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Research Article

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# Computing Efficiency for Decision Making Units with Negative and Interval Data

M. Piri[1], SH. Givehchi[1], M. Keshavarz[1*]

*(1) Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran.*



**Abstract**

Data Envelopment Analysis (DEA) is a nonparametric method for identifying sources and estimating the mount of inefficiencies contained in inputs and outputs produced by Decision Making Units (DMUs). DEA requires that the data for all inputs and outputs should be known exactly, but under many qualifications, exact data are inadequate to model real-life situations. So these data may have different structures such as bounded data, interval data, and fuzzy data.

Moreover, the main assumption in all DEA is that input and output values are positive, but we confront many cases that discount this condition producing negative data. The purpose of this paper is to compute efficiency for DMUs, which permits the presence of intervals which can take both negative and positive values.

**Keywords**: Data Envelopment Analysis, Efficiency, Interval data, Positive and Negative data, SORM Model.

## 1 Introduction

Data envelopment analysis (DEA) is used to identify best practices and efficient frontier decision making units (DMUs) in the presence of multiple inputs and outputs (Charnes et al., [1]). DEA provides not only efficiency scores for inefficient DMUs but also frontier projections for such units onto an full-efficient frontier. The old DEA models did not deal with imprecise data assuming that all input and output data are exactly known. In real world situations, this assumption may not always be true. If such imprecise data information is integrated into the standard linear CCR model, the resulting DEA model is a nonlinear and non convex program, called imprecise DEA (IDEA) (Cooper et al. [2, 3]). In addition to Lee et al. [4], Zhu [5], Thompson et al. [6] investigated IDEA. Recently, upper and lower bounds for the efficiency scores of the DMUs with imprecise data have been calculated by Despotits and Smirlis [7].

In Jahanshahloo et al. [8] the radius of stability for the DMUs with interval data is calculated. Also, in Jahanshahloo et al. [9] Ranking DMUs with interval data using interval super efficiency index is extended. Moreover, the main assumption in all DEA is that input and output values are positive, but we encounter many cases that violate this term ultimately yielding negative inputs and outputs. Among the proposed methods of dealing with negative data, the following models could be provided.

* Corresponding Author. Email address: keshavarz_masome@yahoo.com

Seiford and Zhu. [10] considered a positive and very small value of negative output. Another method was proposed by Halme et al. [11] and modified slack-based measure model, called MSBM was represented by Sharp et al [12].

However, the latest method of behavior with negative data was provided by Emrouznejad et al [13, 14], which is based on SORM model where some variables are considered which are both negative and positive for DMUs. Consequently, radial methods of DEA were modified for the evaluation of the efficiency of units by negative data. Data Envelopment Analysis (DEA) with integer and negative inputs and outputs was proposed by Jahnshahloo and Piri [15]. The main objective of this paper is to decide how to deal with decision making units that have negative and interval inputs and outputs.

This paper is organized as follows. In section 2 we calculate efficiency of decision making units with interval input and output and also with negative and positive input and output. Section 3 discusses how we can calculate efficiency for decision making units using both negative and interval input and output. A numerical example is provided in section 4 and the paper concludes in section 5.

## 2 Efficiency of Decision Making Units with Interval Input and Output.

Now, suppose we have n DMUs which utilize m inputs $x_{ij}\,(i=1,\dots,m)$ to produce s outputs $y_{rj}\,(r=1,\dots,s)$. Also, assume that, input and output levels of each DMU are not precisely known. Let $x_{ij}\in\left[x_{ij}^{L},x_{ij}^{U}\right]$ and $y_{rj}\in\left[y_{rj}^{L},y_{rj}^{U}\right]$ ; $x_{ij}^{L}>0$ , $y_{rj}^{L}>0$ where lower and upper bounds are precisely known, i.e., positive and finite. For $DMU_p$ the following model provides an upper limit of interval efficiency:

$$
\begin{aligned}
&\theta_p^U := \mathrm{Min}\ \theta \\
&\text{s.t} \quad \textstyle\sum_{j=1,\,j\neq p}^{n}\lambda_j x_{ij}^{U}+\lambda_p x_{ip}^{L} \le \theta x_{ip}^{L} && i=1,\dots,m \\
&\qquad \textstyle\sum_{j=1,\,j\neq p}^{n}\lambda_j y_{rj}^{L}+\lambda_p y_{rp}^{U} \ge y_{rp}^{U} && r=1,\dots,s \qquad (2.1)\\
&\qquad \textstyle\sum_{j=1}^{n}\lambda_j=1 \\
&\qquad \lambda_j\ge 0 && j=1,\dots,n
\end{aligned}
$$

We denote by $\theta_p^U$ the efficiency score attained by $DMU_p$ in model (2.1). In this case, $DMU_p$ has the best conditions and other DMUs are in the worst. Moreover, the following model provides a lower limit of interval efficiency score for $DMU_p$:

$$
\begin{aligned}
&\theta_p^L := \mathrm{Min}\ \theta \\
&\text{s.t} \quad \textstyle\sum_{j=1,\,j\neq p}^{n}\lambda_j x_{ij}^{L}+\lambda_p x_{ip}^{U} \le \theta x_{ip}^{U} && i=1,\dots,m \\
&\qquad \textstyle\sum_{j=1,\,j\neq p}^{n}\lambda_j y_{rj}^{U}+\lambda_p y_{rp}^{L} \ge y_{rp}^{L} && r=1,\dots,s \qquad (2.2)\\
&\qquad \textstyle\sum_{j=1}^{n}\lambda_j=1 \\
&\qquad \lambda_j\ge 0 && j=1,\dots,n
\end{aligned}
$$

The efficiency $\theta_p^L$ attained by $DMU_p$ in model (2.2) serves as a lower bound of its possible efficiency scores. In this case, $DMU_p$ has the worst conditions and others DMUs are in the best conditions. Models (2.1) and (2.2) provide each DMU with a bounded interval $\left[\theta_p^L, \theta_p^U\right]$ in which possible efficiency scores lie from the worst to the best case. Considering (2.1) and (2.2), it is evident that $\theta_p^L \leq \theta_p^U$. On the basis of the above efficiency score intervals, DMUs can be classified in three subsets as follows:

$$
\begin{aligned}
&E^{++} = \{ j \in J \mid \theta_j^L = 1\} \\
&E^{+} = \{ j \in J \mid \theta_j^L < 1 \text{ and } \theta_j^U = 1\} \qquad (2.3) \\
&E^{-} = \{ j \in J \mid \theta_j^U < 1\}
\end{aligned}
$$

**Definition 2.1.** $DMU_p$ *is strongly efficient if* $\theta_p^{*L} = 1$ and $\theta_p^{*U} = 1$ *and* $DMU_p$ *is efficient if* $\theta_p^{*L} < 1$ and $\theta_p^{*U} = 1$ *and if* $\theta_p^{*U} < 1$ *then* $DMU_p$ *is inefficient.*

**Theorem 2.1.** *If* $\theta_p^{*U}$ , $\theta_p^{*L}$ *are the optimal solutions for models (2.1) and (2.2), respectively, then we have* $\theta_p^{*L} \leq \theta_p^{*U}$.

*Proof. It is evident.*

**2.1. Efficiency of Decision Making Units with Negative and Positive Input and Output**

In this section, we shall treat each variable that has positive values for some and negative for other DMUs as consisting of the sum of two variables as follows. Let us assume we have n DMUs ($DMU_j \;\; j = 1, \ldots, n$) each associated with m inputs; $x_{ij}\,(i = 1, \ldots, m)$ and s outputs $y_{rj}\,(r = 1, \ldots, s)$;

Also, let

$$
\begin{aligned}
&I = \left\{ i \in \{1, \ldots, m\} : x_{ij} \geq 0 \text{ , } j = 1, \ldots, n \right\} \\
&L = \left\{ l \in \{1, \ldots, m\} : \exists\, j \in \{1, \ldots, n\} \text{ ; for which } x_{lj} < 0 \right\} \\
&R = \left\{ r \in \{1, \ldots, s\} : y_{rj} \geq 0 \;\; j = 1, \ldots, n \right\} \qquad (2.4) \\
&K = \left\{ k \in \{1, \ldots, s\} : \exists\, j \in \{1, \ldots, n\} \text{ ; for which } y_{kj} < 0 \right\} \\
&I \cup L = \{1, \ldots, m\} \quad , \quad R \cup K = \{1, \ldots, s\} \quad , \quad I \cap L = \varnothing \text{ ,} \quad R \cap K = \varnothing
\end{aligned}
$$

That is, the set of index of inputs with nonnegative values is indicated by I while L denotes the set of index of inputs which have negative value in at least one DMU. Similarly, R is the set of index of the outputs with nonnegative values and K is the set of index of outputs which have a negative value in at least one observation. Let us take an output variable $y_k$ which is positive for some DMUs and negative for others. Let us define two variables $y_k^1$ and $y_k^2$ which for the j-th DMU take values $y_{kj}^1$ and $y_{kj}^2$ such that

$$\forall \; k \in K \qquad y_{kj}^{1} = \begin{cases} y_{kj} & \text{if} \quad y_{kj} \geq 0 \\ 0 & \text{if} \quad y_{kj} < 0 \end{cases}$$

$$\forall \; k \in K \qquad y_{kj}^{2} = \begin{cases} 0 & \text{if} \quad y_{kj} \geq 0 \\ -y_{kj} & \text{if} \quad y_{kj} < 0 \end{cases} \tag{2.5}$$

Note that we have $y_{kj} = y_{kj}^1 - y_{kj}^2$ for each $k \in K$ where $y_{kj}^2 \geq 0$ , $y_{kj}^1 \geq 0$, $(\mathrm{j} = 1, \dots, \mathrm{n})$. Similarly, we define two variables $x_l^1$ and $x_l^2$ which for the j-th DMU take values $x_{lj}^1$ and $x_{lj}^2$ such that

$$\forall \; l \in L \qquad x_{lj}^{1} = \begin{cases} x_{lj} & \text{if} \;\; x_{lj} \geq 0 \\ 0 & \text{if} \;\; x_{lj} < 0 \end{cases}$$

$$\forall \; l \in L \qquad x_{lj}^{2} = \begin{cases} 0 & \text{if} \;\; x_{lj} \geq 0 \\ -x_{lj} & \text{if} \;\; x_{lj} < 0 \end{cases} \tag{2.6}$$

We have $x_{lj} = x_{lj}^1 - x_{lj}^2$ for each $l \in L$ where $x_{lj}^2 \geq 0$ , $x_{lj}^1 \geq 0$ , $(\mathrm{j} = 1, \dots, \mathrm{n})$.
Model (2.7) represents the general case for an input oriented VRS DEA model which has both inputs and outputs which take positive values for some DMUs and negative for others.

$$\begin{array}{lll}
\tilde{\theta}_p := \mathrm{Min} & \theta & \\
\text{s.t} & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{ip} & i \in I \\
& \sum_{j=1}^{n} \lambda_j x_{lj}^{1} \leq \theta x_{lp}^{1} & l \in L \\
& \sum_{j=1}^{n} \lambda_j x_{lj}^{2} \geq (2-\theta) x_{lp}^{2} & l \in L \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{rp} & r \in R \\
& \sum_{j=1}^{n} \lambda_j y_{kj}^{1} \geq y_{kp}^{1} & k \in K \\
& \sum_{j=1}^{n} \lambda_j y_{kj}^{2} \leq y_{kp}^{2} & k \in K \\
& \sum_{j=1}^{n} \lambda_j = 1 & \\
& \lambda_j \geq 0 & j = 1, \dots, n
\end{array} \tag{2.7}$$

Based on this optimal solution, we define a DMU as being SORM-Efficient as follows.
**Definition 2.2.** *(SORM - Efficient).* $\mathrm{DMU}_p$ *is SORM - Efficient, if* $\tilde{\theta}_p^* = 1$.

### 3 Efficiency for Decision Making Unit with Negative and Interval Data.

In this section, we discuss how to calculate the efficiency for decision making units both negative and interval input and output. Now suppose we have n DMUs where input and output levels of each DMU are not known exactly. Let $x_{ij} \in \left[ x_{ij}^L, x_{ij}^U \right]$ and $y_{rj} \in \left[ y_{rj}^L, y_{rj}^U \right]$ , where lower and upper bounds are exactly known, finite, positive or/ and negative.

At first, we divide the inputs and outputs into two groups, as follows:

$$
\begin{aligned}
&I=\left\{i\in\{1,\dots,m\}: x_{ij}^{L}\geq 0, x_{ij}^{U}\geq 0, j=1,\dots,n\right\}\\
&L=\left\{l\in\{1,\dots,m\}: \exists j\in\{1,\dots,n\};\ (x_{lj}^{L}<0\ \&\ x_{lj}^{U}<0)\ \text{or}\ (x_{lj}^{L}<0\ \&\ x_{ij}^{U}\geq 0)\right\}\\
&R=\left\{r\in\{1,\dots,s\}: y_{rj}^{L}\geq 0, y_{rj}^{U}\geq 0, j=1,\dots,n\right\}\\
&K=\left\{k\in\{1,\dots,s\}: \exists j\in\{1,\dots,n\};\ (y_{kj}^{L}<0\ \&\ y_{kj}^{U}<0)\ \text{or}\ (y_{kj}^{L}<0\ \&\ y_{kj}^{U}\geq 0)\right\}\\
&I\cup L=\{1,\dots,m\},\quad R\cup K=\{1,\dots,s\},\quad I\cap L=\varnothing,\quad R\cap K=\varnothing
\end{aligned}
\tag{3.8}
$$

That is, the set of index of inputs with nonnegative values is represented by I whereas L denotes the set of index of inputs which have negative values in at least one DMU. In the same way, R is the set of index of the outputs with nonnegative values and K is the set of index of outputs with a negative value in at least one observation.

Thus, for $DMU_p$ the following model provides an upper limit of interval efficiency when we encounter negative and interval data.

$$
\begin{aligned}
\theta_p^L := \ & \text{Min}\quad \theta\\
\text{s.t}\quad & \sum_{j=1, j\neq p}^{n}\lambda_j x_{ij}^{U}+\lambda_p x_{ip}^{L}\leq \theta x_{ip}^{L} && i\in I\\
& \sum_{j=1, j\neq p}^{n}\lambda_j x_{lj}^{1U}+\lambda_p x_{lp}^{1L}\leq \theta x_{lp}^{1L} && l\in L\\
& \sum_{j=1, j\neq p}^{n}\lambda_j x_{lj}^{2U}+\lambda_p x_{lp}^{2L}\leq (2-\theta) x_{lp}^{2L} && l\in L\\
& \sum_{j=1, j\neq p}^{n}\lambda_j y_{rj}^{L}+\lambda_p y_{rp}^{U}\geq y_{rp}^{U} && r\in R\\
& \sum_{j=1, j\neq p}^{n}\lambda_j y_{kj}^{1L}+\lambda_p y_{kp}^{1U}\geq y_{rp}^{1U} && k\in K\\
& \sum_{j=1, j\neq p}^{n}\lambda_j y_{kj}^{2L}+\lambda_p y_{kp}^{2U}\geq y_{kp}^{2U} && k\in K\\
& \sum_{j=1}^{n}\lambda_j=1\\
& \lambda_j\geq 0 && j=1,\dots,n
\end{aligned}
\tag{3.9}
$$

In this case, $DMU_p$ has the best conditions and other DMUs are in the worst conditions. Also, the following model provides a lower limit of interval efficiency score for $DMU_p$ when we encounter negative and interval data.

$$
\begin{aligned}
\theta_p^L := \text{Min}\ \ & \theta \\
\text{s.t}\ \ & \sum_{j=1, j\neq p}^{n} \lambda_j x_{ij}^{L} + \lambda_p x_{ip}^{U} \leq \theta x_{ip}^{U} && i \in I \\
& \sum_{j=1, j\neq p}^{n} \lambda_j x_{lj}^{1\,L} + \lambda_p x_{lp}^{1\,U} \leq \theta x_{lp}^{1U} && l \in L \\
& \sum_{j=1, j\neq p}^{n} \lambda_j x_{lj}^{2\,L} + \lambda_p x_{lp}^{2\,U} \leq (2-\theta) x_{lp}^{2\,U} && l \in L \\
& \sum_{j=1, j\neq p}^{n} \lambda_j y_{rj}^{U} + \lambda_p y_{rp}^{L} \geq y_{rp}^{L} && r \in R \\
& \sum_{j=1, j\neq p}^{n} \lambda_j y_{kj}^{1\,U} + \lambda_p y_{kp}^{1\,L} \geq y_{rp}^{1\,L} && k \in K \\
& \sum_{j=1, j\neq p}^{n} \lambda_j y_{kj}^{2\,U} + \lambda_p y_{kp}^{2\,L} \geq y_{kp}^{2\,L} && k \in K \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda_j \geq 0 && j = 1, \ldots, n
\end{aligned}
\tag{3.10}
$$

**Theorem 3.1.** *If* $\tilde{\theta}_p^{*}$ , $\theta_p^{*U}$ , $\theta_p^{*L}$ *are the optimal solutions for models (2.7), (3.9) and (3.10) respectively then we have* $\theta_p^{*L} \leq \tilde{\theta}_p^{*} \leq \theta_p^{*U}$.

*Proof.* Assume $(\tilde{\lambda}_p^{*}, \tilde{\theta}_p^{*})$ , $(\lambda_p^{*U}, \theta_p^{*U})$ , $(\lambda_p^{*L}, \theta_p^{*L})$ are the optimal solutions for models (2.7), (3.9) and (3.10), respectively. At first, we prove $\tilde{\theta}_p^{*} \leq \theta_p^{*U}$

Because $(\lambda_p^{*U}, \theta_p^{*U})$ is an optimal solution, it is sufficient to prove that $(\lambda_p^{*U}, \theta_p^{*U})$ for model (3.10) is a feasible solution. Thus we have:

$$
\begin{cases}
\sum_{j=1, j\neq p}^{n} \lambda_j^{*U} x_{ij}^{U} + \lambda_p^{*U} x_{ip}^{L} \leq \theta_p^{*U} x_{ip}^{L} & i \in I \\
\sum_{j=1, j\neq p}^{n} \lambda_j^{*U} x_{lj}^{1U} + \lambda_p^{*U} x_{lp}^{1\,L} \leq \theta_p^{*U} x_{lp}^{1\,L} & l \in L \\
\sum_{j=1, j\neq p}^{n} \lambda_j^{*U} x_{lj}^{2\,U} + \lambda_p^{*U} x_{lp}^{2\,L} \leq (2-\theta_p^{*U}) x_{lp}^{2\,L} & l \in L \\
\sum_{j=1, j\neq p}^{n} \lambda_j^{*U} y_{rj}^{L} + \lambda_p^{*U} y_{rp}^{U} \geq y_{rp}^{U} & r \in R \\
\sum_{j=1, j\neq p}^{n} \lambda_j^{*U} y_{kj}^{1\,L} + \lambda_p^{*U} y_{kp}^{1\,U} \geq y_{rp}^{1\,U} & k \in K \\
\sum_{j=1, j\neq p}^{n} \lambda_j^{*U} y_{kj}^{2\,L} + \lambda_p^{*U} y_{kp}^{2\,U} \geq y_{kp}^{2\,U} & k \in K
\end{cases}
\tag{3.11}
$$

We know $x_{ij}^{L} \leq \tilde{x}_{ij} \leq x_{ij}^{U}$ and $y_{rj}^{L} \leq \tilde{y}_{rj} \leq y_{rj}^{U}$

$$\begin{cases} \sum_{j=1,j\neq p}^{n}\lambda_j^{*U}\tilde{x}_{ij} \le \sum_{j=1,j\neq p}^{n}\lambda_j^{*U}x_{ij}^{U} \le (\theta_p^{*U}-\lambda_p^{*U})x_{ip}^{L} \le (\theta_p^{*U}-\lambda_p^{*U})\,\tilde{x}_{ip} & i\in I \\ \sum_{j=1,j\neq p}^{n}\lambda_j^{*U}\tilde{x}_{lj}^{1} \le \sum_{j=1,j\neq p}^{n}\lambda_j^{*U}x_{lj}^{1U} \le (\theta_p^{*U}-\lambda_p^{*U})x_{lp}^{1L} \le (\theta_p^{*U}-\lambda_p^{*U})\tilde{x}_{lp}^{1} & l\in L \\ \sum_{j=1,j\neq p}^{n}\lambda_j^{*U}\tilde{x}_{lj}^{2} \ge \sum_{j=1,j\neq p}^{n}\lambda_j^{*U}x_{lj}^{2U} \ge ((2-\theta_p^{*U})-\lambda_p^{*U})x_{lp}^{2L} \ge ((2-\theta_p^{*U})-\lambda_p^{*U})\tilde{x}_{lp}^{2} & l\in L \\ \sum_{j=1,j\neq p}^{n}\lambda_j^{*U}\tilde{y}_{rj} \ge \sum_{j=1,j\neq p}^{n}\lambda_j^{*U}y_{rj}^{L} \ge (1-\lambda_p^{*U})y_{rp}^{U} \ge (1-\lambda_p^{*U})\tilde{y}_{rp} & r\in R \\ \sum_{j=1,j\neq p}^{n}\lambda_j^{*U}\tilde{y}_{kj}^{1} \ge \sum_{j=1,j\neq p}^{n}\lambda_j^{*U}y_{kj}^{1L} \ge (1-\lambda_p^{*U})y_{kp}^{1U} \ge (1-\lambda_p^{*U})\tilde{y}_{rp}^{1} & k\in K \\ \sum_{j=1,j\neq p}^{n}\lambda_j^{*U}\tilde{y}_{kj}^{2} \le \sum_{j=1,j\neq p}^{n}\lambda_j^{*U}y_{kj}^{2L} \le (1-\lambda_p^{*U})y_{kp}^{2U} \le (1-\lambda_p^{*U})\tilde{y}_{kp}^{2} & k\in K \end{cases} \quad (3.12)$$

$$\begin{cases} \sum_{j=1,j\neq p}^{n}\lambda_j^{*U}\tilde{x}_{ij} \le \theta_p^{*U}\tilde{x}_{ip}-\lambda_p^{*U}\tilde{x}_{ip} & i\in I \\ \sum_{j=1,j\neq p}^{n}\lambda_j^{*U}\tilde{x}_{lj}^{1} \le \theta_p^{*U}\tilde{x}_{lp}^{1}-\lambda_p^{*U}\tilde{x}_{lp}^{1} & l\in L \\ \sum_{j=1,j\neq p}^{n}\lambda_j^{*U}\tilde{x}_{lj}^{2} \ge (2-\theta_p^{*U})\tilde{x}_{lp}^{2}-\lambda_p^{*U}\tilde{x}_{lp}^{2} & l\in L \\ \sum_{j=1,j\neq p}^{n}\lambda_j^{*U}\tilde{y}_{rj} \ge \tilde{y}_{rp}-\lambda_p^{*U}\tilde{y}_{rp} & r\in R \\ \sum_{j=1,j\neq p}^{n}\lambda_j^{*U}\tilde{y}_{kj}^{1} \ge \tilde{y}_{rp}^{1}-\lambda_p^{*U}\tilde{y}_{rp}^{1} & k\in K \\ \sum_{j=1,j\neq p}^{n}\lambda_j^{*U}\tilde{y}_{kj}^{2} \le \tilde{y}_{kp}^{2}-\lambda_p^{*U}\tilde{y}_{kp}^{2} & k\in K \end{cases}$$

(3.13)

$$\begin{cases} \sum_{j=1,j\neq p}^{n}\lambda_j^{*U}\tilde{x}_{ij} \le \theta_p^{*U}\tilde{x}_{ip} & i\in I \\ \sum_{j=1,j\neq p}^{n}\lambda_j^{*U}\tilde{x}_{lj}^{1} \le \theta_p^{*U}\tilde{x}_{lp}^{1} & l\in L \\ \sum_{j=1,j\neq p}^{n}\lambda_j^{*U}\tilde{x}_{lj}^{2} \ge (2-\theta_p^{*U})\tilde{x}_{lp}^{2} & l\in L \\ \sum_{j=1,j\neq p}^{n}\lambda_j^{*U}\tilde{y}_{rj} \ge \tilde{y}_{rp} & r\in R \\ \sum_{j=1,j\neq p}^{n}\lambda_j^{*U}\tilde{y}_{kj}^{1} \ge \tilde{y}_{rp}^{1} & k\in K \\ \sum_{j=1,j\neq p}^{n}\lambda_j^{*U}\tilde{y}_{kj}^{2} \le \tilde{y}_{kp}^{2} & k\in K \end{cases} \quad (3.14)$$

So, $(\lambda_p^{*U},\theta_p^{*U})$ is a feasible solution for model (3.10) and we know $(\tilde{\lambda}_p^{*},\tilde{\theta}_p^{*})$ is an optimal solution for this model. Because the problem is minimizing, the proof is complete. For proving $\theta_p^{*L} \le \tilde{\theta}_p^{*}$ the procedure is similar.

## 4 A numerical example

In this section, the model used in a numerical example attempts to measure the efficiency of 10 DMUs. Suppose that there are 10 DMUs with two interval inputs and outputs shown in Table (1). The second input and output have an interval value which does not have any sign. It means that there is a positive value for some of DMUs and a negative value for some others.

Table 1: DMUs with primary inputs and outputs.

| | In I1 | In L1 | Out R1 | Out K1 |
|---|---|---|---|---|
| DMU(1) | (3,8) | (-1,5) | (3,4) | (-5,-2) |
| DMU(2) | (4,4) | (3,7) | (1,5) | (-1,3) |
| DMU(3) | (5,9) | (-6,-2) | (6,8) | (-2,1) |
| DMU(4) | (4,5) | (-1,4) | (4,4) | (1,7) |
| DMU(5) | (3,6) | (-5,-2) | (3,7) | (2,5) |
| DMU(6) | (6,8) | (1,3) | (2,6) | (-2,4) |
| DMU(7) | (1,4) | (6,8) | (3,5) | (5,8) |
| DMU(8) | (7,7) | (-3,2) | (1,2) | (-5,-2) |
| DMU(9) | (5,8) | (-2,3) | (7,9) | (-1,3) |
| DMU(10) | (2,5) | (2,6) | (5,7) | (-3,5) |

Considering what was mentioned above, variables with no sign are converted to two variables shown in Table (2).Using the interval DEA models (3.9) and (3.10), we obtain an upper and lower limit of interval efficiency score for $DMU_p$.

Table 2: Unsigned variables for DMUs converted to two variables.

| | In I1 | In L1 | In L2 | Out R1 | Out K1 | Out K2 |
|---|---|---|---|---|---|---|
| DMU(1) | (3,8) | (0,5) | (0,1) | (3,4) | (0,0) | (2,5) |
| DMU(2) | (4,4) | (3,7) | (0,0) | (1,5) | (0,3) | (0,1) |
| DMU(3) | (5,9) | (0,0) | (2,6) | (6,8) | (0,1) | (0,2) |
| DMU(4) | (4,5) | (0,4) | (0,1) | (4,4) | (1,7) | (0,0) |
| DMU(5) | (3,6) | (0,0) | (2,5) | (3,7) | (2,5) | (0,0) |
| DMU(6) | (6,8) | (1,3) | (0,0) | (2,6) | (0,4) | (0,2) |
| DMU(7) | (1,4) | (6,8) | (0,0) | (3,5) | (5,8) | (0,0) |
| DMU(8) | (7,7) | (0,2) | (0,3) | (1,2) | (0,0) | (2,5) |
| DMU(9) | (5,8) | (0,3) | (0,2) | (7,9) | (0,3) | (0,1) |
| DMU(10) | (2,5) | (2,6) | (0,0) | (5,7) | (0,5) | (0,3) |

Table 3: Lower and Upper Efficiency for DMUs.

| | Lower Eff | Upper Eff |
|---|---|---|
| DMU(1) | 1 | 1 |
| DMU(2) | 0.32752411 | 0.89486615 |
| DMU(3) | 1 | 1 |
| DMU(4) | 0.47619048 | 1 |
| DMU(5) | 0.54761905 | 1 |
| DMU(6) | 0.72321873 | 1 |
| DMU(7) | 0.5 | 0.93465124 |
| DMU(8) | 0.81120345 | 1 |
| DMU(9) | 0.72352218 | 0.93332617 |
| DMU(10) | 1 | 1 |

So far, as we have showed that DMU1, DMU3, DMU10 are strongly efficient, that DMU2, DMU5, DMU6, DMU8 are efficient and that DMU2, DMU7, DMU9 are inefficient.

## 5 Conclusion

The standard DEA model cannot be used for efficiency assessment of decision making units with negative and interval data. In this paper we have developed a new pair of interval DEA models to deal with imprecise data such as interval and negative data. Then we calculated the efficiency of these decision making units.

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