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The Work of Charles Fefferman

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There was a period, in the 1940s and 1950s, when classical analysis was considered dead and the hope for the future of analysis was considered to be in the abstract branches, specializing in generalization. As is now apparent, the rumour of the death of classical analysis was greatly exaggerated and during the 1960s and 1970s the field has been one of the most successful in all of mathematics. Briefly, I think that one can say that the reasons for this are the unification of methods from harmonic analysis, complex variables and differential equations, the discovery of the correct generalizations to several variables and finally the realization that in many problems complications cannot be avoided and that intricate combinatorial arguments rather than polished theories often are in the centre.

This general description of classical analysis also summarizes the work of Charles Fefferman. In an eminent way he masters these techniques and has contributed to the success of our common field and it is with real joy and pride, as a friend and as co-worker in the field, I shall try to sketch some lines in the development with emphasis on certain of Fefferman's many contributions.

It is natural to start with the Hardy spaces H^p, i.e. functions $f(z)$ holomorphic in $|z|<1$ and belonging L^p on the boundary of $|z|=1$. Through the work of Marcel Riesz we know that H^p is the dual of H^q for conjugate exponents p and q for $1<p<\infty$, and the theory becomes similar to the L^p, L^q-theory. There was, however, no analogy to the L^1, L^∞-duality and special methods were necessary for every situation for H^1. It was therefore a great sensation when Fefferman in 1971 showed that the dual of H^1 was a space that had been used a few years earlier by John and Nirenberg, the space BMO of functions of bounded mean oscillation. This is the space of functions which on every interval differs in the mean from its mean value by a bounded quantity. A canonical non-bounded example is the logarithm of the absolute value. Many problems for H^1 now become concrete, constructive problems for this class. As a simple illustration of the force of the method, consider Hardy's theorem that if $f(z)=\sum_0 c_n z^n \in H^1$ then $\sum |c_n| n^{-1} < \infty$. Dually this means that

$$\sum \varepsilon_n n^{-1} e^{in\theta} \in \text{BMO}, \quad |\varepsilon_n| = 1,$$

and this is essentially trivial to verify.

The idea, however, carries much further. The result from 1971 by Gundy, Burkholder and Silverstein that a harmonic function $u(z)$ in $|z|<1$ is the real part of an H^1-function if and only if

$$\sup_{z \in V_\theta} u(z) \in L^1$$

where V_θ is the Stolz angle at $e^{i\theta}$, also gets a natural explanation as a representation problem for BMO. The interesting result appears that we need only take the sup over the radius. It is clear how this generalizes to several dimensions and we have in this theory one of the most rapidly expanding branches of analysis. In particular, I should like to mention the recent theory of Muckenhaupt, Wheeden and others, where also Fefferman has contributed essentially, generalizing the L^p-theory of conjugate functions to weighted L^p-spaces. The culmination is Calderón's recent work on singular integrals on C^1-curves which you will hear more about during the congress.

In the centre of this development is the theory of singular integrals and different maximal versions of these integrals. In particular, the maximal partial sum operator for a Fourier series is essentially such a maximal operator

$$S^*(f) = \sup_n S_n(f)(x) = \sup_n \int \frac{f(t)e^{-int}}{x-t}\, dt.$$

Fefferman has given a direct combinatorial proof in the spirit of Kolmogorov that $S_{n(x)}(x)$ for arbitrary choice of $n(x)$ is uniformly bounded on L^2 and hence a new proof of the a.e. convergence of the Fourier series of a continuous function. We have of course similar formulas in several variables. It is remarkable that Fefferman was the first to find a counterexample showing that no similar result holds for rectangular partial sums in several variables, even if we make strong restrictions on the ratio of the sides of the rectangles. This is a result that should have been proved 100 years ago!

We have seen here how the interplay between ideas in real and complex analysis has given striking and deep new results and that singular integrals and Fourier analysis were the main tools. These tools are also, as you all know, closely tied to partial differential equations with constant coefficients because of the algebraic way in which Fourier transform reflects derivations. In a similar way one can also treat differential equations with variable coefficients — the idea is to introduce in the Fourier transform a function p — the symbol — which also depends on the space variables:

$$(Pu)(x) = \iint e^{i(x-y)\cdot\xi} p(x, \xi, y) u(y)\, dy\, d\xi.$$

We can make the theory still more general by replacing $(x-y)\cdot\xi$ in the exponent by more general functions. This will be important for us later.

The theory becomes highly technical and a careful classification of symbols p in terms of estimates of derivatives is necessary. In joint work with R. Beals, Fefferman has introduced a new weighted classification. The main application is a new proof of a result of Nirenberg and Trèves for the local solvability of a partial differential equation of principal type. Using these methods Fefferman and Phong recently showed a best possible version of the sharp Gårding inequality, i.e. if $p(x, \xi) \geqslant 0$ and is at most of second order in ξ then

$$(p(x, D)u, u) \geqslant -C\|u\|_2^2.$$

There is a natural connection back from partial differential equations to complex analysis in the classical Cauchy–Riemann equations. In several variables, these are really a system of equations and an important difference between one and several complex variables is that certain of these equations $\overline{\partial_b}f=0$ also make sense on the boundary of the domain Ω. In particular, if f is given on the boundary there is a natural L^2-projection on solutions of these equations. This projection is realized by a kernel, the Szegö kernel. Similarly, the projection corresponding to L^2 for the volume of Ω is given by the Bergman kernel $K(z, \zeta)$. Clearly, in the centre of interest, we have the regularity of K as the points approach the boundary. It was shown by Kerzman that for strictly pseudo convex domains, the singularities appear as $z \to \zeta$ and the case $z=\zeta$ is particularly interesting.

If Ω is a strictly pseudo convex domain given by a smooth plurisubharmonic function ψ so that $\Omega: \psi<0$, then Hörmander proved in 1965 that for $z_0 \in \partial\Omega$

$$\lim_{z\to z_0} \psi(z)^{n+1}K(z, z) = \frac{n!}{\pi^n}\det\begin{pmatrix} \frac{\partial\psi}{\partial z_1} & & \\ \vdots & \tilde{\Delta}\psi = \left(\frac{\partial^2\psi}{\partial z_\nu \partial \overline{z}_\mu}\right) & \\ \frac{\partial\psi}{\partial z_n} & & \\ \psi & \frac{\partial\psi}{\partial \overline{z}_1} \cdots \frac{\partial\psi}{\partial \overline{z}_n} & \end{pmatrix} = \frac{n!}{\pi^n} L(\psi).$$

By a direct very ingenious construction Fefferman (1974) obtained a complete asymptotic formula which to everybody's surprise contained a logarithmic singularity:

$$(-\psi)^{n+1}K(z, z) = F(z)+G(z)(-\psi)^{n+1}\log(-\psi)$$

with F and G smooth. As was shown later by Boutet de Monvel and Sjöstrand this singularity can be best understood in the context of Fourier integral operators. Let $\psi(z, \zeta)$ be a convenient, explicit continuation of $\psi(z, z)=\psi(z)$ from the diagonal in Ω to $C^n \times C^n$. Then $K(z, \zeta)$ is essentially a Laplace transform of the type

$$\int_0^\infty e^{t\psi(z,\zeta)} k(z, \zeta; t)\, dt$$

where k has a singularity at $t=\infty$ of the type t^n which just produces singularities of Fefferman's type at the boundary where $\psi=0$.

Fefferman's interest in the Bergman kernel originated in his desire to show the regularity of biholomorphic mappings up to the boundary of smooth domains. The metric

$$ds^2 = \sum \frac{\partial^2 \log K}{\partial z_\nu \partial \overline{z_\mu}} dz_\nu d\overline{z_\mu}$$

is invariant under the mapping and its geodesics G are therefore natural bases for a geometric description of the correspondence between the boundaries of the domains, *i.e.* if we can show that G approaches a definite boundary point and that directions correspond smoothly to the boundary. Since we have a precise knowledge of the behavior of the metric at $\partial\Omega$ all becomes concrete differential geometric problems. The difficulties are however very serious because of the singular behavior of K at $\partial\Omega$ but were mastered by Fefferman in a remarkable way.

Through Fefferman's result we now have a foundation for a discussion of the mapping on the boundary. The fundamental problem is to classify domains which are biholomorphically equivalent or locally so. We shall only consider the local problem and then need a set of local invariants. In the case $n=2$ all local invariants were formed already by Eli Cartan and Chern and Moser gave a complete theory. In particular they found certain invariant curves; the chains. You will hear more about this in Jürgen Moser's lecture. Fefferman has given a differential geometric description of the Chern–Moser-chains derived from certain geodesics for a metric, which is related to the Bergman kernel function. He has also started the big program to find algebraic descriptions of the local invariants. In all probability we are here at the beginning of a completely new theory in several complex variables.

I should like to finish by pointing to an alternative approach which is very attractive to a classical analyst. The most important tool in one complex variable is the harmonic functions. The class is invariant under conformal mappings because $\Delta(u \circ f) = |f'|^2 \Delta u$. The natural analogue in several variables is the Hessian determinant

$$|\tilde{\Delta}| u = \det\left(\frac{\partial^2 u}{\partial z_\nu \partial \overline{z_\mu}}\right).$$

We have already seen $\tilde{\Delta}$ in Hörmander's formula Lu for $K(z, z)$ and actually $\tilde{\Delta}$ and L are related by the change $u \to \log u$. The study of the equation $\tilde{\Delta} u = \varphi$ has started but basic estimates are still missing. Fefferman has made the important observation that the equation $Lu=1$, $u=0$ on $\partial\Omega$, can be solved approximately, again with regularity until the critical singularities enter for the nth derivative. I am sure that we see here another example of the beginning of an important theory.

I hope this brief survey has convinced you of the vitality of classical analysis and of the great contributions of Charles Fefferman.

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