# Notes on SU(3) and the Quark Model

## Contents


**1. SU(3) and the Quark Model** **2**

1.1 Raising and Lowering Operators: The Weight Diagram 4

1.1.1 Triangular Weight Diagrams (I) 6

1.1.2 Triangular Weight Diagrams (II) 8

1.1.3 Hexagonal Weight Diagrams 10

1.1.4 Dimension of Irreducible Representations 13

1.1.5 The Complex Conjugate Representation 14

1.2 Some Low-Dimensional Irreducible Representations of $\mathcal{L}(SU(3))$ 15

1.2.1 The Singlet 15

1.2.2 3-dimensional Representations 15

1.2.3 Eight-Dimensional Representations 18

1.3 Tensor Product Representations 18

1.3.1 $\mathbf{3} \otimes \mathbf{3}$ decomposition. 19

1.3.2 $\mathbf{3} \otimes \bar{\mathbf{3}}$ decomposition 21

1.3.3 $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$ decomposition. 23

1.4 The Quark Model 26

1.4.1 Meson Multiplets 26

1.4.2 Baryon Multiplets 27

1.4.3 Quarks: Flavour and Colour 28

## 1. SU(3) and the Quark Model

The Lie algebra of $SU(3)$ consists of the traceless antihermitian $3\times 3$ complex matrices and is eight-dimensional. A generalisation of the Pauli matrices are the *Gell-Mann matrices*

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$
$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$
$$\lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{1.1}$$

These are traceless and hermitian and satisfy

$$Tr(\lambda_a\lambda_b) = 2\delta_{ab} \tag{1.2}$$

The antiehermitian traceless generators of $SU(3)$ can be taken to be

$$T_a = -\frac{i}{2}\lambda_a \tag{1.3}$$

with the structure constants defined by

$$[T_a, T_b] = f_{ab}{}^c T_c \tag{1.4}$$

The following sets of Gell-Mann matrices:

i) $\lambda_1, \lambda_2, \lambda_3$

ii) $\lambda_4, \lambda_5, \frac{1}{2}(\sqrt{3}\lambda_8 + \lambda_3)$

iii) $\lambda_6, \lambda_7, \frac{1}{2}(\sqrt{3}\lambda_8 - \lambda_3)$

each have the same algebraic properties as the Pauli matrices and so determine three natural $\mathcal{L}(SU(2))$ subalgebras.

It will be convenient to use a different set of matrices. We define the following matrices

$$h_1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad h_2 = \begin{pmatrix} \frac{1}{2\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{2\sqrt{3}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{3}} \end{pmatrix}$$

$$
e^1_+ = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad e^1_- = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$
$$
e^2_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{pmatrix} \qquad e^2_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}
$$
$$
e^3_+ = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad e^3_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix} \tag{1.5}
$$

Then $ih_1$, $ih_2$ and $i(e^m_+ + e^m_-)$, $e^m_+ - e^m_-$ for $m = 1, 2, 3$ form a basis for the antihermitian traceless $3 \times 3$ matrices (over $\mathbb{R}$), and hence are a basis for $\mathcal{L}(SU(3))$.

Suppose that $d$ is the irreducible representation of $\mathcal{L}(SU(3))$ acting on a complex vector space $V$ which is induced from an irreducible representation of $SU(3)$ acting on $V$. It is convenient to set

$$
H_1 = d(h_1), \quad H_2 = d(h_2), \quad E^m_\pm = d(e^m_\pm) \text{ for } m = 1, 2, 3 \tag{1.6}
$$

Then we find the following commutators:

$$
\begin{aligned}
[H_1, H_2] &= 0 \\
[H_1, E^1_\pm] &= \pm E^1_\pm, \qquad [H_1, E^2_\pm] = \mp\frac{1}{2}E^2_\pm, \qquad [H_1, E^3_\pm] = \pm\frac{1}{2}E^3_\pm \\
[H_2, E^1_\pm] &= 0, \qquad [H_2, E^2_\pm] = \pm\frac{\sqrt{3}}{2}E^2_\pm, \qquad [H_2, E^3_\pm] = \pm\frac{\sqrt{3}}{2}E^3_\pm
\end{aligned} \tag{1.7}
$$

and

$$
\begin{aligned}
[E^1_+, E^1_-] &= H_1 \\
[E^2_+, E^2_-] &= \frac{\sqrt{3}}{2}H_2 - \frac{1}{2}H_1 \\
[E^3_+, E^3_-] &= \frac{\sqrt{3}}{2}H_2 + \frac{1}{2}H_1
\end{aligned} \tag{1.8}
$$

The remaining commutation relations are

$$
\begin{aligned}
[E^1_+, E^2_+] &= \frac{1}{\sqrt{2}}E^3_+, \qquad [E^1_-, E^2_-] = -\frac{1}{\sqrt{2}}E^3_- \\
[E^1_+, E^3_-] &= -\frac{1}{\sqrt{2}}E^2_-, \quad [E^1_-, E^3_+] = \frac{1}{\sqrt{2}}E^2_+ \\
[E^2_+, E^3_-] &= \frac{1}{\sqrt{2}}E^1_-, \qquad [E^2_-, E^3_+] = -\frac{1}{\sqrt{2}}E^1_+
\end{aligned} \tag{1.9}
$$

and

$$
[E^1_+, E^2_-] = [E^1_-, E^2_+] = [E^1_+, E^3_+] = [E^1_-, E^3_-] = [E^2_+, E^3_+] = [E^2_-, E^3_-] = 0 \tag{1.10}
$$

Note in particular that $H_1$, $H_2$ commute. The subalgebra of $\mathcal{L}(SU(3))$ spanned by $ih_1$ and $ih_2$ is called the Cartan subalgebra. It is the maximal commuting subalgebra of $\mathcal{L}(SU(3))$.

### 1.1 Raising and Lowering Operators: The Weight Diagram

The Lie algebra of $\mathcal{L}(SU(3))$ can be used to obtain three sets of $\mathcal{L}(SU(2))$ algebras. In particular, we find that

$$[H_1, E^1_\pm] = \pm E^1_\pm, \qquad [E^1_+, E^1_-] = H_1 \tag{1.11}$$

and

$$[\frac{\sqrt{3}}{2}H_2 - \frac{1}{2}H_1, E^2_\pm] = \pm E^2_\pm, \qquad [E^2_+, E^2_-] = \frac{\sqrt{3}}{2}H_2 - \frac{1}{2}H_1 \tag{1.12}$$

and

$$[\frac{\sqrt{3}}{2}H_2 + \frac{1}{2}H_1, E^3_\pm] = \pm E^3_\pm, \qquad [E^3_+, E^3_-] = \frac{\sqrt{3}}{2}H_2 + \frac{1}{2}H_1 \tag{1.13}$$

In particular, there are three pairs of raising and lowering operators $E^m_\pm$.

For simplicity, consider a representation $d$ of $\mathcal{L}(SU(3))$ obtained from a unitary representation $\mathcal{D}$ of $SU(3)$ such that $d$ is an anti-hermitian representation- so that $H_1$ and $H_2$ are hermitian, and hence diagonalizable with real eigenvalues. Hence, $H_1$ and $\frac{\sqrt{3}}{2}H_2 \pm \frac{1}{2}H_1$, can be simultaneously diagonalized, and the eigenvalues are real. (In fact the same can be shown without assuming unitarity!)

Suppose then that $|\phi\rangle$ is an eigenstate of $H_1$ with eigenvalue $p$ and also an eigenstate of $H_2$ with eigenvalue $q$. It is convenient to order the eigenvalues as points in $\mathbb{R}^2$ with position vectors $(p, q)$ where $p$ is the eigenvalue of $H_1$ and $q$ of $H_2$. $(p, q)$ is then referred to as a weight.

From the commutation relations we have the following properties

i) Either $E^1_\pm |\phi\rangle = 0$ or $E^1_\pm |\phi\rangle$ is an eigenstate of $H_1$ and $H_2$ with eigenvalue $(p, q) \pm (1, 0)$

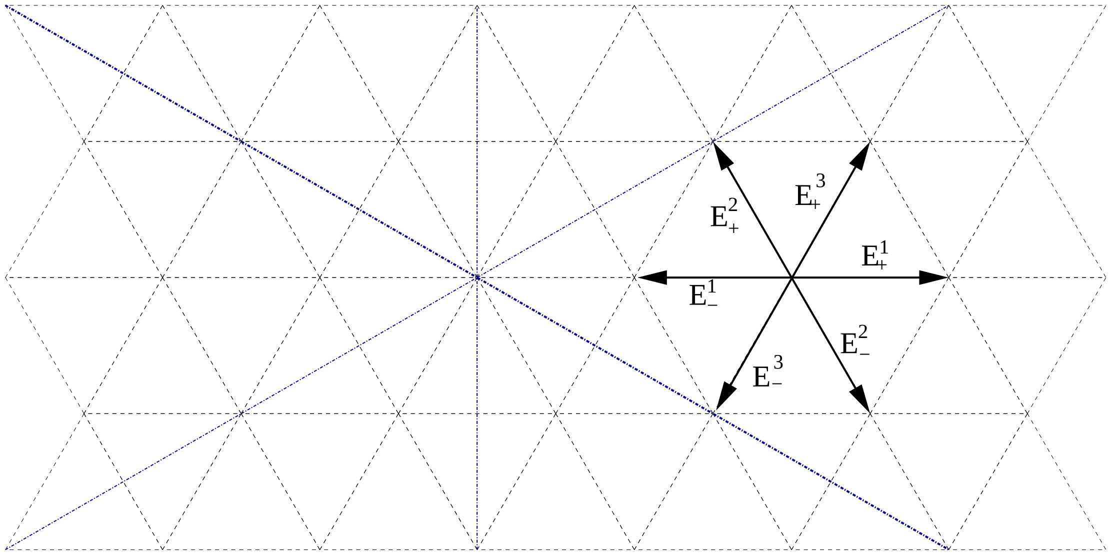
ii) Either $E^2_\pm |\phi\rangle = 0$ or $E^2_\pm |\phi\rangle$ is an eigenstate with eigenvalue $(p, q) \pm (-\frac{1}{2}, \frac{\sqrt{3}}{2})$

iii) Either $E^3_\pm |\phi\rangle = 0$ or $E^3_\pm |\phi\rangle$ is an eigenstate with eigenvalue $(p, q) \pm (\frac{1}{2}, \frac{\sqrt{3}}{2})$

Moreover, from the properties of $\mathcal{L}(SU(2))$ representations we know that

$$2p = m_1, \quad \sqrt{3}q - p = m_2, \quad \sqrt{3}q + p = m_3 \tag{1.14}$$

for $m_1$, $m_2$, $m_3 \in \mathbb{Z}$. It follows that $2\sqrt{3}q \in \mathbb{Z}$. It is particularly useful to plot the sets of eigenvalues $(p, q)$ as points in the plane. The resulting plot is known as the weight diagram. As the representation is assumed to be irreducible, there can only be finitely many points on the weight diagram, though it is possible that a particular weight may correspond to more than one state. Moreover, as $2p \in \mathbb{Z}$, $2\sqrt{3}q \in \mathbb{Z}$, the weights are constrained to lie on the points of a lattice. From the effect of the raising and lowering operators on the eigenvalues, it is straightforward to see that this lattice is formed by the tessalation of the plane by equilateral triangles of side 1. This is illustrated in Figure 1, where the effect of the raising and lowering operators is given (in this diagram $(0, 0)$ is a weight, though this need not be the case generically).




The weight diagram has three axes of symmetry. To see this, recall that if $m$ is a weight of a state in an irreducible representation of $\mathcal{L}(SU(2))$ then so is $-m$. In the context of the three $\mathcal{L}(SU(2))$ algebras contained in $\mathcal{L}(SU(3))$ this means that from the properties of the algebra in (1.11), if $(p, q)$ is a weight then so is $(-p, q)$, i.e. the diagram is reflection symmetric about the line $\theta = \frac{\pi}{2}$ passing through the origin. Also, due to the symmetry of the $\mathcal{L}(SU(2))$ algebra in (1.12), the weight diagram is reflection symmetric about the line $\theta = \frac{\pi}{6}$ passing through the origin: so if $(p, q)$ is a weight then so is $(\frac{1}{2}(p+\sqrt{3}q), \frac{1}{2}(\sqrt{3}p-q))$. And due to the symmetry of the $\mathcal{L}(SU(2))$ algebra in ((1.13) the weight diagram is reflection symmetric about the line $\theta = \frac{5\pi}{6}$ passing through the origin: so if $(p, q)$ is a weight then so is $(\frac{1}{2}(p - \sqrt{3}q), \frac{1}{2}(-\sqrt{3}p - q))$.

Using this symmetry, it suffices to know the structure of the weight diagram in the sector of the plane $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}$. The remainder is fixed by the reflection symmetry.

Motivated by the treatment of $SU(2)$ we make the definition:

**Definition 1.** *$|\psi\rangle$ is called a highest weight state if $|\psi\rangle$ is an eigenstate of both $H_1$ and $H_2$, and $E_+^m |\psi\rangle = 0$ for $m = 1, 2, 3$.*

Note that there must be a highest weight state, for otherwise one could construct infinitely many eigenstates by repeated application of the raising operators $E_+^m$. Given a highest weight state, let $V'$ be the vector space spanned by $|\psi\rangle$ and states obtained by acting with all possible products of lowering operators $E_-^m$ on $|\psi\rangle$. As there are only finitely many points on the weight diagram, there can only be finitely many such terms. Then, by making use of the commutation relations, it is clear that $V'$ is an invariant subspace of $V$. As the representation is irreducible on $V$, this implies that $V' = V$, i.e. $V$ is spanned by $|\psi\rangle$ and a finite set of states obtained by acting with lowering operators on $|\psi\rangle$. Suppose that $(p, q)$ is the weight of $|\psi\rangle$. Then $V$ is spanned by a basis of eigenstates of $H_1$ and $H_2$ with weights confined to the sector given by $\pi \leq \theta \leq \frac{5\pi}{3}$ relative to $(p, q)$- all points on the weight diagram must therefore lie in this sector.

**Lemma 1.** *The highest weight state is unique.*

**Proof** Suppose that $|\psi\rangle$ and $|\psi'\rangle$ are two highest weight states with weights $(p,q)$, $(p',q')$ respectively. Then $(p',q')$ must make an angle $\pi \leq \theta \leq \frac{5\pi}{3}$ relative to $(p,q)$ and $(p,q)$ must make an angle $\pi \leq \theta \leq \frac{5\pi}{3}$ relative to $(p',q')$. This implies that $p = p'$, $q = q'$.

Next suppose that $|\psi_1\rangle$ and $|\psi_2\rangle$ are two linearly independent highest weight states (both with weight $(p,q)$). Let $V_1$ and $V_2$ be the vector spaces spanned by the states obtained by acting with all possible products of lowering operators $E^m_-$ on $|\psi_1\rangle$ and $|\psi_2\rangle$ respectively; one therefore obtains bases for $V_1$ and $V_2$ consisting of eigenstates of $H_1$ and $H_2$. By the reasoning given previously, as $V_1$ and $V_2$ are invariant subspaces of $V$ and the representation is irreducible on $V$, it must be the case that $V_1 = V_2 = V$. In particular, we find that $|\psi_2\rangle \in V_1$. However, the only basis element of $V_1$ which has weight $(p,q)$ is $|\psi_1\rangle$, hence we must have $|\psi_2\rangle = c\,|\psi_1\rangle$ for some constant $c$, in contradiction to the assumption that $|\psi_1\rangle$ and $|\psi_2\rangle$ are linearly independent. ■

Having established the existence of a unique highest weight state $|\psi\rangle$, we can proceed to obtain the generic form for the weight diagram.

Suppose that $|\psi\rangle$ has weight $(p,q)$. We have shown that all other states must have weights making an angle $\pi \leq \theta \leq \frac{5\pi}{3}$ relative to $(p,q)$. This implies that $(p,q)$ must lie in the sector $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}$ relative to $(0,0)$, or at the origin. Denote this portion of the plane by $S$.

To see this, note that if $q < 0$ then all weights must lie in the lower half plane, so there are no weights in $S$. But from the reflection symmetry of the weight diagram, this then implies that there are no weights at all. Next, note that if $p < 0$, then from the properties of the $\mathcal{L}(SU(2))$ algebra corresponding to (1.11), the state $E^1_+\,|\psi\rangle$ is non-vanishing, in contradiction to the definition of the highest weight state. Hence, we must have $p \geq 0$ and $q \geq 0$. Next suppose that $(p,q)$ lies in the sector $0 \leq \theta < \frac{\pi}{6}$. By the properties of the $\mathcal{L}(SU(2))$ algebra corresponding to (1.12), the state $E^2_+\,|\psi\rangle$ is non-vanishing, again in contradiction to the definition of the highest weight state. Hence the only remaining possibility if for $(p,q)$ to lie in the sector $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}$ relative to $(0,0)$, or at the origin.

**Lemma 2.** *If the highest weight is* $(0,0)$*, then there is only one state in the representation, which is called the singlet.*

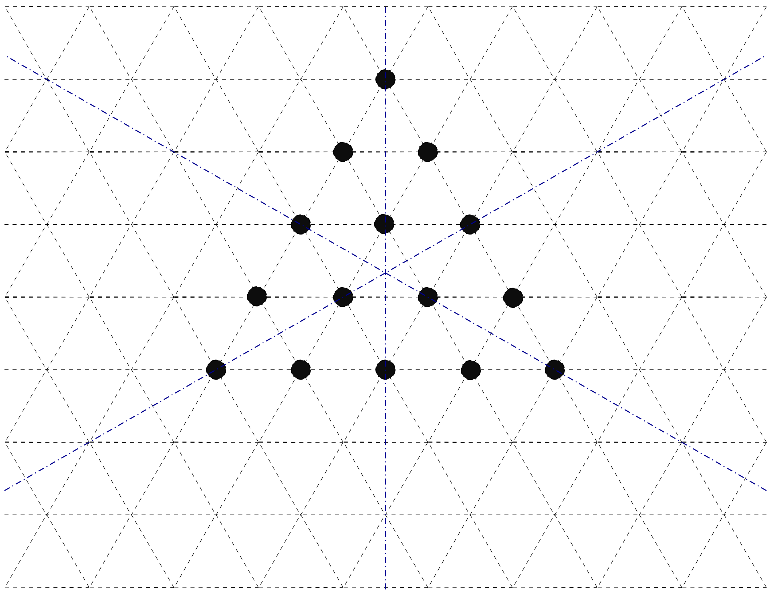
**Proof**. Let $|\psi\rangle$ be the highest weight state with weight $(0,0)$. Suppose that $E^m_-\,|\psi\rangle \neq 0$ for some $m$. Then by the reflection symmetry of the weight diagram, it follows that $E^m_+\,|\psi\rangle \neq 0$, in contradiction to the fact that $E^i_+\,|\psi\rangle = 0$ for $i = 1,2,3$, as $|\psi\rangle$ is the highest weight state. Hence $E^m_\pm\,|\psi\rangle = 0$ for $m = 1,2,3$. Also $H_1\,|\psi\rangle = H_2\,|\psi\rangle = 0$. It follows that the 1-dimensional subspace $V'$ spanned by $|\psi\rangle$ is an invariant subspace of $V$, and therefore $V = V'$ as the representation is irreducible. ■

There are then three possible locations for the highest weight state $|\psi\rangle$.

### 1.1.1 Triangular Weight Diagrams (I)

Suppose that the highest weight lies on the line $\theta = \frac{\pi}{2}$. In this case, by applying powers of $E^2_-$ the states of the $\mathcal{L}(SU(2))$ representation given in (1.12) are generated. These form a line orthogonal to the axis of reflection $\theta = \frac{\pi}{6}$, about which they are symmetric, and there are no states outside this line, as these points cannot be reached by applying lowering operators. Then, by using the reflection symmetry, it follows that the outermost states from an equilateral triangle with horizontal base. Each lattice point inside the triangle corresponds to (at least) one state which has this weight, because each lattice point in the triangle lies at some possible weight within the $\mathcal{L}(SU(2))$ representation given in (1.11), and from the properties of $\mathcal{L}(SU(2))$ representations, we know that this has a state with this weight (i.e. as the $\mathcal{L}(SU(2))$ weight diagram has no "holes" in it, neither does the $\mathcal{L}(SU(3))$ weight diagram).

This case is illustrated by



**Proposition 1.** *Each weight in this triangle corresponds to a unique state.*

**Proof**. Note that all of the states on the right edge of the triangle correspond to unique states, because these weights correspond to states which can only be obtained by acting on $|\psi\rangle$ with powers of $E^2_-$. It therefore follows by the reflection symmetry that all of the states on the edges of the triangle have multiplicity one.

Now note the commutation relation

$$[E^1_-, E^2_-] = -\frac{1}{\sqrt{2}}E^3_- \tag{1.15}$$

This implies that products of lowering operators involving $E^3_-$ can be rewritten as linear combinations of products of operators involving only $E^1_-$ and $E^2_-$ (in some order). In particular, we find

$$\begin{aligned}(E^1_-)(E^2_-)^n|\psi\rangle &= [E^1_-, E^2_-](E^2_-)^{n-1}|\psi\rangle + E^2_-E^1_-(E^2_-)^{n-1}|\psi\rangle \\ &= -\frac{1}{\sqrt{2}}E^3_-(E^2_-)^{n-1}|\psi\rangle + E^2_-E^1_-(E^2_-)^{n-1}|\psi\rangle \\ &\dots \\ &= -\frac{n}{\sqrt{2}}E^3_-(E^2_-)^{n-1}|\psi\rangle \end{aligned} \tag{1.16}$$

by simple induction, where we have used the fact that $E^1_-|\psi\rangle = 0$ and $[E^2_-, E^3_-] = 0$.

A generic state of some fixed weight in the representation can be written as a linear combination of products of $E^2_-$ and $E^1_-$ lowering operators acting on $|\psi\rangle$ of the form

$$\Pi(E^1_-, E^2_-)\,|\psi\rangle \tag{1.17}$$

where $\Pi(E^1_-, E^2_-)$ contains $m$ powers of $E^2_-$ and $\ell$ powers of $E^1_-$ where $m$, $\ell$ are uniquely determined by the weight of the state- only the order of the operators is unfixed.

Using (1.16), commute the $E^1_-$ states in this product to the right as far as they will go. Then either one finds that the state vanishes (due to an $E^1_-$ acting directly on $|\psi\rangle$), or one can eliminate all of the $E^1_-$ terms and is left with a term proportional to

$$(E^2_-)^{m-\ell}(E^3_-)^{\ell}\,|\psi\rangle \tag{1.18}$$

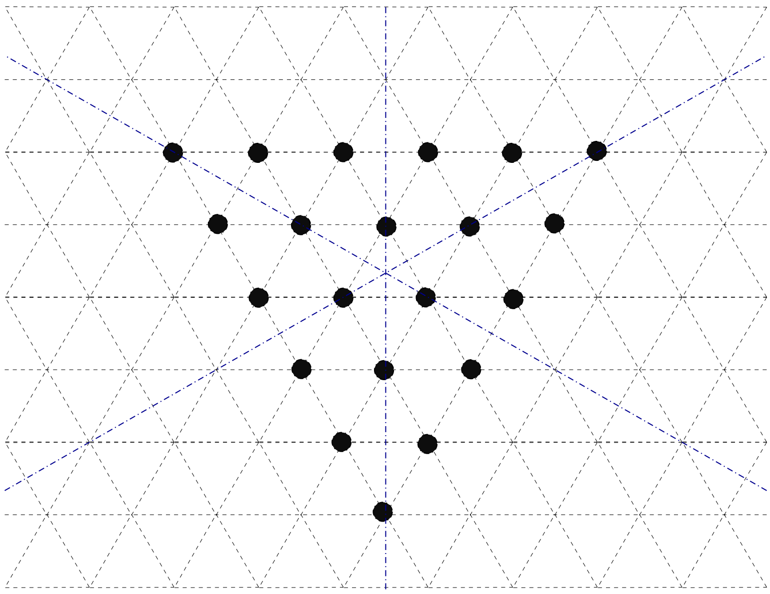
where we have used the commutation relations $[E^2_-, E^3_-] = [E^1_-, E^3_-] = 0$.

Hence, it follows that all weights in the diagram can have at most multiplicity 1. However, from the property of the $\mathcal{L}(SU(2))$ representations, as the weights in the outer layers have multiplicity 1, it follows that all weights in the interior have multiplicity at least 1. Hence, all the weights must be multiplicity 1. ■

### 1.1.2 Triangular Weight Diagrams (II)

Suppose that the highest weight lies on the line $\theta = \frac{\pi}{6}$. In this case, by applying powers of $E^1_-$ the states of the $\mathcal{L}(SU(2))$ representation given in (1.11) are generated. These form a horizontal line orthogonal to the axis of reflection $\theta = \frac{\pi}{2}$, about which they are symmetric, and there are no states outside this line, as these points cannot be reached by applying lowering operators. Then, by using the reflection symmetry, it follows that the outermost states from an inverted equilateral triangle with horizontal upper edge. Each lattice point inside the triangle corresponds to (at least) one state which has this weight, because each lattice point in the triangle lies at some possible weight within the $\mathcal{L}(SU(2))$ representation given in (1.11), and from the properties of $\mathcal{L}(SU(2))$ representations, we know that this has a state with this weight (i.e. as the $\mathcal{L}(SU(2))$ weight diagram has no "holes" in it, neither does the $\mathcal{L}(SU(3))$ weight diagram).

This case is illustrated by



**Proposition 2.** *Each weight in this triangle corresponds to a unique state.*
**Proof**. Note that all of the states on the horizontal top edge of the triangle correspond to unique states, because these weights correspond to states which can only be obtained by acting on $|\psi\rangle$ with powers of $E^1_-$. It therefore follows by the reflection symmetry that all of the states on the edges of the triangle have multiplicity one.

Now, using (1.15) it is straightforward to show that

$$E^2_-(E^1_-)^n\,|\psi\rangle = \frac{n}{\sqrt{2}}E^3_-(E^1_-)^{n-1}\,|\psi\rangle \tag{1.19}$$

for $n \geq 1$, where we have used $E^2_-\,|\psi\rangle = 0$. Next consider a state of some fixed weight in the representation; this can be written as a linear combination of terms of the form

$$\Pi(E^1_-, E^2_-)\,|\psi\rangle \tag{1.20}$$

where $\Pi(E^1_-, E^2_-)$ contains $m$ powers of $E^1_-$ and $\ell$ powers of $E^2_-$ in an appropriate order, where $m$ and $\ell$ are determined uniquely by the weight of the state in question. Using (1.19), commute the $E^2_-$ states in this product to the right as far as they will go. Then either one finds that the state vanishes (due to an $E^2_-$ acting directly on $|\psi\rangle$), or one can eliminate all of the $E^1_-$ terms and is left with a term proportional to

$$(E^1_-)^{m-\ell}(E^3_-)^{\ell}\,|\psi\rangle \tag{1.21}$$

where we have used the commutation relations $[E^2_-, E^3_-] = [E^1_-, E^3_-] = 0$.

Hence, it follows that all weights in the diagram can have at most multiplicity 1. However, from the property of the $\mathcal{L}(SU(2))$ representations, as the weights in the outer layers have multiplicity 1, it follows that all weights in the interior have multiplicity at least 1.

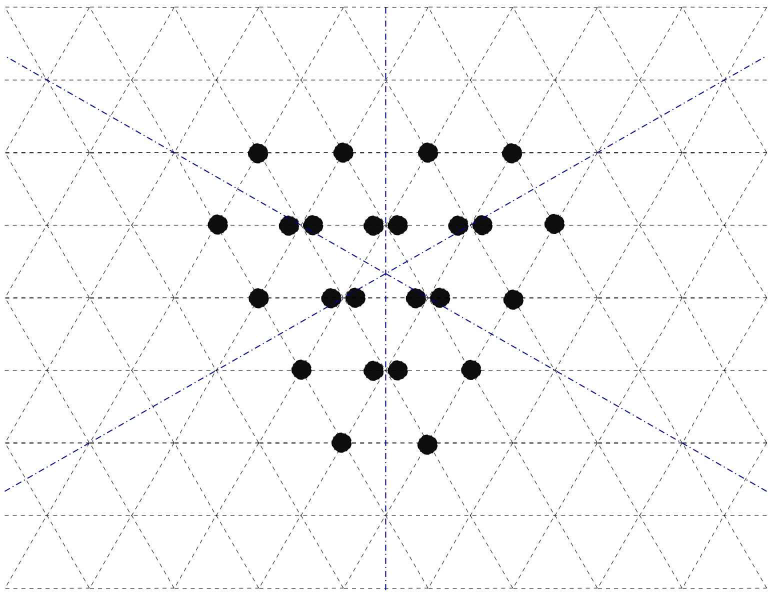
Hence, all the weights must be multiplicity 1. ■

### 1.1.3 Hexagonal Weight Diagrams

Suppose that the highest weight lies in the sector $\frac{\pi}{6} < \theta < \frac{\pi}{2}$. In this case, by applying powers of $E^1_-$ the states of the $\mathcal{L}(SU(2))$ representation given in (1.11) are generated. These form a horizontal line extending to the left of the maximal weight which is orthogonal to the line $\theta = \frac{\pi}{2}$, about which they are symmetric, There are no states above, as these points cannot be reached by applying lowering operators. Also, by applying powers of $E^2_-$ the states of the $\mathcal{L}(SU(2))$ representation given in (1.12) are generated. These form a line extending to the right of the maximal weight which is orthogonal to the axis of reflection $\theta = \frac{\pi}{6}$, about which they are symmetric, and there are no states to the right of this line, as these points cannot be reached by applying lowering operators.

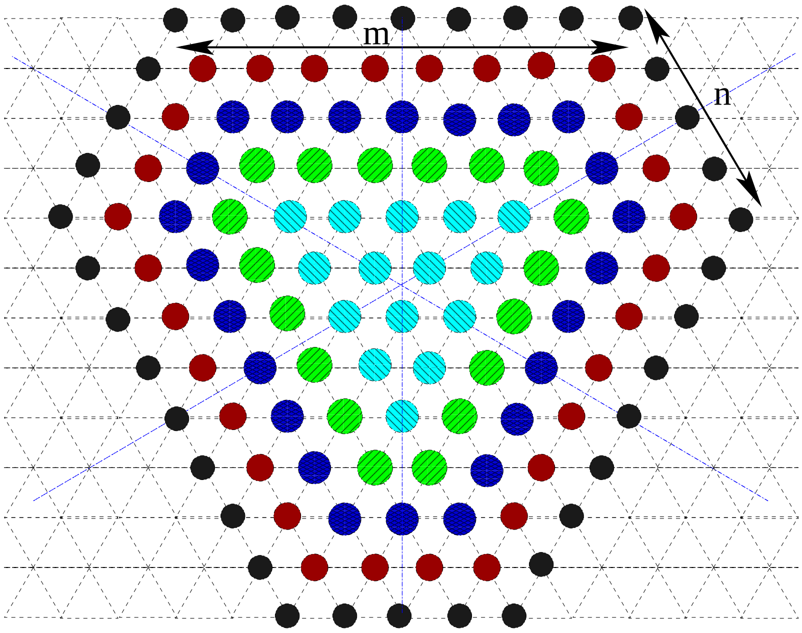
Then, by using the reflection symmetry, it follows that the outermost states form a hexagon. Each lattice point inside the hexagon corresponds to (at least) one state which has this weight, because each lattice point in the hexagon lies at some possible weight within the $\mathcal{L}(SU(2))$ representation given in (1.11), and from the properties of $\mathcal{L}(SU(2))$ representations, we know that this has a state with this weight (i.e. as the $\mathcal{L}(SU(2))$ weight diagram has no "holes" in it, neither does the $\mathcal{L}(SU(3))$ weight diagram).

This case is illustrated by



The multiplicities of the states for these weight diagrams are more complicated than for the triangular diagrams. In particular, the weights on the two edges of the hexagon leading off from the highest weight have multiplicity 1, because these states can only be constructed as $(E^1_-)^n \left|\psi\right\rangle$ or $(E^2_-)^m \left|\psi\right\rangle$. So by symmetry, all of the states on the outer layer of the hexagon have multiplicity 1. However, if one proceeds to the next layer, then the multiplicity of all the states increases by 1. This happens until the first triangular layer is reached, at which point all following layers have the same multiplicity as the first triangular layer.

Suppose that the top horizontal edge leading off the maximal weight is of length $m$, and that the other outer edge is of length $n$, with $m \geq n$. This situation is illustrated below




The highest weight is then at $(\frac{m}{2}, \frac{1}{2\sqrt{3}}(m+2n))$. The outer $n$ layers are hexagonal, whereas the $n+1$-th layer is triangular, and all following layers are also triangular. As one goes inwards through the outer $n+1$ layers the multiplicity of the states in the layers increases from 1 in the first outer layer to $n+1$ in the $n+1$-th layer. Then all the states in the following triangular layers have multiplicity $n+1$ as well.

We will prove this in several steps.

**Proposition 3.** *States with weights on the $k$-th hexagonal layer for $k = 1, \ldots, n$ or the $k = n+1$-th layer (the first triangular layer) have multiplicity not exceeding $k$.*

**Proof**. In order to prove this, consider first a state on the upper horizontal edge of the $k$-th layer for $k \leq n+1$. The length of this edge is $m - k + 1$. A general state on this edge is obtained via

$$\Pi(E_-^2, E_-^1) \, |\psi\rangle \tag{1.22}$$

where $\Pi(E_-^2, E_-^1)$ contains (in some order) $k-1$ powers of $E_-^2$ and $\ell$ powers of $E_-^1$ for $\ell = k-1, \ldots, m$.

Now use the commutation relation (1.15) to commute the powers of $E_-^2$ to the right as far as they will go. Then the state can be written as a linear combination of the $k$ vectors

$$|v_i\rangle = (E_-^3)^{i-1} (E_-^1)^{\ell-i+1} (E_-^2)^{k-i} \, |\psi\rangle \tag{1.23}$$

for $i = 1, \ldots, k$. It follows that this state has multiplicity $\leq k$.

Next consider a state again on the $k$-th level, but now on the edge leading off to the right of the horizontal edge which we considered above; this edge is parallel to the outer edge of length $n$. Take $k \leq n+1$, so the edge has length $n-k+1$. A state on this edge is obtained via

$$\hat{\Pi}(E^1_-, E^2_-)\,|\psi\rangle \tag{1.24}$$

where $\hat{\Pi}(E^1_-, E^2_-)$ contains (in some order) $k-1$ powers of $E^1_-$ and $\ell$ powers of $E^2_-$ where $\ell = k-1, \dots, n$. Now use the commutation relation (1.15) to commute the powers of $E^1_-$ to the right as far as they will go. Then the state can be written as a linear combination of the $k$ vectors

$$|w_i\rangle = (E^3_-)^{i-1}(E^2_-)^{\ell-i+1}(E^1_-)^{k-i}\,|\psi\rangle \tag{1.25}$$

for $i = 1, \dots, k$.

So these states also have multiplicity $\leq k$. By using the reflection symmetry, it follows that the all the states on the $k$-th hexagonal layer have multiplicity $k$. ■

We also have the

**Proposition 4.** *The states with weights in the triangular layers have multiplicity not exceeding* $n+1$.

**Proof**. Consider a state on the $k$-th row of the weight diagram for $m+1 \geq k \geq n+1$ which lies inside the triangular layers. Such a state can also be written as

$$\Pi(E^2_-, E^1_-)\,|\psi\rangle \tag{1.26}$$

where $\Pi(E^2_-, E^1_-)$ contains (in some order) $k-1$ powers of $E^2_-$ and $\ell$ powers of $E^1_-$ for $\ell = k-1, \dots, m$. and hence by the reasoning above, it can be rewritten as a linear combination of the $k$ vectors $|v_i\rangle$ in (1.23), however for $i < k-n$, $|v_i\rangle = 0$ as $(E^2_-)^{k-i}\,|\psi\rangle = 0$. The only possible non-vanishing vectors are the $n+1$ vectors $|v_{k-n}\rangle, |v_{k-n+1}\rangle, \dots, |v_k\rangle$. Hence these states have multiplicity $\leq n+1$. ■

Next note the lemma

**Lemma 3.** *Define* $|w_{i,k}\rangle = (E^3_-)^{i-1}(E^1_-)^{k-i}(E^2_-)^{k-i}\,|\psi\rangle$ *for* $i = 1, \dots, k$, $k = 1, \dots, n+1$. *Then the sets* $S_k = \{|w_{1,k}\rangle, \dots, |w_{k,k}\rangle\}$ *are linearly independent for* $k = 1, \dots, n+1$.

**Proof**. By using the commutation relations, it is straightforward to prove the identities

$$\begin{aligned}
E^3_+\,|w_{i,k}\rangle &= (i-1)\big(\frac{\sqrt{3}}{2}q + \frac{1}{2}p + \frac{i}{2} + 1 - k\big)\,|w_{i-1,k-1}\rangle \\
&\quad - \frac{1}{\sqrt{2}}(k-i)^2\big(\frac{\sqrt{3}}{2}q - \frac{1}{2}p + \frac{i}{2} + \frac{1}{2} - \frac{k}{2}\,|w_{i,k-1}\rangle \\
E^2_+\,|w_{i,k}\rangle &= E^1_-\big(\frac{1}{\sqrt{2}}(i-1)\,|w_{i-1,k-1}\rangle \\
&\quad + (k-i)\big(\frac{\sqrt{3}}{2}q - \frac{1}{2}p - \frac{1}{2}(k-i-1)\big)\,|w_{i,k-1}\rangle\,\big)
\end{aligned} \tag{1.27}$$

(with obvious simplifications in the cases when $i = 1$ or $i = k$)

Note that $S_1 = \{|\psi\rangle\}$ is linearly independent. Suppose that $S_{k-1}$ is linearly independent ($k \geq 2$). Consider $S_k$. Suppose

$$\sum_{i=1}^{k} c_i \left| w_{i,k} \right\rangle = 0 \tag{1.28}$$

for some constants $c_i$. Applying $E^3_+$ to (1.28) and using the linear independence of $S_{k-1}$ we find the relation

$$i(\frac{\sqrt{3}}{2} + \frac{1}{2}p + \frac{i}{2} + \frac{3}{2} - k)c_{i+1} - \frac{1}{\sqrt{2}}(k-i)^2(\frac{\sqrt{3}}{2}q - \frac{1}{2}p + \frac{i}{2} + \frac{1}{2} - \frac{1}{2}k)c_i = 0 \tag{1.29}$$

for $i = 1, \dots, k-1$. Applying $E^2_+$ to (1.28) another recursion relation is obtained

$$\frac{1}{\sqrt{2}}ic_{i+1} + (k-i)(\frac{\sqrt{3}}{2}q - \frac{1}{2}p + \frac{i}{2} + \frac{1}{2} - \frac{1}{2}k)c_i = 0 \tag{1.30}$$

Combining these relations we find $c_{i+1} = 0$ for $i = 1, \dots, k-1$. If $\frac{\sqrt{3}}{2}q - \frac{1}{2}p + \frac{i}{2} + \frac{1}{2} - \frac{1}{2}k \neq 0$ when $i = 1$ then one also has $c_1 = 0$. This holds if $k \leq n+1$, however if $k = n+2$ then $c_1$ is not fixed by these equations. The induction stops at this point. ■

These results are sufficient to fix the multiplicity of all the states. This is because the vectors in $S_k$ for $1 \leq k \leq k+1$ correspond to states with weight $(p,q) - (k-1)(\frac{1}{2}, \frac{\sqrt{3}}{2})$ which are at the top right hand corner of the $k$-th hexagonal (or outermost triangular for $k = n+1$) layer. We have shown therefore that these weights have multiplicity both less than or equal to, and greater than or equal to $k$. Hence these weights have multiplicity $k$. Next consider the states on the level $k$ edges which are obtained by acting with the $\mathcal{L}(SU(2))$ lowering operators $E^1_-$ and $E^2_-$ on the "corner weight" states. Observe the following lemma, whose proof is left as an exercise:

**Lemma 4.** *Let $d$ be a representation of $\mathcal{L}(SU(2))$ on $V$ be such that a particular $\mathcal{L}(SU(2))$ weight $m > 0$ has multiplicity $p$. Then all weights $m'$ such that $|m'| \leq m$ have multiplicity $\geq p$*

By this lemma, all the states on the $k$-th layer obtained in this fashion have multiplicity $k$ also. Then the reflection symmetry implies that all states on the $k$-th layer have multiplicity $k$. In particular, the states on the outer triangular layer have multiplicity $n+1$. We have shown that the states on the triangular layers must have multiplicity not greater than $n+1$, but by the lemma above together with the reflection symmetry, they must also have multiplicity $\geq n+1$. Hence the triangular layer weights have multiplicity $n+1$, and the proof is complete.

This was rather long-winded. There exist general formulae constraining multiplicities of weights in more general Lie group representations, but we will not discuss these here.

### 1.1.4 Dimension of Irreducible Representations

Using the multiplicity properties of the weight diagram, it is possible to compute the dimension of the representation. We consider first the hexagonal weight diagram for $m \geq n$.

Then there are $1 + \cdots + (m-n) + (m-n+1) = \frac{1}{2}(m-n+1)(m-n+2)$ weights in the interior triangle. Each of these weights has multiplicity $(n+1)$ which gives $\frac{1}{2}(n+1)(m-$

$n+1)(m-n+2)$ linearly independent states corresponding to weights in the triangle. Consider next the $k$-th hexagonal layer for $k=1,\dots,n$. This has $3((m+1-(k-1))+(n+1-(k-1))-2)=3(m+n+2-2k)$ weights in it, and each weight has multiplicity $k$, which gives $3k(m+n+2-2k)$ linearly independent states in the $k$-th hexagonal layer.

The total number of linearly independent states is then given by

$$\frac{1}{2}(n+1)(m-n+1)(m-n+2)+\sum_{k=1}^{n}3k(m+n+2-2k)=\frac{1}{2}(m+1)(n+1)(m+n+2) \quad (1.31)$$

This formula also applies in the case for $m \leq n$ and also for the triangular weight diagrams by taking $m=0$ or $n=0$. The lowest dimensional representations are therefore 1,3,6,8,10...

### 1.1.5 The Complex Conjugate Representation

**Definition 2.** *Let $d$ be a representation of a Lie algebra $\mathcal{L}(G)$ acting on $V$. If $v \in \mathcal{L}(G)$, then viewing $d(v)$ as a matrix acting on $V$, the complex representation $\bar{d}$ is defined by*

$$\bar{d}(v)u=(d(v))^* u \quad (1.32)$$

*for $u \in V$, where $*$ denotes matrix complex conjugation.*

Note that as $d(v)$ is linear in $v$ over $\mathbb{R}$, it follows that $(d(v))^*$ is also linear in $v$ over $\mathbb{R}$. Also, as

$$d([v,w])=d(v)d(w)-d(w)d(v) \quad (1.33)$$

for $v,w \in \mathcal{L}(G)$, so taking the complex conjugate of both sides we find

$$\bar{d}([v,w])=\bar{d}(v)\bar{d}(w)-\bar{d}(w)\bar{d}(v) \quad (1.34)$$

i.e. $\bar{d}$ is indeed a Lie algebra representation. Suppose that $T_a$ are the generators of $\mathcal{L}(G)$ with structure constants $c_{ab}{}^c$. Then as $d$ is a representation,

$$[d(T_a),d(T_b)]=c_{ab}{}^c d(T_c) \quad (1.35)$$

Taking the complex conjugate, and recalling that $c_{ab}{}^c$ are real, we find

$$[\bar{d}(T_a),\bar{d}(T_b)]=c_{ab}{}^c \bar{d}(T_c) \quad (1.36)$$

i.e. the $d(T_a)$ and $\bar{d}(T_a)$ satisfy the same commutation relations.

In the context of representations of $\mathcal{L}(SU(3))$, the conjugate operators to $iH_1$, $iH_2$, $i(E^m_+ + E^m_-)$ and $E^m_+ - E^m_-$ are denoted by $i\bar{H}_1$, $i\bar{H}_2$, $i(\bar{E}^m_- + \bar{E}^m_+)$, and $\bar{E}^m_+ - \bar{E}^m_-$ respectively and are given by

$$\begin{aligned} i\bar{H}_1 &= (iH_1)^* \\ i\bar{H}_2 &= (iH_2)^* \\ i(\bar{E}^m_- + \bar{E}^m_+) &= (i(E^m_+ + E^m_-))^* \\ \bar{E}^m_+ - \bar{E}^m_- &= (E^m_+ - E^m_-)^* \end{aligned} \quad (1.37)$$

which implies

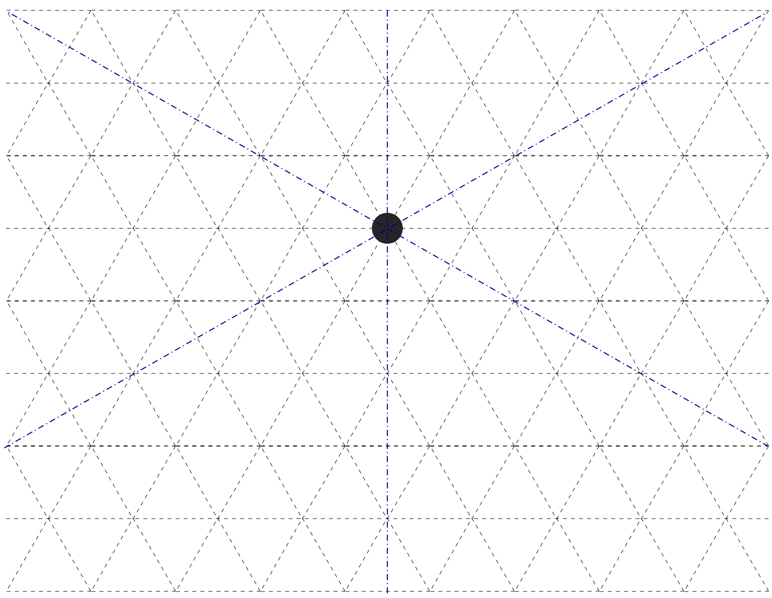
$$\bar{H}_1 = -(H_1)^*, \quad \bar{H}_2 = -(H_2)^*, \quad \bar{E}^m_{\pm} = -(E^m_{\mp})^* \tag{1.38}$$

Then $\bar{H}_1$, $\bar{H}_2$ and $\bar{E}^m_{\pm}$ satisfy the same commutation relations as the unbarred operators, and also behave in the same way under the hermitian conjugate. One can therefore plot the weight diagram associated with the conjugate representation $\bar{d}$, the weights being the (real) eigenvalues of $\bar{H}_1$ and $\bar{H}_2$. But as $\bar{H}_1 = -(H_1)^*$ and $\bar{H}_2 = -(H_2)^*$ it follows that if $(p, q)$ is a weight of the representation $d$, then $(-p, -q)$ is a weight of the representation $\bar{d}$. So the weight diagram of $\bar{d}$ is obtained from that of $d$ by inverting all the points $(p, q) \to -(p, q)$. Note that this means that the equilateral triangular weight diagrams ▲ and ▼ of equal length sides are conjugate to each other.

## 1.2 Some Low-Dimensional Irreducible Representations of $\mathcal{L}(SU(3))$

### 1.2.1 The Singlet

The simplest representation has only one state, which is the highest weight state with weight $(0, 0)$. This representation is denoted **1**.

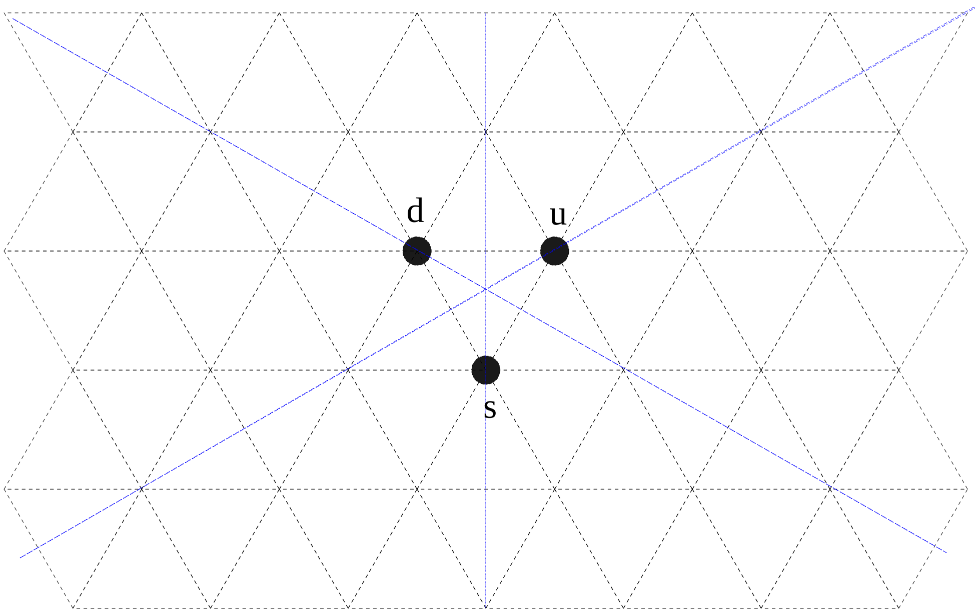


### 1.2.2 3-dimensional Representations

Take the fundamental representation. Then as $h_1$ and $h_2$ are already diagonalized, it is straightforward to compute the eigenstates and weights.

| *State* | *Weight* |
|---|---|
| $\begin{pmatrix}1\\0\\0\end{pmatrix}$ | $(\frac{1}{2}, \frac{1}{2\sqrt{3}})$ |
| $\begin{pmatrix}0\\1\\0\end{pmatrix}$ | $(-\frac{1}{2}, \frac{1}{2\sqrt{3}})$ |
| $\begin{pmatrix}0\\0\\1\end{pmatrix}$ | $(0, -\frac{1}{\sqrt{3}})$ |

The state of highest weight is $\begin{pmatrix}1\\0\\0\end{pmatrix}$ which has weight $(\frac{1}{2}, \frac{1}{2\sqrt{3}})$. The weight diagram is



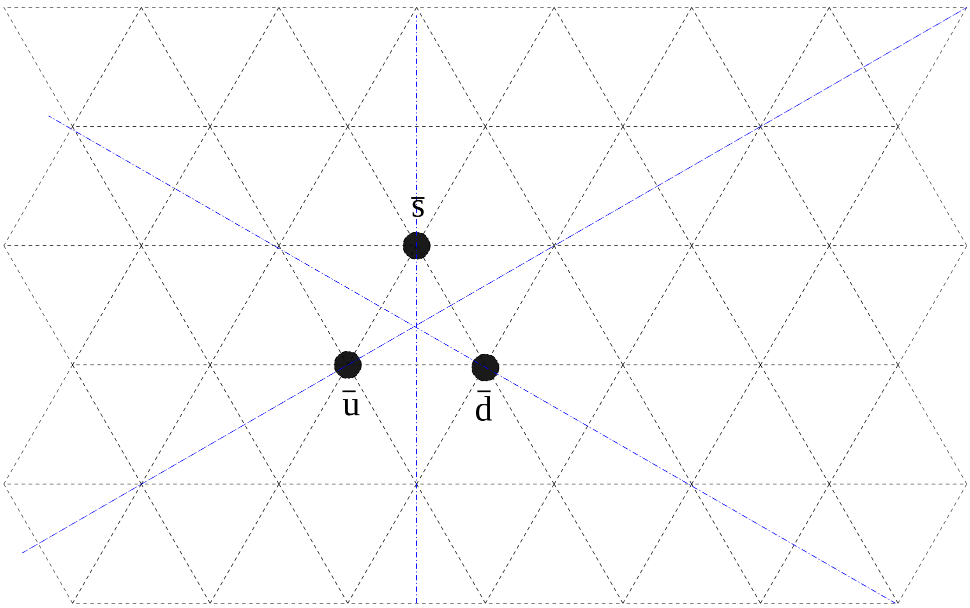

This representation is denoted **3**. It will be convenient to define the following states in the **3** representation.

$$u = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad d = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad s = \begin{pmatrix}0\\0\\1\end{pmatrix} \tag{1.39}$$

so that $u$ has weight $(\frac{1}{2}, \frac{1}{2\sqrt{3}})$, $d$ has weight $(-\frac{1}{2}, \frac{1}{2\sqrt{3}})$ and $s$ has weight $(0, -\frac{1}{\sqrt{3}})$. The lowering operators have the following effect: $d = \sqrt{2}e^1_- u$, $s = \sqrt{2}e^3_- u$ and $s = \sqrt{2}e^2_- d$. The complex conjugate of this representation is called $\bar{\mathbf{3}}$ and the weights are obtained by multiplying the weights of the **3** representation by $-1$.

| *State* | *Weight* |
|---|---|
| $\begin{pmatrix}1\\0\\0\end{pmatrix}$ | $(-\frac{1}{2}, -\frac{1}{2\sqrt{3}})$ |
| $\begin{pmatrix}0\\1\\0\end{pmatrix}$ | $(\frac{1}{2}, -\frac{1}{2\sqrt{3}})$ |
| $\begin{pmatrix}0\\0\\1\end{pmatrix}$ | $(0, \frac{1}{\sqrt{3}})$ |

The state of highest weight is $\begin{pmatrix}0\\0\\1\end{pmatrix}$ which has weight $(0, \frac{1}{\sqrt{3}})$. The weight diagram is




It will be convenient to define the following states in the $\bar{\mathbf{3}}$ representation.

$$\bar{u} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \bar{d} = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \bar{s} = \begin{pmatrix}0\\0\\1\end{pmatrix} \tag{1.40}$$

so that $\bar{u}$ has weight $(-\frac{1}{2}, -\frac{1}{2\sqrt{3}})$, $\bar{d}$ has weight $(\frac{1}{2}, -\frac{1}{2\sqrt{3}})$ and $\bar{s}$ has weight $(0, \frac{1}{\sqrt{3}})$. The lowering operators have the following effect: $\bar{u} = -\sqrt{2}\bar{e}^3_-\bar{s}$, $\bar{d} = -\sqrt{2}\bar{e}^2_-\bar{s}$ and $\bar{u} = -\sqrt{2}\bar{e}^1_-\bar{d}$; where $\bar{e}^m_\pm = -(e^m_\mp)^*$.

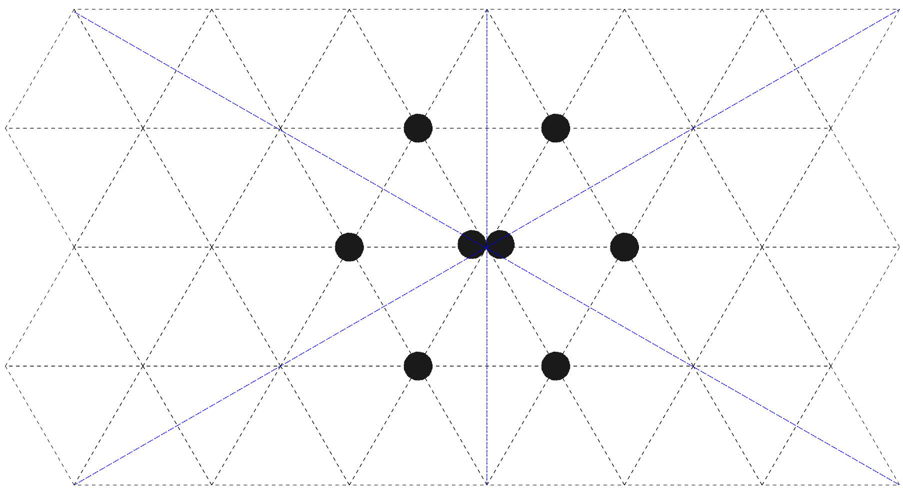
<u>**Exercise:**</u> Verify that all other lowering operators $\bar{e}^m_-$ (except those given above) annihilate $\bar{u}$, $\bar{d}$, $\bar{s}$. Also compute the effect of the raising operators $\bar{e}^m_+$.

### 1.2.3 Eight-Dimensional Representations

Consider the adjoint representation defined on the *complexified* Lie algebra $\mathcal{L}(SU(3))$, i.e. $\mathrm{ad}(v)w = [v, w]$. Then the weights of the states can be computed by evaluating the commutators with $h_1$ and $h_2$:

| *State* $v$ | $[h_1, v]$ | $[h_2, v]$ | *Weight* |
|---|---|---|---|
| $h_1$ | $0$ | $0$ | $(0, 0)$ |
| $h_2$ | $0$ | $0$ | $(0, 0)$ |
| $e^1_+$ | $e^1_+$ | $0$ | $(1, 0)$ |
| $e^1_-$ | $-e^1_-$ | $0$ | $(-1, 0)$ |
| $e^2_+$ | $-\frac{1}{2}e^2_+$ | $\frac{\sqrt{3}}{2}e^2_+$ | $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ |
| $e^2_-$ | $\frac{1}{2}e^2_-$ | $-\frac{\sqrt{3}}{2}e^2_-$ | $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$ |
| $e^3_+$ | $\frac{1}{2}e^3_+$ | $\frac{\sqrt{3}}{2}e^3_+$ | $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ |
| $e^3_-$ | $-\frac{1}{2}e^3_-$ | $-\frac{\sqrt{3}}{2}e^3_-$ | $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$ |

The highest weight state is $e^3_+$ with weight $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. All weights have multiplicity 1 except for $(0, 0)$ which has multiplicity 2. The weight diagram is a regular hexagon:



## 1.3 Tensor Product Representations

Suppose that $d_1$, $d_2$ are irreducible representations of $\mathcal{L}(SU(3))$ acting on $V_1$, $V_2$ respectively. Then let $V = V_1 \bigotimes V_2$ and $d = d_1 \otimes 1 + 1 \otimes d_2$ be the tensor product representation of $\mathcal{L}(SU(3))$ on $V$. In general $d$ is not irreducible on $V$, so we want to decompose $V$ into a direct sum of invariant subspaces on which the restriction of $d$ is irreducible.

To do this, recall that one can choose a basis of $V_1$ which consists entirely of eigenstates of both $d_1(h_1)$ and $d_1(h_2)$. Similarly, one can also choose a basis of $V_2$ which consists entirely of eigenstates of both $d_2(h_1)$ and $d_2(h_2)$. Then the tensor product of the basis eigenstates produces a basis of $V_1 \bigotimes V_2$ which consists of eigenstates of $d(h_1)$ and $d(h_2)$.

Explicitly, suppose that $|\phi_i\rangle \in V_i$ is an eigenstate of $d_i(h_1)$ and $d_i(h_2)$ with weight $(p_i, q_i)$ (i.e. $d_i(h_1)\,|\phi_i\rangle = p_i\,|\phi_i\rangle$ and $d_i(h_2)\,|\phi_i\rangle = q_i\,|\phi_i\rangle$) for $i = 1, 2$. Define $|\phi\rangle =$

$|\phi_1\rangle \otimes |\phi_2\rangle$. Then

$$\begin{aligned} d(h_1)\,|\phi\rangle &= (d_1(h_1)\,|\phi_1\rangle) \otimes |\phi_2\rangle + |\phi_1\rangle \otimes (d_2(h_1)\,|\phi_2\rangle) \\ &= (p_1\,|\phi_1\rangle) \otimes |\phi_2\rangle + |\phi_1\rangle \otimes (p_2\,|\phi_2\rangle) \\ &= (p_1 + p_2)\,|\phi\rangle \end{aligned} \tag{1.41}$$

and similarly

$$d(h_2)\,|\phi\rangle = (q_1 + q_2)\,|\phi\rangle \tag{1.42}$$

So the weight of $|\phi\rangle$ is $(p_1+p_2, q_1+q_2)$; the weights add in the tensor product representation.

Using this, one can plot a weight diagram consisting of the weights of all the eigenstates in the tensor product basis of $V$, the points in the weight diagram are obtained by adding the pairs of weights from the weight diagrams of $d_1$ and $d_2$ respectively, keeping careful track of the multiplicities (as the same point in the tensor product weight diagram may be obtained from adding weights from different states in $V_1 \bigotimes V_2$.)

Once the tensor product weight diagram is constructed, pick a highest weight, which corresponds to a state which is annihilated by the tensor product operators $E^m_+$ for $m = 1, 2, 3$. (Note that as the representation is finite-dimensional such a state is guaranteed to exist, though as the representation is no longer irreducible, it need not be unique). If there are multiple highest weight states corresponding to the same highest weight, one can without loss of generality take them to be mutually orthogonal. Picking one of these, generate further states by acting on a highest weight state with all possible combinations of lowering operators. The span of these (finite number) of states produces an invariant subspace $W_1$ of $V$ on which $d$ is irreducible. Remove these weights from the tensor product weight diagram. If the multiplicity of one of the weights in the original tensor product diagram is $k$, and the multiplicity of the same weight in the $W_1$ weight diagram is $k'$ then on removing the $W_1$ weights, the multiplicity of that weight must be reduced from $k$ to $k-k'$. Repeat this process until there are no more weights left. This produces a decomposition $V = W_1 \bigoplus ... \bigoplus W_k$ of $V$ into invariant subspaces $W_j$ on which $d$ is irreducible.

Note that one could also perform this process on triple (and higher order) tensor products e.g. $V_1 \bigotimes V_2 \bigotimes V_3$. In this case, one would construct the tensor product weight diagram by adding triplets of weights from the weight diagrams of $d_1$ on $V_1$, $d_2$ on $V_2$ and $d_3$ on $V_3$ respectively.

This process can be done entirely using the weight diagrams, because we have shown that for irreducible representations, the location of the highest weight fixes uniquely the shape of the weight diagram and the multiplicities of its states.

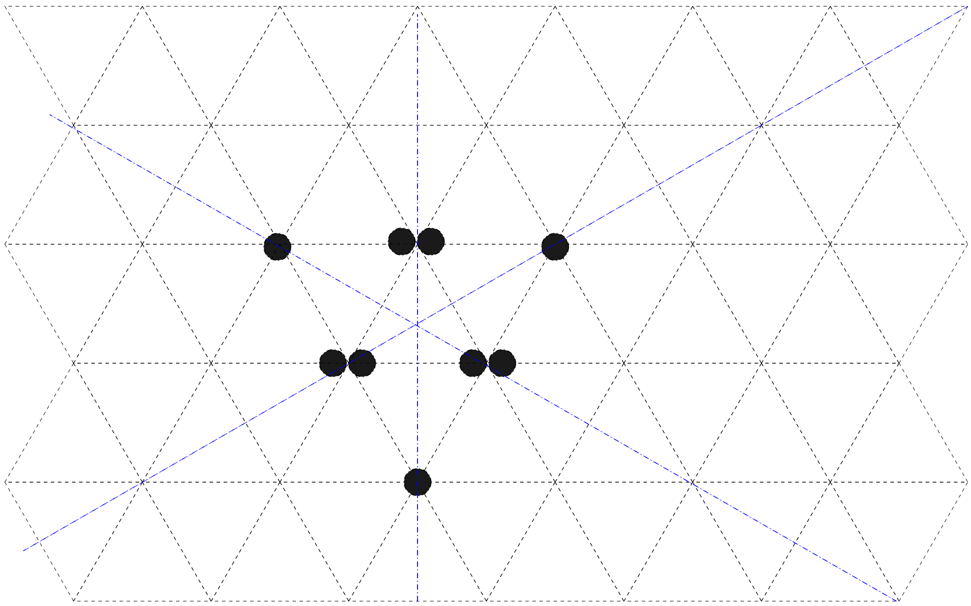
We will see how this works for some simple examples:

### 1.3.1 $\mathbf{3} \otimes \mathbf{3}$ decomposition.

Consider the $\mathbf{3} \otimes \mathbf{3}$ tensor product. Adding the weights together one obtains the following table of quark content and associated weights

| Quark content and weights for $\mathbf{3} \otimes \mathbf{3}$ | |
|---|---|
| *Quark Content* | *Weight* |
| $u \otimes u$ | $(1, \frac{1}{\sqrt{3}})$ |
| $d \otimes d$ | $(-1, \frac{1}{\sqrt{3}})$ |
| $s \otimes s$ | $(0, -\frac{2}{\sqrt{3}})$ |
| $u \otimes d,\ d \otimes u$ | $(0, \frac{1}{\sqrt{3}})$ |
| $u \otimes s,\ s \otimes u$ | $(\frac{1}{2}, -\frac{1}{2\sqrt{3}})$ |
| $d \otimes s,\ s \otimes d$ | $(-\frac{1}{2}, -\frac{1}{2\sqrt{3}})$ |

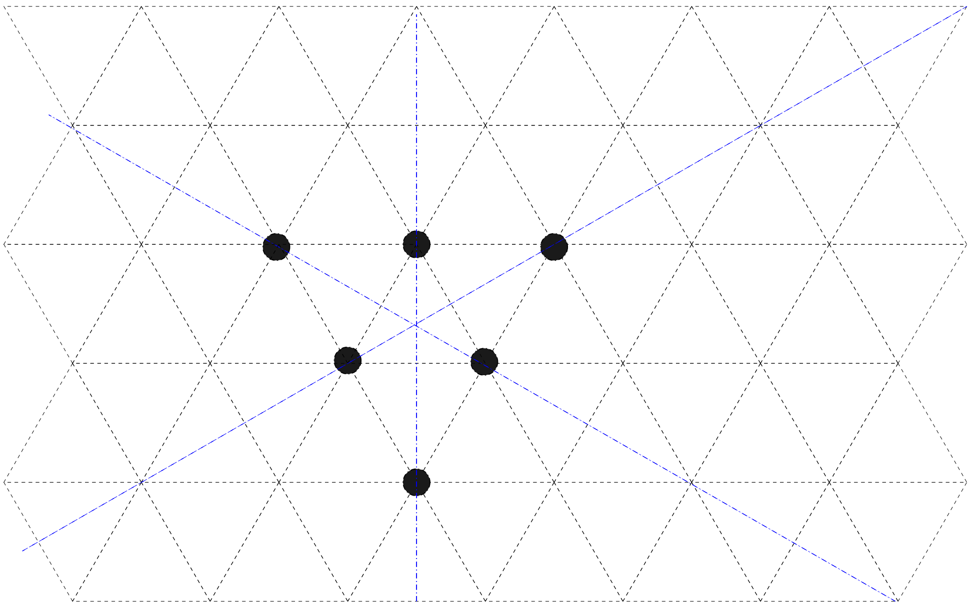
Plotting the corresponding weight diagram gives



The raising and lowering operators are $E^m_{\pm} = e^m_{\pm} \otimes 1 + 1 \otimes e^m_{\pm}$. The highest weight state is $u \otimes u$ with weight $(1, \frac{1}{\sqrt{3}})$. Applying lowering operators to $u \otimes u$ it is clear that a six-dimensional irreducible representation is obtained. The (unit-normalized) states and weights are given by

| States and weights for the $\mathbf{6}$ in $\mathbf{3} \otimes \mathbf{3}$ | |
|---|---|
| *State* | *Weight* |
| $u \otimes u$ | $(1, \frac{1}{\sqrt{3}})$ |
| $d \otimes d$ | $(-1, \frac{1}{\sqrt{3}})$ |
| $s \otimes s$ | $(0, -\frac{2}{\sqrt{3}})$ |
| $\frac{1}{\sqrt{2}}(d \otimes u + u \otimes d)$ | $(0, \frac{1}{\sqrt{3}})$ |
| $\frac{1}{\sqrt{2}}(u \otimes s + s \otimes u)$ | $(\frac{1}{2}, -\frac{1}{2\sqrt{3}})$ |
| $\frac{1}{\sqrt{2}}(d \otimes s + s \otimes d)$ | $(-\frac{1}{2}, -\frac{1}{2\sqrt{3}})$ |

which has the following weight diagram



This representation is called **6**. Removing the (non-vanishing) span of these states from the tensor product space, one is left with a 3-dimensional vector space. The new highest weight is at $(0, \frac{1}{\sqrt{3}})$ with corresponding state $\frac{1}{\sqrt{2}}(d \otimes u - u \otimes d)$ (this is the unique linear combination- up to overall scale- of $d \otimes u$ and $u \otimes d$ which is annihilated by all the raising operators). This generates a $\bar{\mathbf{3}}$. The states and their weights are

| States and weights for the $\bar{\mathbf{3}}$ in $\mathbf{3} \otimes \mathbf{3}$ | |
|---|---|
| *State* | *Weight* |
| $\frac{1}{\sqrt{2}}(d \otimes u - u \otimes d)$ | $(0, \frac{1}{\sqrt{3}})$ |
| $\frac{1}{\sqrt{2}}(d \otimes s - s \otimes d)$ | $(-\frac{1}{2}, -\frac{1}{2\sqrt{3}})$ |
| $\frac{1}{\sqrt{2}}(s \otimes u - u \otimes s)$ | $(\frac{1}{2}, -\frac{1}{2\sqrt{3}})$ |

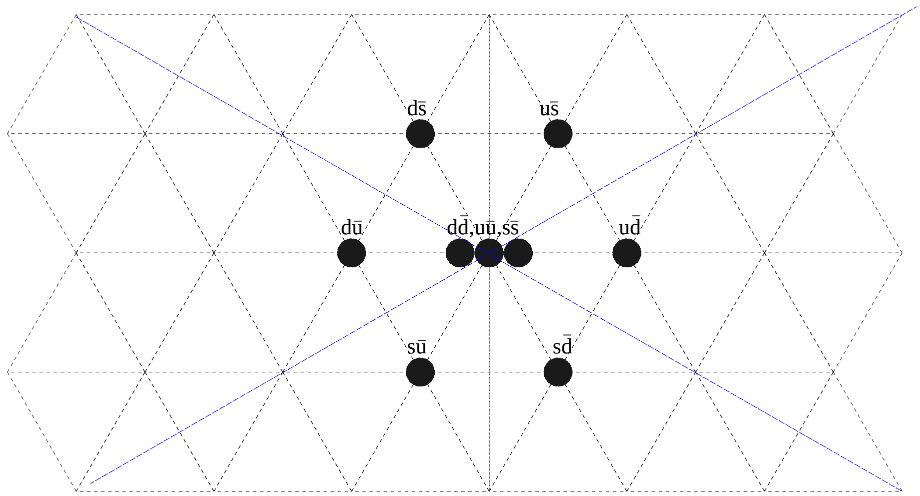
Hence $\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \bar{\mathbf{3}}$. The states in the **6** are symmetric, whereas those in the $\bar{\mathbf{3}}$ are antisymmetric.

### 1.3.2 $\mathbf{3} \otimes \bar{\mathbf{3}}$ decomposition

For this tensor product the quark content/weight table is as follows:

| Quark content and weights for $\mathbf{3} \otimes \bar{\mathbf{3}}$ | |
|---|---|
| *Quark Content* | *Weight* |
| $u \otimes \bar{s}$ | $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ |
| $u \otimes \bar{d}$ | $(1, 0)$ |
| $d \otimes \bar{s}$ | $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ |
| $u \otimes \bar{u}$, $d \otimes \bar{d}$, $s \otimes \bar{s}$ | $(0, 0)$ |
| $d \otimes \bar{u}$ | $(-1, 0)$ |
| $s \otimes \bar{u}$ | $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$ |
| $s \otimes \bar{d}$ | $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$ |

with weight diagram




The raising and lowering operators are $E^m_\pm = e^m_\pm \otimes 1 + 1 \otimes \bar{e}^m_\pm$ All weights have multiplicity 1, except for $(0, 0)$ which has multiplicity 3. The highest weight state is $u \otimes \bar{s}$ with weight $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. Acting on this state with all possible lowering operators one obtains an **8** with the following states and weights

| States and weights for the $\mathbf{8}$ in $\mathbf{3} \otimes \bar{\mathbf{3}}$ | |
|---|---|
| *State* | *Weight* |
| $u \otimes \bar{s}$ | $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ |
| $u \otimes \bar{d}$ | $(1, 0)$ |
| $d \otimes \bar{s}$ | $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ |
| $\frac{1}{\sqrt{2}}(d \otimes \bar{d} - u \otimes \bar{u})$, $\frac{1}{\sqrt{6}}(d \otimes \bar{d} + u \otimes \bar{u} - 2s \otimes \bar{s})$ | $(0, 0)$ |
| $d \otimes \bar{u}$ | $(-1, 0)$ |
| $s \otimes \bar{u}$ | $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$ |
| $s \otimes \bar{d}$ | $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$ |

Removing these weights from the weight diagram, one is left with a singlet **1** with weight $(0,0)$, corresponding to the state

$$\frac{1}{\sqrt{3}}(u \otimes \bar{u} + s \otimes \bar{s} + d \otimes \bar{d}) \tag{1.43}$$

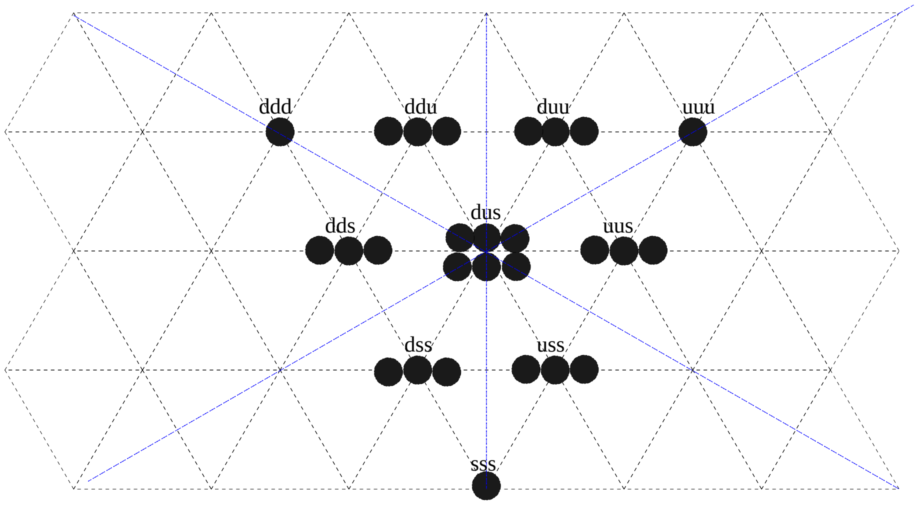
which is the unique linear combination- up to an overall scale- of $u \otimes \bar{u}$, $s \otimes \bar{s}$ and $d \otimes \bar{d}$ which is annihilated by the raising operators $E^m_+$. Hence we have the decomposition $\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$.

### 1.3.3 $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$ decomposition.

For this tensor product the quark content/weight table is as follows:

| Quark content and weights for $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$ | |
|---|---|
| *Quark Content* | *Weight* |
| $u \otimes u \otimes u$ | $(\frac{3}{2}, \frac{\sqrt{3}}{2})$ |
| $s \otimes s \otimes s$ | $(0, -\sqrt{3})$ |
| $d \otimes d \otimes d$ | $(-\frac{3}{2}, \frac{\sqrt{3}}{2})$ |
| $u \otimes u \otimes s$, $u \otimes s \otimes u$, $s \otimes u \otimes u$ | $(1, 0)$ |
| $u \otimes u \otimes d$, $u \otimes d \otimes u$, $d \otimes u \otimes u$ | $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ |
| $s \otimes s \otimes u$, $s \otimes u \otimes s$, $u \otimes s \otimes s$ | $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$ |
| $s \otimes s \otimes d$, $s \otimes d \otimes s$, $d \otimes s \otimes s$ | $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$ |
| $d \otimes d \otimes s$, $d \otimes s \otimes d$, $s \otimes d \otimes d$ | $(-1, 0)$ |
| $d \otimes d \otimes u$, $d \otimes u \otimes d$, $u \otimes d \otimes d$ | $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ |
| $u \otimes d \otimes s$, $u \otimes s \otimes d$, $d \otimes u \otimes s$, $d \otimes s \otimes u$, $s \otimes u \otimes d$, $s \otimes d \otimes u$ | $(0, 0)$ |

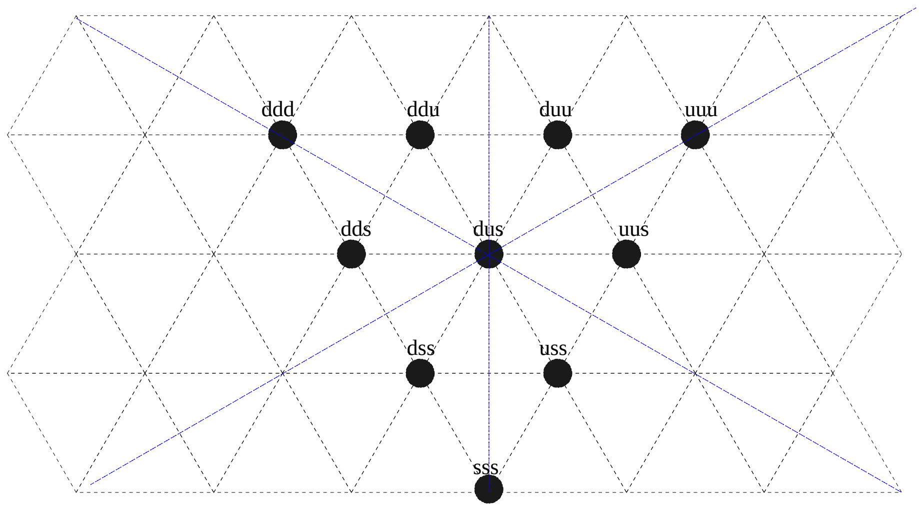
with weight diagram



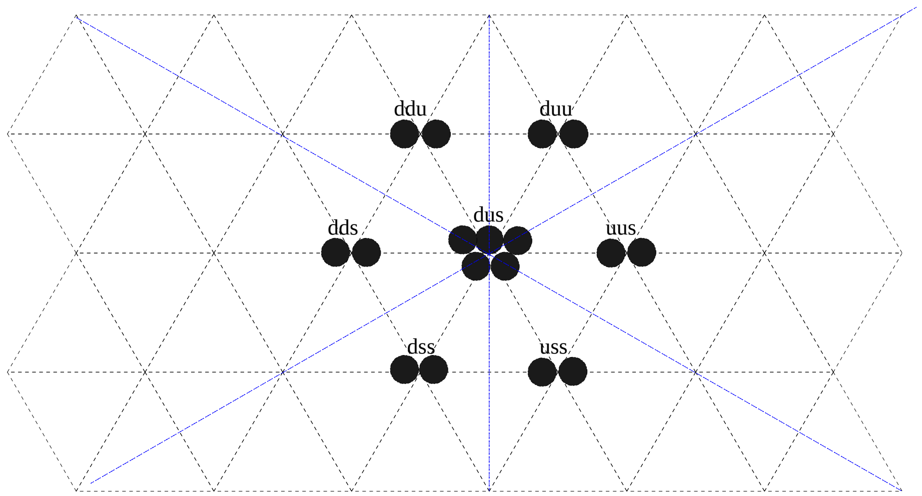

The raising and lowering operators are $E^m_\pm = e^m_\pm \otimes 1 \otimes 1 + 1 \otimes e^m_\pm \otimes 1 + 1 \otimes 1 \otimes e^m_\pm$. There are six weights of multiplicity 3, and the weight $(0,0)$ has multiplicity 6. The highest weight is $u \otimes u \otimes u$ with weight $(\frac{3}{2}, \frac{\sqrt{3}}{2})$. By applying lowering operators to this state, one obtains a triangular 10-dimensional irreducible representation denoted by **10**, which has normalized states and weights:

| States and weights for $\mathbf{10}$ in $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$ | |
|---|---|
| *State* | *Weight* |
| $u \otimes u \otimes u$ | $(\frac{3}{2}, \frac{\sqrt{3}}{2})$ |
| $s \otimes s \otimes s$ | $(0, -\sqrt{3})$ |
| $d \otimes d \otimes d$ | $(-\frac{3}{2}, \frac{\sqrt{3}}{2})$ |
| $\frac{1}{\sqrt{3}}(u \otimes u \otimes s + u \otimes s \otimes u + s \otimes u \otimes u)$ | $(1, 0)$ |
| $\frac{1}{\sqrt{3}}(u \otimes u \otimes d + u \otimes d \otimes u + d \otimes u \otimes u)$ | $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ |
| $\frac{1}{\sqrt{3}}(s \otimes s \otimes u + s \otimes u \otimes s + u \otimes s \otimes s)$ | $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$ |
| $\frac{1}{\sqrt{3}}(s \otimes s \otimes d + s \otimes d \otimes s + d \otimes s \otimes s)$ | $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$ |
| $\frac{1}{\sqrt{3}}(d \otimes d \otimes s + d \otimes s \otimes d + s \otimes d \otimes d)$ | $(-1, 0)$ |
| $\frac{1}{\sqrt{3}}(d \otimes d \otimes u + d \otimes u \otimes d + u \otimes d \otimes d)$ | $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ |
| $\frac{1}{\sqrt{6}}(u \otimes d \otimes s + u \otimes s \otimes d + d \otimes u \otimes s +$ $d \otimes s \otimes u + s \otimes u \otimes d + s \otimes d \otimes u)$ | $(0, 0)$ |

The **10** weight diagram is




Removing the (non-vanishing) span of these states from the tensor product space, one is left with a 17-dimensional vector space. The new weight diagram is




Note that the highest weight is now $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. This weight has has multiplicity 2. It should be noted that the subspace consisting of linear combinations of $d \otimes u \otimes u$, $u \otimes d \otimes u$ and $u \otimes u \otimes d$ which is annihilated by all raising operators $E^m_+$ is two-dimensional and is spanned by the two orthogonal states $\frac{1}{\sqrt{6}}(d \otimes u \otimes u + u \otimes d \otimes u - 2u \otimes u \otimes d)$ and $\frac{1}{\sqrt{2}}(d \otimes u \otimes u - u \otimes d \otimes u)$. By acting on these two states with all possible lowering operators, one obtains two **8** representations whose states are mutually orthogonal.

The states and weights of these two **8** representations are summarized below:

| States and weights for an **8** in $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$ | |
|---|---|
| *State* | *Weight* |
| $\frac{1}{\sqrt{6}}(d \otimes u \otimes u + u \otimes d \otimes u - 2u \otimes u \otimes d)$ | $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ |
| $\frac{1}{\sqrt{6}}(s \otimes u \otimes u + u \otimes s \otimes u - 2u \otimes u \otimes s)$ | $(1, 0)$ |
| $\frac{1}{\sqrt{6}}(2d \otimes d \otimes u - d \otimes u \otimes d - u \otimes d \otimes d)$ | $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ |
| $\frac{1}{2\sqrt{3}}(s \otimes d \otimes u + s \otimes u \otimes d + d \otimes s \otimes u$ $+u \otimes s \otimes d - 2d \otimes u \otimes s - 2u \otimes d \otimes s)$, $\frac{1}{2\sqrt{3}}(2s \otimes d \otimes u + 2d \otimes s \otimes u - s \otimes u \otimes d$ $-d \otimes u \otimes s - u \otimes s \otimes d - u \otimes d \otimes s)$ | $(0, 0)$ |
| $\frac{1}{\sqrt{6}}(s \otimes d \otimes d + d \otimes s \otimes d - 2d \otimes d \otimes s)$ | $(-1, 0)$ |
| $\frac{1}{\sqrt{6}}(2s \otimes s \otimes u - s \otimes u \otimes s - u \otimes s \otimes s)$ | $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$ |
| $\frac{1}{\sqrt{6}}(2s \otimes s \otimes d - s \otimes d \otimes s - d \otimes s \otimes s)$ | $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$ |

| States and weights for another $\mathbf{8}$ in $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}$ | |
|---|---|
| *State* | *Weight* |
| $\frac{1}{\sqrt{2}}(d\otimes u\otimes u - u\otimes d\otimes u)$ | $(\frac{1}{2},\frac{\sqrt{3}}{2})$ |
| $\frac{1}{\sqrt{2}}(s\otimes u\otimes u - u\otimes s\otimes u)$ | $(1,0)$ |
| $\frac{1}{\sqrt{2}}(d\otimes u\otimes d - u\otimes d\otimes d)$ | $(-\frac{1}{2},\frac{\sqrt{3}}{2})$ |
| $\frac{1}{2}(s\otimes d\otimes u + s\otimes u\otimes d - d\otimes s\otimes u - u\otimes s\otimes d)$,<br>$\frac{1}{2}(s\otimes u\otimes d + d\otimes u\otimes s - u\otimes s\otimes d - u\otimes d\otimes s)$ | $(0,0)$ |
| $\frac{1}{\sqrt{2}}(s\otimes d\otimes d - d\otimes s\otimes d)$ | $(-1,0)$ |
| $\frac{1}{\sqrt{2}}(s\otimes u\otimes s - u\otimes s\otimes s)$ | $(\frac{1}{2},-\frac{\sqrt{3}}{2})$ |
| $\frac{1}{\sqrt{2}}(s\otimes d\otimes s - d\otimes s\otimes s)$ | $(-\frac{1}{2},-\frac{\sqrt{3}}{2})$ |

Removing these weights from the weight diagram, we are left with a singlet $\mathbf{1}$ with weight $(0,0)$. The state corresponding to this singlet is

$$\frac{1}{\sqrt{6}}(s\otimes d\otimes u - s\otimes u\otimes d + d\otimes u\otimes s - d\otimes s\otimes u + u\otimes s\otimes d - u\otimes d\otimes s) \tag{1.44}$$

which is the only linear combination-up to overall scale- of $u\otimes d\otimes s$, $u\otimes s\otimes d$, $d\otimes u\otimes s$, $d\otimes s\otimes u$, $s\otimes u\otimes d$ and $s\otimes d\otimes u$ which is annihilated by all the raising operators.

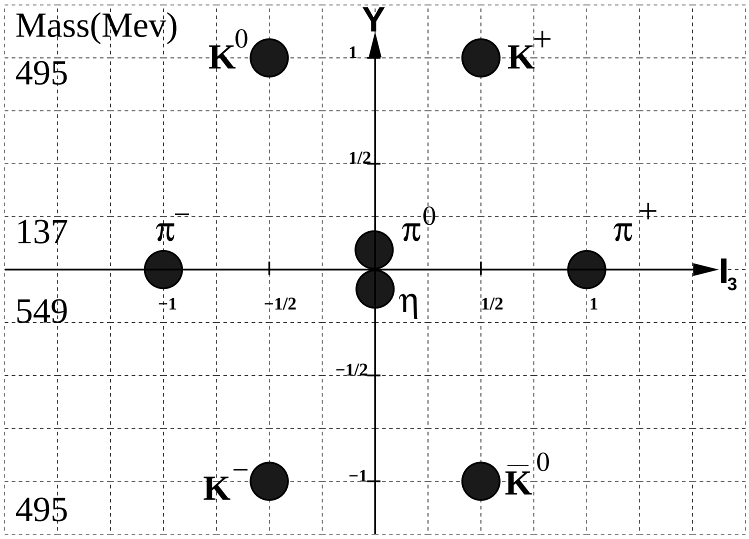
Hence we have the decomposition $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3} = \mathbf{10}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{1}$ where the states in $\mathbf{10}$ are symmetric, but the state in $\mathbf{1}$ is antisymmetric. The $\mathbf{8}$ states have mixed symmetry.

### 1.4 The Quark Model

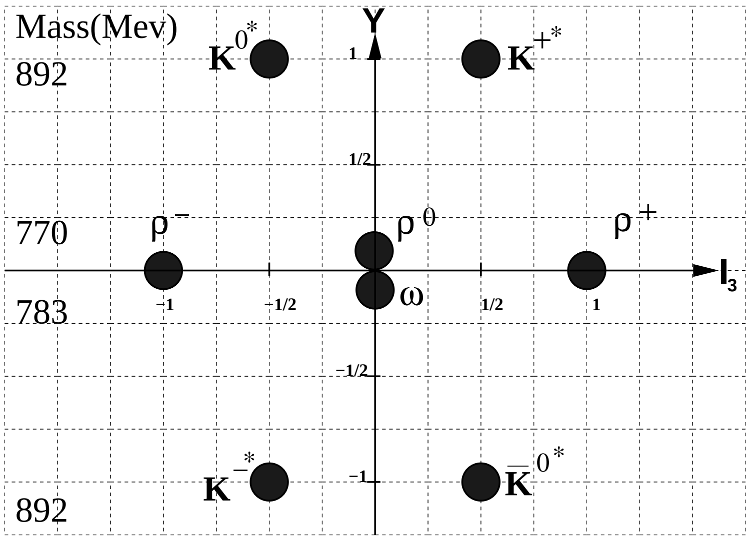
It is possible to arrange the baryons and the mesons into $SU(3)$ multiplets; i.e. the states lie in Hilbert spaces which are tensor products of vector spaces equipped with irreducible representations of $\mathcal{L}(SU(3))$. To see examples of this, it is convenient to group hadrons into multiplets with the same baryon number and spin. We plot the hypercharge $Y = S + B$ where $S$ is the strangeness and $B$ is the baryon number against the isospin eigenvalue $I_3$ for these particles.

#### 1.4.1 Meson Multiplets

The pseudoscalar meson octet has $B = 0$ and $J = 0$. The $(I_3, Y)$ diagram is



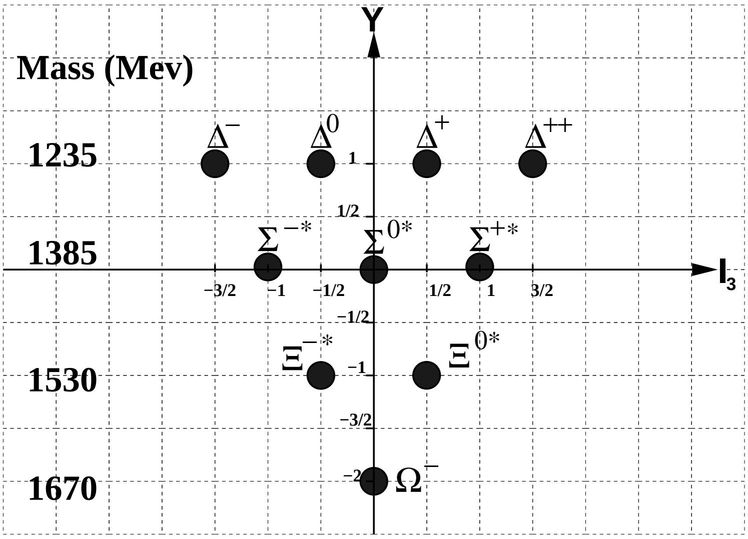

There is also a $J = 0$ meson singlet $\eta'$. The vector meson octet has $B = 0$ and $J = 1$. The $(I_3, Y)$ diagram is



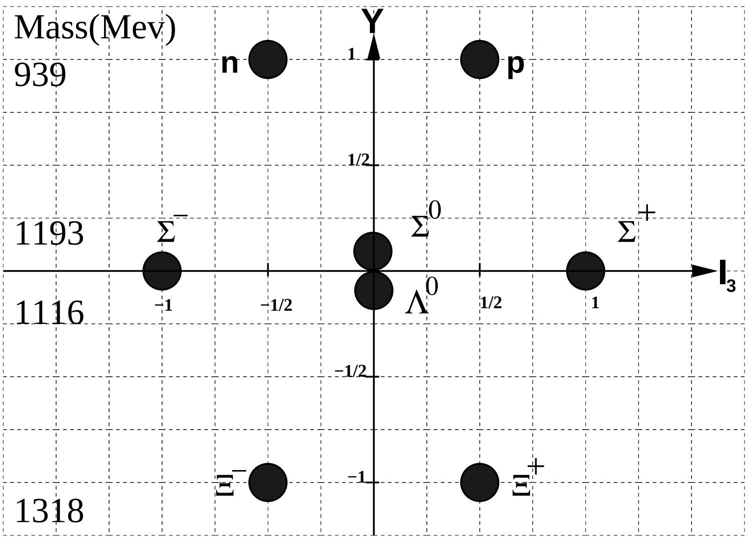

There is also a $J = 1$ meson singlet, $\phi$.

### 1.4.2 Baryon Multiplets

The baryon decuplet has $B = 1$ and $J = \frac{3}{2}$ with $(I_3, Y)$ diagram




There is also an antibaryon decuplet with $(I_3, Y) \to -(I_3, Y)$. The baryon octet has $B = 1$, $J = \frac{1}{2}$ with $(I_3, Y)$ diagram




and there is also a $J = \frac{1}{2}$ baryon singlet $\Lambda^{0*}$.

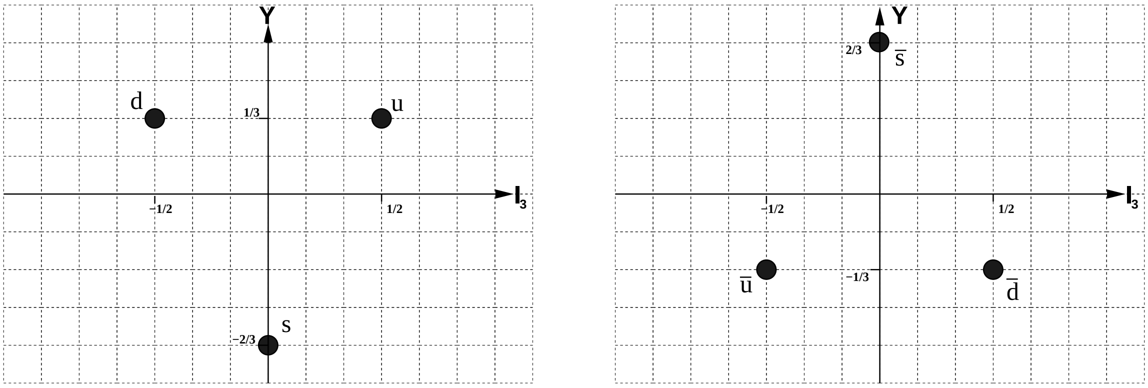
### 1.4.3 Quarks: Flavour and Colour

On making the identification $(p, q) = (I_3, \frac{\sqrt{3}}{2}Y)$ the points on the meson and baryon octets and the baryon decuplet can be matched to points on the weight diagrams of the $\mathbf{8}$ and $\mathbf{10}$ of $\mathcal{L}(SU(3))$.

Motivated by this, it is consistent to consider the (light) meson states as lying within a $\mathbf{3} \otimes \bar{\mathbf{3}}$; as $\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$, the meson octets are taken to correspond to the $\mathbf{8}$ states, and the meson singlets correspond to the singlet $\mathbf{1}$ states. The light baryon states lie within a $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$; the baryon decuplet corresponds to the $\mathbf{10}$ in $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}$; the baryon octet corresponds to appropriate linear combinations of elements in the $\mathbf{8}$ irreps, and the baryon singlet corresponds to the $\mathbf{1}$.

In this model, the fundamental states in the $\mathbf{3}$ are quarks, with basis states $u$ (up), $d$ (down) and $s$ (strange). The basis labels $u, d, s$ are referred to as the *flavours* of the quarks. The $\bar{\mathbf{3}}$ states are called antiquarks with basis $\bar{u}$, $\bar{d}$, $\bar{s}$. Baryons are composed of bound states of three quarks $qqq$, mesons are composed of bound states of pairs of quarks and antiquarks $q\bar{q}$. The quarks have $J = \frac{1}{2}$ and $B = \frac{1}{3}$ whereas the antiquarks have $J = \frac{1}{2}$ and $B = -\frac{1}{3}$ which is consistent with the values of $B$ and $J$ for the baryons and mesons.

The quark and antiquark flavours can be plotted on the $(I_3, Y)$ plane:




We have shown that mesons and baryons can be constructed from $q\bar{q}$ and $qqq$ states respectively. But why do $qq$ particles not exist? This problem is resolved using the notion of colour. Consider the $\Delta^{++}$ particle in the baryon decuplet. This is a $u \otimes u \otimes u$ state with $J = \frac{3}{2}$. The members of the decuplet are the spin $\frac{3}{2}$ baryons of lowest mass, so we assume that the quarks have vanishing orbital angular momentum. Then the spin $J = \frac{3}{2}$ is obtained by having all the quarks in the spin up state, i.e. $u \uparrow \otimes u \uparrow \otimes u \uparrow$. However, this violates the Pauli exclusion principle. To get round this problem, it is conjectured that quarks possess additional labels other than flavour. In particular, quarks have additional charges called *colour* charges- there are three colour basis states associated with quarks called $r$ (red), $g$ (green) and $b$ (blue). The quark state wave-functions contain colour factors which lie in a $\mathbf{3}$ representation of $SU(3)$ which describes their colour; the colour of antiquark states corresponds to a $\bar{\mathbf{3}}$ representation of $SU(3)$ (colour). This colour $SU(3)$ is independent of the flavour $SU(3)$.

These colour charges are also required to remove certain discrepancies (of powers of 3) between experimentally observed processes such as the decay $\pi^0 \to 2\gamma$ and the cross section ratio between the processes $e^+e^- \to$ hadrons and $e^+e^- \to \mu^+\mu^-$ and theoretical predictions. However, although colour plays an important role in these processes, it seems that one cannot measure colour directly experimentally- all known mesons and baryons are $SU(3)$ colour singlets (so colour is confined). This principle excludes the possibility of having $qq$ particles, as there is no singlet state in the $SU(3)$ (colour) tensor product decomposition $\mathbf{3} \otimes \mathbf{3}$, though there is in $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$ and $\mathbf{3} \otimes \bar{\mathbf{3}}$. Other products of $\mathbf{3}$ and $\bar{\mathbf{3}}$ can also be ruled out in this fashion.

Nevertheless, the decomposition of $\mathbf{3} \otimes \mathbf{3}$ is useful because it is known that in addition to the $u$, $d$ and $s$ quark states, there are also $c$ (charmed), $t$ (top) and $b$ (bottom) quark flavours. However, the $c$, $t$ and $b$ quarks are heavier than the $u$, $d$ and $s$ quarks, and are unstable- they decay into the lighter quarks. The $SU(3)$ symmetry cannot be meaningfully extended to a naive $SU(6)$ symmetry because of the large mass differences which break the symmetry. In this context, meson states formed from a heavy antiquark and a light quark can only be reliably put into **3** multiplets, whereas baryons made from one heavy and two light quarks lie in $\mathbf{3} \otimes \mathbf{3} = 6 \oplus \bar{\mathbf{3}}$ multiplets.