ON THE ASYMPTOTIC PROPERTIES OF A CERTAIN CLASS OF TCHEBYCHEFF POLYNOMIALS

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1. Denote by

$$\phi_n(p;\,x)=a_n(p)(x^n-S_n(p)x^{n-1}+\dots)\qquad (n=0,\,1,\,2\dots;\,a_n>0) \tag{1}$$

a system of orthogonal and normal "*Tchebycheff polynomials*" corresponding to a given interval $(a,\,b)$ with the "*characteristic function*" $p(x)$ integrable and not negative in $(a,\,b)$. These polynomials are uniquely determined by means of the relations:

$$\int_a^b p(x)\phi_m(p;x)\phi_n(p;x)dx=\begin{cases}0, & m\neq n,\\ 1, & m=n.\end{cases} \tag{2}$$

In the case where $(a,\,b)$ is *finite* and $p(x)=(x-a)^{\alpha-1}(b-x)^{\beta-1}(\alpha,\,\beta>0)$ we get Jacobi's polynomials.

Tchebycheff polynomials, it is well known, are closely connected with the continued fractions:

$$\int_a^b \frac{p(y)}{x-y}dy\sim\cfrac{\lambda_1(p)}{x-c_1-\cfrac{\lambda_2(p)}{x-c_2-\cdots}}. \tag{3}$$

$$\int_a^b \frac{p(y)}{x-y}dy\sim\cfrac{1}{l_1x+\cfrac{1}{l_2+\cfrac{1}{l_3x+\cdots}}}=\cfrac{b_1(p)}{x-\cfrac{b_2(p)}{1-\cfrac{b_3(p)}{x-\cdots}}}\qquad (0\leq a<b). \tag{4}$$

2. Consider two characteristic functions $p(x)$ and $p_1(x)$, where

$$p_1(x)=\Pi(x)p(x)\equiv(cx^s+\dots)p(x)=c\prod_{i=1}^{s}(x-a_i)p(x), \tag{5}$$

the polynomial $\Pi(x)$ having real non-negative coefficients in $(a,\,b)$.

We then have

$$\Pi(x)\phi_n(p_1;x)=\sum_{i=0}^{s}h_{n+i}^{(n)}\phi_{n+i}(p;\,x); \tag{6}$$

(7) $$\Pi(x)\phi_n(p_1;x)=\frac{(-1)^s h_{n+s}^{(n)}}{\Delta_{n,s}}\begin{vmatrix}\phi_n(p;x)\,\phi_{n+1}(p;x)\ldots\phi_{n+s}(p;x)\\ \phi_n(p;a_1)\ldots\ldots\ldots\ldots\phi_{n+s}(p;a_1)\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \phi_n(p;a_s)\ldots\ldots\ldots\ldots\phi_{n+s}(p;a_s)\end{vmatrix};$$

(8) $$\Delta_{n,i}=\begin{vmatrix}\phi_n(p;\,a_1)\ldots\phi_{n+i-1}(p;\,a_1)\;\;\phi_{n+i+1}(p;\,a_1)\ldots\phi_{n+s}(p;\,a_1)\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \phi_n(p;\,a_s)\ldots\phi_{n+i-1}(p;\,a_s)\;\;\phi_{n+i+1}(p;\,a_s)\ldots\phi_{n+s}(p;\,a_s)\end{vmatrix};$$
$$(i=0,\,1,\,2,\ldots,s).$$

[If, for example, $a_k=a_{k+1}$, then the row $\phi_n(p;\,a_{k+1})\ldots$ is to be replaced by $\phi_n'(p;\,a_k)\ldots$etc.].

Formula (8) was given by Christoffel* for Legendre's polynomials $[(a,\,b)$ finite; $p(x)\equiv 1]$. Therefore, we write:

(9) $$\frac{h_{n+i}^{(n)}}{h_{n+s}^{(n)}}=\frac{(-1)^{s-i}\Delta_{n,i}}{\Delta_{n,s}},\quad(i=0,\,1,\,2,\,\ldots,s).$$

In particular, using (2), we get:

(10) $$h_n^{(n)}=\frac{a_n(p)}{a_n(p_1)},\;h_{n+s}^{(n)}=\frac{ca_n(p_1)}{a_{n+s}(p)}.$$

3. Formula (7) will be the starting point in our discussion. Consider some special cases.

1° $s=1$; $\Pi(x)=c(x-\xi)$, ξ real$<a$ or$>b$, $c=\pm 1$ (according to the value of ξ). Formula (7) gives:

(11) $$\frac{\phi_{n+1}(p;\,\xi)}{\phi_n(p;\,\xi)}=\mp\frac{a_n(p)\,a_{n+1}(p)}{a_n^2(p_1)}\quad\begin{pmatrix}-,\;\xi<a\\+,\;\xi>b\end{pmatrix}.$$

This formula holds for any real ξ not belonging to $(a,\,b)$ and for any characteristic function $p(x)$.

2° $s=2$. $\Pi(x)=(x-\xi)\,(x-\bar{\xi})$, ξ, $\bar{\xi}$—conjugate complex numbers. Using (7) we get:

(12) $$\begin{cases}\dfrac{K_{n+1}(p;\,\xi)}{K_n(p;\,\xi)}=\dfrac{a_{n+1}^2(p)}{a_n^2(p_1)},\\[2ex] \dfrac{|\phi_{n+1}(p;\,\xi)|^2}{K_n(p;\,\xi)}=\dfrac{a_{n+1}^2(p)}{a_n^2(p_1)}-1,\\[2ex] K_n(p;\,\xi)\equiv\sum_{i=0}^{n}|\phi_i(p;\,\xi)|^2.\end{cases}$$

***Ueber die Gaussche Quadratur* . . ., Journal für Math. 55 (1858), 61-82, s. 77. See also G. Szegö: *Ueber die Entwickelung einer analytischen Funktion* . . ., Mathematische Annalen, Bd. 81-82 (1920-21), 188-212, s. 191.

3° $s=2$; $\Pi(x)=(x-\xi)^2$, ξ—*any real number whatever.* Making use of Darboux's formula

(13) $$K_n(p,x)\equiv\sum_0^n{}_i\phi_i^2(p;x)=\frac{a_n(p)}{a_{n+1}(p)}[\phi'_{n+1}(x)\,\phi_n(x)-\phi'_n(x)\,\phi_{n+1}(x)]^*,$$

which is a particular case of the formula

(14) $$K_n(p;x,y)\equiv\sum_0^n{}_i\phi_i(p;x)\phi_i(p;y)=\frac{a_n(p)}{a_{n+1}(p)}\frac{\phi_{n+1}(x)\phi_n(y)-\phi_n(x)\phi_{n+1}(y)}{x-y},$$

we obtain:

(15) $$\begin{cases}\dfrac{\phi_{n+1}^2(p;\xi)}{K_n(p;\xi)}=\dfrac{a_{n+1}^2(p)}{a_n^2(p_1)}-1\equiv\alpha_n,\\[2mm] \dfrac{\phi_{n+1}^2(p;\xi)}{\phi_n^2(p;\xi)}=\dfrac{\alpha_n}{\alpha_{n-1}}\dfrac{a_n^2(p)}{a_{n-1}^2(p_1)},\\[2mm] \left[\dfrac{\phi_{n+1}(p;x)}{\phi_n(p;x)}\right]'_{x=\xi}=\dfrac{a_{n+1}(p)}{a_n(p_1)}\cdot\dfrac{1}{\alpha_n}.\end{cases}$$

Formulae (15) *hold for any real* ξ *and for any characteristic function* $p(x)$ *and enable us to find the asymptotic expression* (*for* $n\to\infty$) *of* $\dfrac{\phi_{n+1}(p;\xi)}{\phi_n(p;\xi)}$, *also that of* $K_n(p;\xi)$ *in many cases.*

We can apply (15) also to an infinite interval, for instance, to the polynomials of Hermite-Tchebycheff $[p(x)=e^{-x^2};\ (a,b)=(-\infty,+\infty)]$.

4. *From now on we shall assume* (a, b) *finite.* Suppose $p(x)$ satisfies certain general conditions (I°, II°) given in my paper: "Sur le développement de l'intégrale $\int_a^b\frac{p(y)}{x-y}dy$. . . ."† Then‡:

(16) $$\begin{cases}a_n(p)=\left(\dfrac{4}{b-a}\right)^n A(p)\,(1+0(1))\\[2mm] S_n(p)=n\left(\dfrac{b+a}{2}\right)+\sigma(p)+0(1),\end{cases}$$

where $A(p)>0$ and $\sigma(p)$ do not depend upon n.

The proof of (16) is based upon the relations (2) and upon the fact that in the case under consideration

(17) $$a_n(p)\sim\frac{2^{2n}}{(b-a)^n}.$$

*Darboux, *Mémoire sur l'approximation des fonctions des grands nombres . . . ,* Jour. de Math., III sér., t. IV (1878), 5-57, 377-416, p. 413.

†Rend. Circ. Mat. Palermo. t. 47 (1923), 25-46, p. 26.

‡*Ibid.*, p. 37.

In addition to (16), if $0 \leq a < b$, we have also (see (4)):

$$(18) \qquad b_{2n}(p) \to \left(\frac{\sqrt{b}+\sqrt{a}}{2}\right)^2, \; b_{2n+1}(p) \to \left(\frac{\sqrt{b}-\sqrt{a}}{2}\right)^2 \; (n \to \infty).$$

Using formulae (15, 16), we establish the following theorems:

Theorem I. For any real ξ *and for any* $p(x)$ *subjected to the conditions* (I°, II°) *above we have, as* $n \to \infty$:

$$1° \qquad \frac{K_{n+1}(p;\,\xi)}{K_n(p;\,\xi)} \to \begin{cases} \dfrac{(\sqrt{|\xi-a|}+\sqrt{|\xi-b|})^4}{(b-a)^2}, & (\xi \leq a \text{ or } \geq b), \\ 1, & (a \leq \xi \leq b); \end{cases}$$

$$2° \qquad \frac{\phi^2_{n+1}(p;\,\xi)}{K_n(p;\,\xi)} \to \begin{cases} \left(\dfrac{4}{b-a}\right)^2 \sqrt{(\xi-a)(\xi-b)}\,[\sqrt{|\xi-a|}+\sqrt{|\xi-b|}]^2, & (\xi \leq a \text{ or} \geq b), \\ 0, & (a \leq \xi \leq b); \end{cases}$$

$$3° \qquad \frac{\phi_{n+1}(p;\,\xi)}{\phi_n(p;\,\xi)} = \pm \frac{(\sqrt{|\xi-a|}+\sqrt{|\xi-b|})^2}{b-a} = \frac{4}{b-a} z_1, \quad \begin{pmatrix} +, & \xi \geq b \\ -, & \xi \leq a \end{pmatrix},$$

z_1 *being the root of the equation* $z^2 - z\left(\xi - \dfrac{b+a}{2}\right) + \left(\dfrac{b-a}{4}\right)^2 = 0$ *with the larger modulus*;

$$4° \qquad \left[\frac{\phi_{n+1}(p;\,x)}{\phi_n(p;\,x)}\right]'_{x=\xi} \to \frac{1}{b-a} \frac{(\sqrt{|\xi-a|}+\sqrt{|\xi-b|})^2}{\sqrt{(\xi-a)(\xi-b)}}, \quad (\xi < a \text{ or } > b).$$

Formula 3° proves (for ξ real) the famous Poincaré's theorem* *completely*, *i.e.*, it gives exactly the root of the characteristic equation involved. The above results are a particular case of a more general theorem.

Theorem II. In the development (6) *there exists*

$$\lim_{n=\infty} h^{(n)}_{n+i} = h_i \qquad (i = 0, 1, 2, \ldots, s),$$

provided $p(x)$ *satisfies the conditions* (I°, II°).

Making use of formulae (7, 8), from *Theorem II* we derive

Theorem III. Consider s *arbitrary points* $a_1, a_2, \ldots, a_s$ *(real or complex) subjected to the condition that* $\Pi(x) \equiv \pm \prod_{i=1}^{s} (x - a_i)$ *has real coefficients and is not negative in* (a, b). *Form all determinants* $\Delta_{n,i}$, *given by* (8). *Then, as* $n \to \infty$, $\dfrac{\Delta_{n,i}}{\Delta_{n,s}}$ *tends to a certain limit* $(i = 0, 1, 2, \ldots, s-1)$.

**Sur les équations linéaires* . . . , Amer. Jour. Math., v. VII, 1885, 203-285 p. 217.

5. We now use the results obtained by Szegö† and introduce the following *Definition*: *$p(x)$ is a "function (S)" if*

1°. *$p(x) \geq 0$ and is almost everywhere positive in (a, b);*

2°. *$p(x)$ and $\dfrac{\log p(x)}{\sqrt{(x-a)(b-x)}}$ are (IL), i.e., integrable in Lebesgue's sense in (a, b).*

Theorem IV. Formulae (16, 18), *and therefore Theorems I, II, III, hold for any characteristic function (S). Furthermore, under certain general conditions for $p(x)$,*

$$(19) \qquad a_n(p) = \frac{2^{2n}}{(b-a)^n}\sqrt{\frac{2}{(b-a)\pi}}\; e^{-\frac{1}{2\pi}\int_a^b \frac{\log p(x)}{\sqrt{(x-a)(b-x)}}dx}\,(1+0(1)),$$

$$(20) \qquad S_n(p) = \frac{n}{2}(b+a) + \frac{1}{2\pi}\int_a^b \frac{\{x-\frac{1}{2}(b+a)\}\log p(x)}{\sqrt{(x-a)(b-x)}}\,dx + 0\,(1).$$

Formula (19) was given by Szegö† and can be derived, by using (18). *Formula* (20) *seems to be a new formula, giving the asymptotic expression of the sum of the roots of $\phi_n(p;\, x)$ for $n \to \infty$.*

If we take, for example,

$$(a, b) = (-1, 1),\ p(x) = (1+x)^{\alpha-1}(1-x)^{\beta-1}q(x)$$
$$\alpha, \beta > 0,\ q(x) \equiv q(-x),\ \text{for } -1 \leq x \leq 1,$$

we get, using (20):

$$S_n(p) = \tfrac{1}{2}(\alpha-\beta),$$

a result obtained in a different way in my paper previously referred to‡.

It is interesting to notice that the comparison of Szegö's formula (19) with the results given in *Theorem I* enables us to evaluate certain definite integrals. For example:

$$(21) \qquad \int_a^b \frac{\log[(\xi-x)^2]\,dx}{\sqrt{(x-a)(b-x)}} = \begin{cases} 4\pi\log\left\{\dfrac{\sqrt{|\xi-b|}+\sqrt{|\xi-a|}}{2}\right\}, & \xi \leq a \text{ or } \geq b, \\[2ex] 2\pi\log\left(\dfrac{b-a}{4}\right), & a \leq \xi \leq b. \end{cases}$$

6. Using (18) and a linear transformation of the interval we derive the following formulae:

$$(22) \qquad \begin{cases} K_n(p;\, \xi) = \sqrt{b^*_{2n+2}(p^*)}\,\phi_n(p;\, \xi)\phi_n(p_1;\, \xi), \\ p_1(x) = \pm(x-\xi)p(x),\ p^*(x^*) \equiv p(\xi \pm x^*), \qquad (+,\ \xi \leq a),\ (-,\ \xi \geq b). \\ b^*_{2n+2}(p^*) = \left(\dfrac{\sqrt{|a-\xi|}+\sqrt{|b-\xi|}}{2}\right)^2. \end{cases}$$

7. The following theorem gives some properties of the h_i already obtained from *Theorem II:*

†*loc. cit.*, p. 207.
‡*loc. cit.*, p. 39.

Theorem V. Suppose $p_1(x)=\Pi(x)p(x)$, *where* $p(x)$ *is a function* (S) *and* $\Pi(x)=c\prod_{i=1}^{m}(x-a_i)^{2k_i}\prod_{j=1}^{e}(x-a_j)^{l_j}$ $(a<a_i<b;\sum_{i=1}^{m}2k_i+\sum_{j=1}^{e}l_j=s)$. *Consider the polynomial* $\phi(x)=\sum_{i=o}^{s}h_i x^i$ *given in Theorem II. To every* a_j *there corresponds a root of* $\phi(x)$ *of the form*

$$z_j=\frac{2}{b-a}\left[a_j-\frac{b+a}{2}+\sqrt{(a_j-a)(a_j-b)}\right]$$

of multiplicity l_j. *To every* a_i *there correspond two roots of* $\phi(x)$, *each of multiplicity* k_i, *and having the form*

$$z_i^{1,2}=\cos\theta\pm\sqrt{-1}\sin\theta=\frac{2}{b-a}\left[a_i-\frac{b+a}{2}\pm\sqrt{(a_i-a)(a_i-b)}\right],$$

$$\cos\theta=\frac{2}{b-a}\left(a_i-\frac{b+a}{2}\right).$$

8. We apply the general results given above to a special case—*a generaliation of Jacobi's polynomials*. Namely, we take

$$\text{(23)}\qquad \begin{cases} p(x)=(x-a)^{\alpha-1}(b-x)^{\beta-1}\Pi(x), & (\alpha,\beta>0),\\ \Pi(x)=cx^s+\dots-\text{polynomial}; & \Pi(a)\Pi(b)\neq 0.\end{cases}$$

Without loss of generality we assume $(a, b)=(0, 1)$.

Using the properties of Jacobi's polynomials*, we get the following results:

1° $0<\xi<1$, $\Pi(x)=(x-\xi)^{2m}\Pi_1(x)$, $(m\geq 0,\ \Pi_1(\xi)\neq 0)$;

$$\text{(24)}\qquad \begin{cases} \phi_n^{(e)}(p;\ \xi)=n^{m+e}[C_e\cos(n\theta+c_e)+\epsilon_n(\xi)], & (2\xi-1=\cos\theta),\\ K_n(p;\ \xi)=n^{2m+1}C(1+\epsilon_n(\xi)),\end{cases}$$

where C, C_e, c_e do not depend upon n, and e is any finite positive integer or zero, while $\epsilon_n(\xi)$ represents a quantity which $\longrightarrow 0$ as $n\longrightarrow\infty$.

2° $\xi=0,\ +1$;

$$\text{(25)}\qquad \begin{cases} \phi_n^{(e)}(p;\ 0)=(-1)^{n-e}n^{\alpha-\frac{1}{2}+2e}C_e'(1+0(1)), & (C_e'\neq 0),\\ \phi_n^{(e)}(p;\ 1)=n^{\beta-\frac{1}{2}+2e}C_e''(1+0(1)), & (C_e''\neq 0),\\ K_n(p;\ {}^0_1)=C_{1,2}n^{2{}^\alpha_\beta}(1+0(1)), & (C_{1,2}\neq 0)).\end{cases}$$

3° *ξ is any real or complex number not in the interval* $(0, 1)$;

$$\text{(26)}\qquad \begin{cases} \phi_n^{(e)}(p;\ \xi)=n^e z^n f_e(z)(1+\epsilon_n(z)),\\ K_n(p;\ \xi)=|z|^{2n}f(z)(1+\epsilon_n(z)),\\ |z|\equiv|2\xi-1+\sqrt{4\xi^2-4\xi}\,|,\ z\longrightarrow\infty\ \text{as}\ \xi\longrightarrow\infty;\end{cases}$$

*Darboux, *loc. cit.*, pp. 43, 44.

where $f_e(z)$, $f(z)$ do not depend upon n and are $\neq 0$.

Our formulae permit us to find also the asymptotic expression of

$$\sum_{i=0}^{n} \phi_i(p;\, \xi)\phi_i(p;\, \eta),$$

where ξ and η are any two numbers real or complex.

9. In the case under consideration (see formula (23); $a=0$, $b=1$) we are able to make the results concerning b_n, l_n in the continued fractions (4) more precise as follows:

$$\text{(27)}\qquad \begin{cases} b_{2n}=\dfrac{1}{4}+\dfrac{2\alpha-1}{8n}+\delta_n,\ b_{2n+1}=\dfrac{1}{4}-\dfrac{2\alpha-1}{8n}+\delta_n{}',\ (\delta_n, \delta_n{}'\rightarrow 0 \text{ as } n\rightarrow\infty); \\ l_{2n+1}=\phi_n^2(0)=n^{2\alpha-1}C'(1+0(1))\ \ (C'\neq 0),\ \ l_{2n}l_{2n+1}\rightarrow -4 \text{ as } n\rightarrow\infty\,^*. \end{cases}$$

10. In conclusion we consider the $n+1$ zeros of $\phi_{n+1}(p;\, x)$. These zeros, indicated by the notation $x_i^{(n+1)}$ $(i=1,\, \ldots,\, n+1)$, satisfy the inequalities:

$$\text{(28)}\qquad 0<x_1^{(n+1)}<x_2^{(n+1)}\ldots\ <x_{n+1}^{(n+1)}<1.$$

Using formula (13), we get:

$$\text{(29)}\qquad \begin{cases} x_1^{(n+1)}<\dfrac{a_n(p)}{a_{n+1}(p)}\,\dfrac{|\phi_{n+1}(p;\,0)|}{\sqrt{K_n(p;\,0)}}, \\ 1-x_{n+1}^{(n+1)}<\dfrac{a_n(p)}{a_{n+1}(p)}\,\dfrac{\phi_{n+1}(p;\,1)}{\sqrt{K_n(p;\,1)}}. \end{cases}$$

In any arbitrarily small interval $0<\alpha<\beta<1$ there exists at least one point $x=\lambda$ such that $\phi_n(p;\,\lambda)\neq 0$, for any n. Call $\delta_n(\lambda)$ the shortest distance from the point λ to the roots $x_i^{(n+1}(i=1,\, 2,\, \ldots,\, n+1)$. Then:

$$\text{(30)}\qquad \delta_n(\lambda)<\frac{a_n(p)}{a_{n+1}(p)}\,\frac{|\phi_{n+1}(p;\,\lambda)|}{\sqrt{K_n(p;\,\lambda)}}.$$

Formulae (29, 30) *hold for any* $p(x)$ and give, if $p(x)$ is a function (S):

$$\text{(31)}\qquad 1+x_1{}^{(n+1)},\ 1-x_{n+1}^{(n+1)},\ \delta_n(\lambda)\rightarrow 0 \text{ as } n\rightarrow\infty,$$

as is well known.

In the case of $p(x)$ given by (23), we have more precisely:

$$\text{(32)}\qquad \begin{cases} x_1{}^{(n+1)},\ 1-x_{n+1}^{(n+1)}=0\ (n^{-\frac{1}{2}}), \\ \delta_n(\lambda)=0\ (n^{-\frac{1}{2}}). \end{cases}$$

*See my paper in the Rend. Circ. Mat. Palermo, pp. 41-42.

11. Using the results given above we can find the asymptotic expressions for the coefficients of $\phi_n(p; x)$. Thus we find, for example, if $(a, b) = (-1, 1)$:

$$(33)\qquad \begin{cases} \phi_n(p;x) = a_n(p)\ (x^n - S_n(p)x^{n-1} + d_{n,n-2}(p)x^{n-2} + \dots), \\ d_{n,n-3}(p) = n(d(p) + 0(1)), \end{cases}$$

where $d(p)$ does not depend upon n*.

12. In the case $(a, b) = (-1, 1)$, $\alpha = \beta$, $\Pi(x) \equiv \Pi(-x)$ we have, if $\Pi(x) = x^{2m}\Pi_1(x)$ $(\Pi_1(0) \neq 0)$:

$$(34)\qquad \begin{cases} \phi_n^{(l)}(p;\ 0) = (-1)^{\frac{n-l}{2}}\ Cn^{m+l}(1+0(1)), \quad \text{for } n \equiv l \pmod 2, \quad (C \neq 0) \\ \phi_n^{(l)}(p;\ 0) = 0 \text{ for } n \equiv l+1 \pmod 2. \end{cases}$$

The proof of (34) is based upon the following Lemma:

Lemma. Consider the interval $(-h,h)$ *(finite or infinite) with the characteristic function* $p(x)$ *such that*

$$p(x) \equiv p(-x) \text{ for } -h \leq x \leq h.$$

Consider also the interval $(0, h^2)$ *with*

$$p_1(x) \equiv \frac{p(\sqrt{x})}{\sqrt{x}},\ p_2(x) \equiv \sqrt{x}\, p(\sqrt{x}).$$

Then,

$$\phi_{2n}(p;\ x) \equiv \phi_n(p_1;\ x^2);\ \phi_{2n+1}(p;\ x) \equiv x\phi_n(p_2;\ x^2).$$

Using this Lemma we reduce in many cases the investigation of Tchebycheff polynomials corresponding to $(-\infty, \infty)$ to those corresponding to $(0, \infty)$. This is, for example, the case of the two systems of Tchebycheff polynomials: Hermite-Tchebycheff and Laguerre-Tchebycheff.

*See my paper, *loc. cit.*, p. 46.