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The Normality of Products

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By a *space*, we mean a *nondiscrete* Hausdorff space, and by a *map* we mean a continuous function. References to early papers can be found in the papers listed in the bibliography, particularly in [3].

Until paracompactness was defined, we essentially knew nothing about the normality of products. Sorgenfrey's half-open interval topology on the line was the first example of a normal (paracompact) space whose square is not normal. Michael gave a similar example of a metric space and a normal space whose product is not normal. Nonnormal products with one compact and one normal factor have also been known for many years. It was known that $\omega_1 \times (\omega_1 + 1)$ is not normal when Tamano proved that, for completely regular X, $X \times \beta X$ is normal if and only if X is paracompact. Dowker and Katětov independently proved that if I is the closed unit interval (or any compact metric space), then $I \times Y$ is normal if and only if Y is normal and countably paracompact. Extending this theorem, Morita proved that, for infinite cardinals λ, $I^\lambda \times Y$ is normal if and only if Y is normal and λ-paracompact. The problem of finding a Dowker space (normal but not countably paracompact) seemed important partly because of Borsuk's homotopy extension theorem which had been proved to hold for spaces Y where $I \times Y$ is normal.

Closed maps preserve normality, paracompactness, collectionwise normality, λ-paracompactness, and λ-collectionwise normality. For a given space X, let $\mathscr{N}(X)$ denote the class of all spaces Y such that $X \times Y$ is normal. Let $\mathscr{N}$ be the class of all spaces X such that $\mathscr{N}(X)$ is closed under closed maps. The class $\mathscr{N}(X)$ is trivially closed under perfect maps since, if $f: X \to Z$ is perfect, then $(f \times \mathrm{id}_Y): (X \times Y) \to (Z \times Y)$ is perfect. Morita asked if all metric and all compact spaces belong to $\mathscr{N}$.

Four years ago in Nice, Nagami spoke on the normality of products [5]. He stressed the beautiful work which had been done in space classification, particularly in the discovery of a number of useful classes which are preserved under countable products. He called for answers to Morita's questions as well as to others almost all of which have now been answered.

We know that there is a Dowker space [6]; this Dowker space is a subset of a box product of $\{\omega_n\}_{n\in\omega}$ and its cardinal functions are basically bounded below by $\aleph_\omega$. If there is a Souslin line, we know that there is a Dowker space of cardinality $\aleph_1$ which is hereditarily separable. A Souslin tree of cardinality λ, where λ is the successor of a regular cardinal, can be used to construct a Dowker space most of whose cardinal functions are $\leqq \lambda$.

QUESTION 1. *Is the existence of a separable (ccc, 1st countable, cardinality $\aleph_1$) Dowker space independent of the usual axioms for set theory?*

QUESTION 2. *Is there a Dowker space $X \times Y$ such that neither X nor Y is a Dowker space?*

Normality does not imply countable paracompactness nor vice versa. However [8] if X is a metric space and Y is a normal space, then $X \times Y$ is normal if and only if $X \times Y$ is countably paracompact; in addition if $X \times Y$ is normal, then $X \times Y$ is λ-paracompact (λ-collectionwise normal) if and only if Y is. In fact,

THEOREM 1 [8]. *Suppose that X is metric, C is compact, and Y is normal and λ-paracompact. Then the following are equivalent:*

(a) *$X \times Y$ and $X \times C$ are both normal.*

(b) *$X \times Y \times C$ is normal and countably paracompact.*

(c) *$X \times Y \times C$ is normal and λ-paracompact.*

This allows us to answer Morita's questions. Both the class of all metric spaces and the class of all compact spaces are contained in $\mathcal{N}$. In fact,

THEOREM 2 [8]. *If X is a metric space and C is a compact space, $X \times C \times Y$ is normal, and Z is the image of Y under a closed map, then $X \times C \times Z$ is normal.*

All of the questions about normality in products with a metric factor are tied to the countable paracompactness of the *product*. However, when one looks at normality questions for products with a compact factor X, the basic requirement lies between the $w(X)$-collectionwise normality and the $w(X)$-paracompactness; neither condition is both necessary and sufficient in all cases. By $w(X)$ we mean the weight of X or the minimal cardinality of a basis for X.

THEOREM 3 [7], [10]. *Suppose that X is a compact space and that Y is a normal space. If $X \times Y$ is normal, then Y is $w(X)$-collectionwise normal. If Y is $w(X)$-paracompact, then $X \times Y$ is normal.*

Necessary and sufficient conditions for the product of a compact space and a normal space to be normal must of necessity be complicated, but one can give such conditions which together with the following basic lemma are sufficient to prove that all compact spaces belong to $\mathcal{N}$.

THEOREM 4 [7]. *Assume that λ is a cardinal, that Y is a space which is normal and α-collectionwise normal for all $\alpha < \lambda$, that $\mathcal{G}$ is an open cover of Y of cardinality λ,* *and that $\mathscr{H}$ is a hereditarily closure-preserving closed refinement of $\mathscr{G}$. Then $\mathscr{G}$ has a locally finite refinement.*

Starbird proves [8] that $\mathscr{N}$ is not the class of all spaces. However the following basic questions remain unanswered:

QUESTION 3. *Is there a paracompact (or collectionwise normal) space not in $\mathscr{N}$?*

QUESTION 4. *Is there a paracompact p-space not in $\mathscr{N}$?*

The behavior of products with a compact factor leads one to a theory of *test spaces*; a space X is a test space for property P provided a space Y has property P if and only if $X \times Y$ is normal. Besides I^λ, $\lambda + 1$ is a test space for λ-paracompactness. The one-point compactification of a discrete set of cardinality λ is a test space for λ-collectionwise normality [10].

Starbird [9] and Morita [4] independently discovered the following remarkable theorem.

THEOREM 5. *If C is a closed subset of a normal space X, A is any compact or metrizable absolute neighborhood retract, and $f: (C \times I) \cup (X \times \{0\}) \to A$ is a map, then there exists a map extending f to $X \times I$.*

Thus the binormal hypothesis in Borsuk's homotopy extension theorem is unnecessary! Both Starbird and Morita also discovered a related extension theroem:

THEOREM 6. *If X is a compact space, then any map from a closed subset of $C(X)$ into a $w(X)$-collectionwise normal space has an extension to $C(X)$.*

Still unsolved is:

QUESTION 5. *If X is normal, C is a closed subset of X, and f:$(C \times I) \cup (X \times \{0\}) \to Y$ is continuous, can f be extended to $X \times I$ if Y is an* ANR (*normal*)?

The example of a Dowker space in [6] is a subset of a box product. Four years ago we knew nothing about the normality or paracompactness of any box product of infinitely many spaces. For the next theorems we assume that X is a box product of a family $\{X_n\}_{n\in\omega}$ of (nondiscrete) topological spaces. The fact that all of the positive theorems are consistency results and that we have no positive theorems for box products of uncountably many spaces is unfortunate. However, the progress is tremendous even so.

THEOREM 7 [1]. *If X_0 is the set of all irrational numbers with the usual topology; and $X_n = \omega_0 + 1$ for all $n > 0$, then X is not normal.*

So having all factors metric is not sufficient to ensure normality.

THEOREM 8 [2]. *If each $X_n = 2^{(c^+)}$, then X is not normal.*

So having all factors compact is not sufficient to ensure normality. However:

THEOREM 9 [2], [11]. *The continuum hypothesis implies that X is paracompact if each X_n is a compact space which is either scattered or of weight $\leqq c$.*

Compact in this theorem can be replaced by σ-compact and paracompact.

If one assumes the generalized continuum hypothesis, one can decide whether any given box product of ordinals is normal [3], [12]. Assuming Martin's axiom rather than the continuum hypothesis, one can still prove Theorem 9 if *1st countable* is added to the hypothesis [3].

QUESTION 6. *Is the box product of uncountably many copies of I normal?*

QUESTION 7. *Can the set theoretic assumptions be removed from Theorem 9?*

I conjecture that the answer to Questions 6 and 7 is *no*, but that the answer to the remaining questions is *yes*.

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Section 10

Operator Algebras, Harmonic Analysis and Representation of Groups