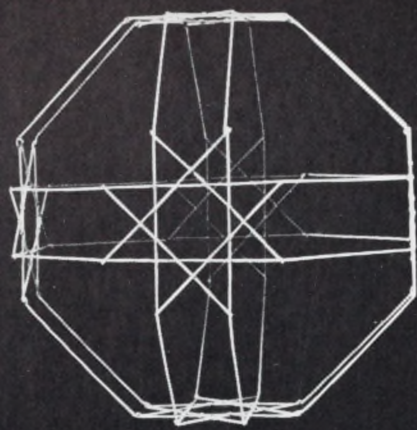
107. $3\ 5\,|\,2,\ \frac{3}{2}\ 5\,|\,5,\ 2\ 5\ {}^{\frac{3}{2}}_{\frac{5}{2}}\big|$.

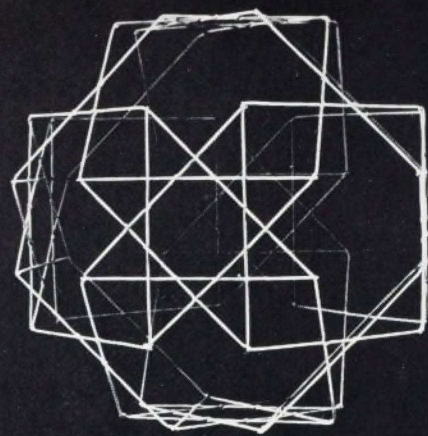
108. $3\ \frac{5}{2}\,|\,\frac{5}{3},\ 3\ \frac{5}{3}\,|\,2,\ 2\ \frac{5}{3}\ {}^{\frac{3}{2}}_{\frac{5}{4}}\big|$.

109. $3\ \frac{5}{2}\,|\,3,\ 3\ \frac{5}{3}\,|\,5,\ 3\ 5\ {}^{\frac{3}{2}}_{\frac{5}{4}}\big|$.

110. $3\ 5\,|\,\frac{5}{3},\ \frac{3}{2}\ 5\,|\,3,\ 3\ \frac{5}{3}\ {}^{\frac{3}{2}}_{\frac{5}{2}}\big|$.

111





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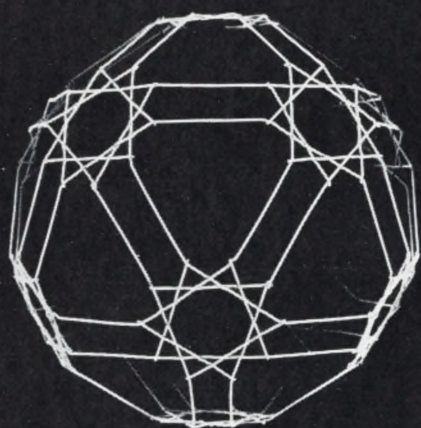

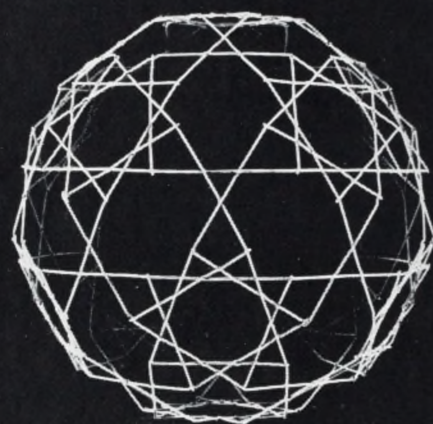

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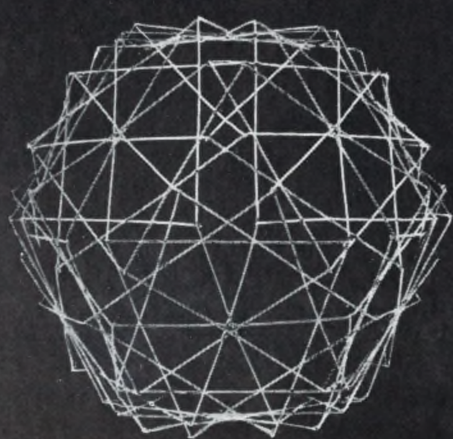
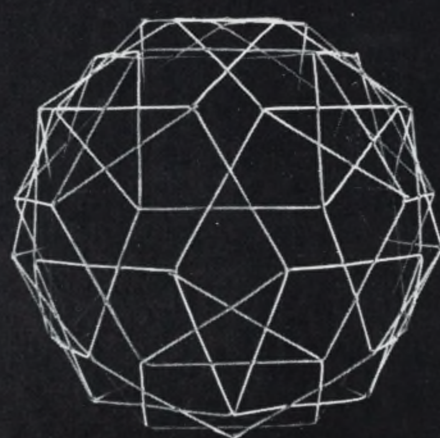




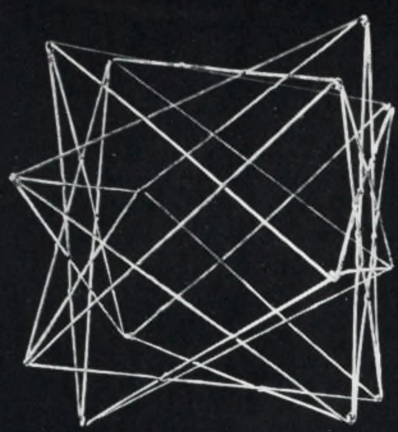
116

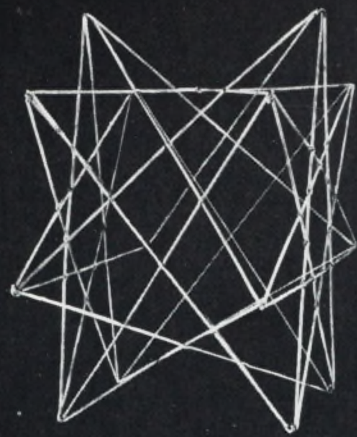
FIGURES 111 to 115. Semi-regular polyhedra.

FIGURE 116. Semi-regular and even-faced polyhedra.

111. $3\,\frac{4}{3}\,4\,|$.　112. $2\,3\,\frac{4}{3}\,|$.　113. $3\,\frac{5}{3}\,5\,|$.

114. $2\,\frac{5}{3}\,5\,|$.　115. $2\,3\,\frac{5}{3}\,|$.　116. $\frac{5}{2}\,5\,|\,2$, $\frac{5}{3}\,5\,|\,3$, $2\,3\,{\textstyle\frac{5}{4} \atop \textstyle\frac{5}{2}}\,|$.

117
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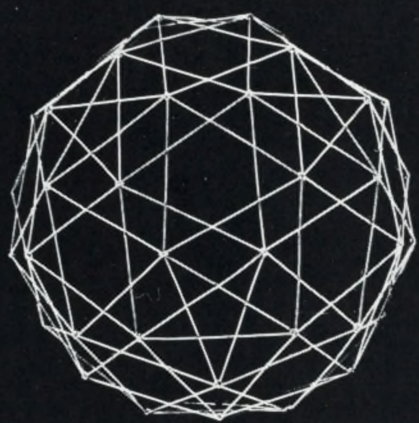
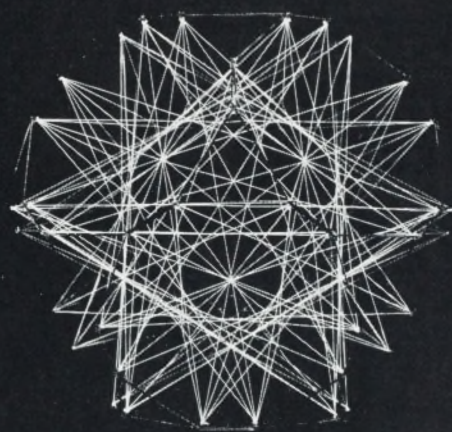
119
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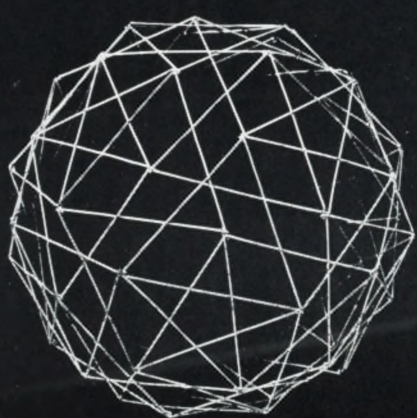
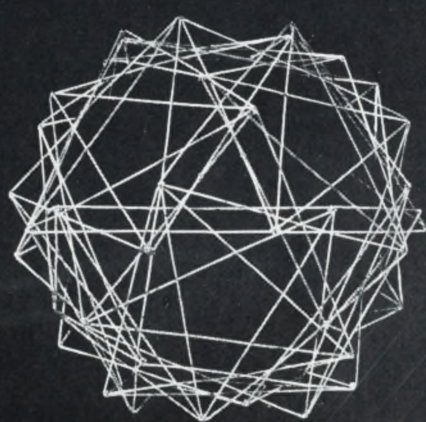



122

FIGURES 117 to 122. Snub polyhedra.

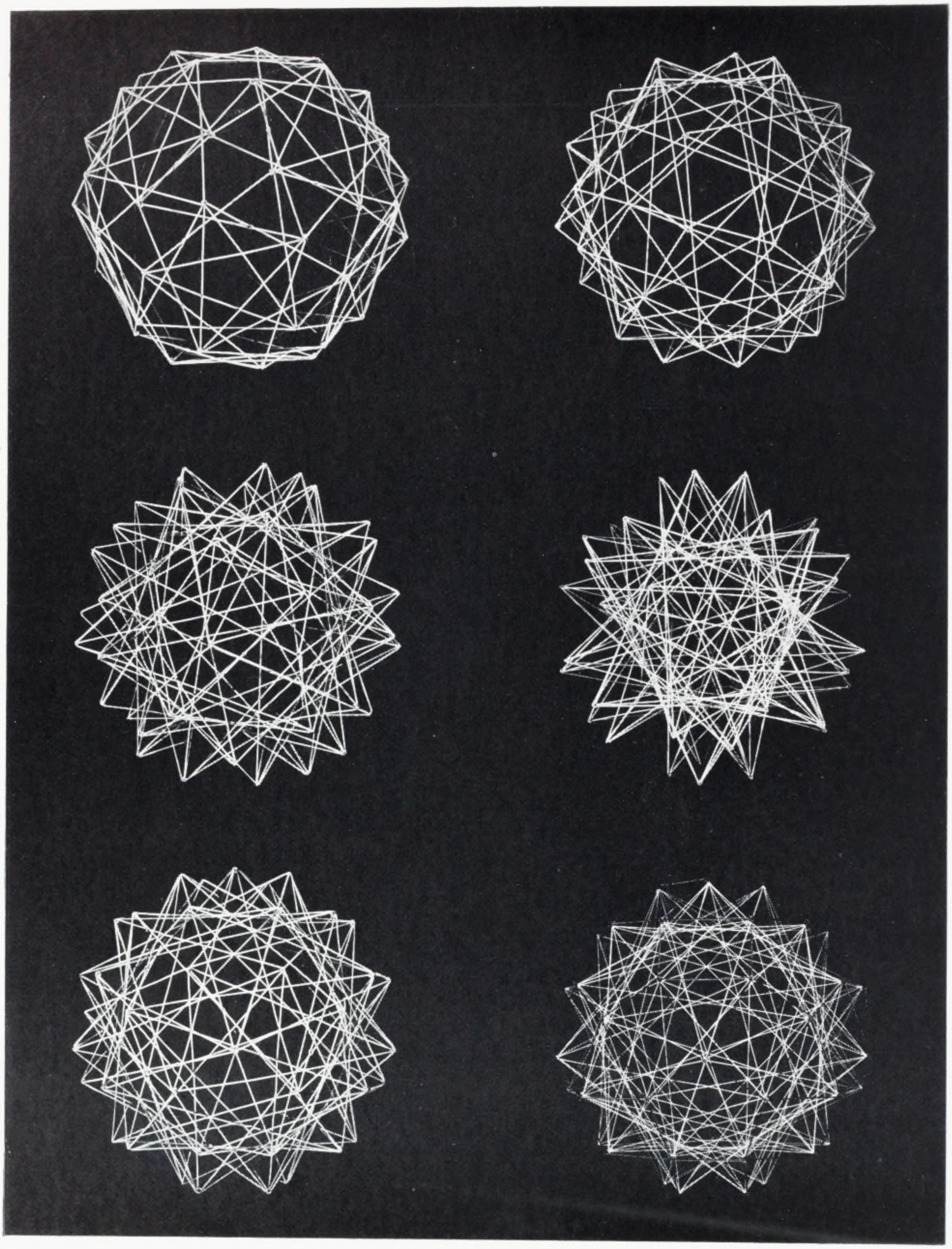
117. $|2\ 2\ \frac{7}{3}$. 118. $|2\ 2\ \frac{7}{4}$. 119. $|3\ 3\ \frac{5}{2}$.

120. $|\frac{3}{2}\ \frac{3}{2}\ \frac{5}{2}$. 121. $|2\ \frac{5}{2}\ 5$. 122. $|2\ \frac{5}{3}\ 5$.

123 124

125 126

127 128



FIGURES 123 to 128. Snub polyhedra.

123. $|\,3\ \tfrac{5}{3}\ 5$. 124. $|\,2\ 3\ \tfrac{5}{2}$. 125. $|\,2\ 3\ \tfrac{5}{3}$.

126. $|\,2\ \tfrac{3}{2}\ \tfrac{5}{3}$. 127. $|\,3\ \tfrac{5}{3}\ \tfrac{5}{2}$. 128. $|\,\tfrac{3}{2}\ \tfrac{5}{3}\ 3\ \tfrac{5}{2}$.

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